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ABSOLUTE SUMMABILITY BY RIESZ MEANS

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In this paper we ensure the absolute Riesz summability of Lebesgue-Fourier series under more liberal conditions imposed upon the generating function of Lebesgue-Fourier series and by taking more general type of Riesz means than whatever the present author has previously taken in proving the corresponding result. Also we give a refinement over the criterion previously proved by author himself.

1. Definitions and notations. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Throughout the paper we suppose that

$$\lambda_n = \mu_0 + \mu_1 + \mu_2 + \cdots + \mu_n \longrightarrow \infty, \text{ as } n \longrightarrow \infty . \tag{1.1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{\lambda_n}\sum_{\nu=0}^{n} \mu_\nu s_\nu , \tag{1.2}$$

defines the Riesz means of sequence $\{s_n\}$ (or the series $\sum_{n=0}^{\infty} a_n$) of the type $\{\lambda_{n-1}\}$ and order unity.[1] If $t_n \to s$, as $n \to \infty$, the sequence $\{s_n\}$ is said to be summable $(R, \lambda_{n-1}, 1)$ to the sum s and if, in addition, $\{t_n\} \in BV$,[2] then it is said to be absolutely summable $(R, \lambda_{n-1}, 1)$, or summable $|R, \lambda_{n-1}, 1|$ and symbolically we write $\sum_{n=0}^{\infty} a_n \in |R, \lambda_{n-1}, 1|$.

The series $\sum_{n=1}^{\infty} a_n \in |R, \lambda_{n-1}, 1|$, if

$$\sum_{n=0}^{\infty} \left| \frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}} \sum_{\nu=0}^{n} \lambda_\nu a_{\nu+1} \right| < \infty .$$

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality the constant term of the Lebesgue-Fourier series of $f(t)$ can be taken to be zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 ,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) .$$

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[1] It is some-times called $(\bar{N}, \mu_n)$ mean, or (R, μ_n) mean, or Riesz's discrete mean of 'type' λ_{n-1} and 'order' unity and is, in fact, equivalent to the usually known $(R, \lambda_{n-1}, 1)$ mean. An explicit proof of it is contained in Iyer [**6**]. Also see Dikshit [**3**].

[2] '$\{t_n\} \in BV$' means $\sum_n |\Delta t_n| < \infty$, when $\Delta t_n = t_n - t_{n+1}$.

We use the following notations:

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} . \tag{1.3}$$

$$\Lambda(t) = \frac{1}{t}\int_0^t u d\phi(u) . \tag{1.4}$$

$$K(n, t) = \sum_{\nu=0}^{n} \frac{\lambda_\nu}{(\nu+1)} \sin(\nu+1)t . \tag{1.5}$$

2. Introduction. Recently the present author [2] has established the following theorem concerning the absolute Riesz summability of Lebesgue-Fourier series of the type $\exp(n^\alpha)(0 < \alpha < 1)$ and order unity.

THEOREM A. *If* (i) $\phi(t) \in BV(0, \pi)$ *and* (ii) $\Lambda(t)(\log k/t)^{1+\varepsilon} \in BV(0, \pi)$, *where* $\varepsilon > 0$ *and* $k \geqq \pi e^2$, *then* $\sum_{n=1}^\infty A_n(x) \in |R, \exp(n^\alpha), 1|(0 < \alpha < 1)$.

By using the technique, which Mohanty [7] used in establishing the criterion for the absolute convergence of a Lebesgue-Fourier series at a point, which is the analogue for absolute convergence of the classical Hardy-Littlewood convergence criterion [4, 5], we have recently established the following:

THEOREM B. *If* (i) $\phi(t) \in BV(0, \pi)$, (ii) $\Lambda(t)(\log k/t)^{1+\varepsilon} \in BV(0, \pi)$, *where* $\varepsilon > 0, k \geqq \pi e^2$ *and* (iii) $\{n^\alpha A_n(x)\} \in BV$, *for* $0 < \alpha < 1$, *then* $\sum_{n=1}^\infty |A_n(x)| < \infty$.

The purpose of this paper is to ensure the absolute Riesz summability of Lebesgue-Fourier series under more liberal condition imposed upon the generating function of Lebesgue-Fourier series and taking more general type of Riesz means.

We first prove the following general theorem.

THEOREM 1. *Let, for* $0 < \alpha < 1$, *the strictly increasing sequences* $\{\lambda_n\}$ *and* $\{g(n)\}$, *of nonnegative terms, tending to infinity with* n, *satisfy the following conditions*:

$$\log\frac{\pi}{t} = O\{g(k/t)\}; \text{ as } t \to 0 , \tag{2.1}$$

$$\{\lambda_n/(n+1)\} \nearrow \text{ with } n \geqq n_0 , \tag{2.2}$$

$$n^{1-\alpha}\Delta\lambda_n = O\{\lambda_{n+1}\}; \text{ as } n \to \infty , \tag{2.3}$$

$$(2.4)\qquad \begin{cases} \text{(i)} & \{x/g(x)\}\nearrow \textit{ with } x\,, \\ \text{(ii)} & x\dfrac{d}{dx}\Big(\dfrac{1}{g(k/x)}\Big)\nearrow \textit{ with } x\,, \\ \text{(iii)} & \dfrac{d}{dx}\Big(\dfrac{1}{g(k/x)}\Big)\searrow \textit{ with } x\,. \end{cases}$$

$$(2.5)\qquad \begin{cases} \text{(i)} & \Big[\dfrac{d}{dt}\Big(\dfrac{1}{g(k/t)}\Big)\Big]_{t=1/n} = O\{n/g(n)\}\,, \\ \text{(ii)} & \sum_{n=1}^{\infty}(ng(n))^{-1} < \infty\,. \end{cases}$$

Then, if $\phi(t)\in BV(0,\pi)$ *and* $\Lambda(t)g(k/t)\in BV(0,\pi)$, *the series*

$$\sum_{n=1}^{\infty} A_n(x)\in |R, \lambda_{n-1}, 1|\,,$$

where k *is a suitable positive constant such that* $g(k/t)>0$ *for* $t>0$.

3. We shall use the following order-estimates, uniformly in $0<t\leqq\pi$.

$$(3.1)\qquad K(n,t) = O\{t^{-1}\lambda_n/(n+1)\}\,.$$

$$(3.2)\qquad \int_0^t \frac{\sin(n+1)u}{ug(k/u)}du = O\{1/g(n+1)\}\,.$$

$$(3.3)\qquad \int_0^t \sin(n+1)u\frac{d}{du}\Big(\frac{1}{g(k/u)}\Big)du = O\{1/g(n+1)\}\,.$$

Proof of 3.1. By using Abel's Lemma and (2.2), the proof follows.

Proof of 3.2. *Case* (I). When $(n+1)^{-1}\leqq t$, we have

$$\int_0^t \frac{\sin(n+1)u}{ug(k/u)}du = \Big(\int_0^{(n+1)^{-1}} + \int_{(n+1)^{-1}}^t\Big)\frac{\sin(n+1)u}{ug(k/u)}du$$
$$= I_1 + I_2,\ \text{say}\,.$$

Now, since $|\sin(n+1)u|\leqq(n+1)u$, we have

$$I_1 = O\Big\{(n+1)\int_0^{(n+1)^{-1}}\frac{1}{g(k/u)}du\Big\} = O\{1/g(n+1)\}\,.$$

And, by the second mean value theorem and (2.4)(i) we have

$$I_2 = O\{1/g(n+1)\}\,.$$

Case (II). When $(n+1)^{-1}>t$, we have

$$\int_0^t \frac{\sin (n+1)u}{ug(k/u)}du = \left(\int_0^{(n+1)^{-1}} - \int_t^{(n+1)^{-1}}\right)\frac{\sin (n+1)u}{ug(k/u)}du$$
$$= I_1 - I_2', \text{ say .}$$

Proceeding as in I_1, for I_2', we obtain

$$I_2' = O\{1/g(n+1)\} .$$

This completes the proof.

Proof of (3.3). In view of (2.4)(ii), (2.4)(iii) and (2.5)(i), the proof runs parallel to that of (3.2).

4. We require the following lemmas, for the proof of the theorems.

LEMMA 1. *If* $F(x) \in BV(a, b)$, *then it can be expressed as* $(F_1(x) - F_2(x))$ *where* $F_1(x)$ *and* $F_2(x)$ *are positive, bounded and monotonic increasing functions in* (a, b) (see Carslaw [1], p. 83).

LEMMA 2 (Pati [8]). *If* (i) $\sum_{n=1}^{\infty} a_n \in |R, \lambda_n, k|(k>0)$, (ii) $\{\lambda_n/\lambda_{n+1}\} \in BV$ *and* (iii) $\{a_n\lambda_n/(\lambda_n - \lambda_{n-1})\} \in BV$, *then* $\sum_{n=1}^{\infty} |a_n| < \infty$.

5. **Proof of Theorem 1.** We have

$$\begin{aligned}
A_n(x) &= \frac{2}{\pi}\int_0^{\pi}\phi(t)\cos ntdt \\
&= \frac{2}{\pi}\left[\frac{\sin nt}{n}\phi(t)\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nt}{n}d\phi(t) \\
&= -\frac{2}{\pi}\int_0^{\pi}\frac{\sin nt}{n}d\phi(t) \\
&= -\frac{2}{\pi}\left[\frac{\sin nt}{n}\Lambda(t)\right]_0^{\pi} + \frac{2}{\pi}\int_0^{\pi}\Lambda(t)t\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt \\
&= \frac{2}{\pi}\int_0^{\pi}\Lambda(t)g(k/t)\frac{t}{g(k/t)}\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt ,
\end{aligned}$$

integrating by parts.

In view of Lemma 1 and second mean value theorem, the series $\sum_{n=1}^{\infty} A_n(x) \in |R, \lambda_{n-1}, 1|$, if

$$\Sigma = \sum_{n=0}^{\infty}\left|\frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}}\sum_{\nu=0}^{n}\frac{\lambda_\nu}{(\nu+1)}\int_0^t\frac{u}{g(k/u)}\frac{\partial}{\partial u}\left(\frac{\sin(\nu+1)u}{u}\right)du\right| = O(1) ,$$

uniformly in $0 < t \leqq \pi$. And, now

$$\int_0^t \frac{u}{g(k/u)} \frac{\partial}{\partial u}\left(\frac{\sin(\nu+1)u}{u}\right)du = \frac{\sin(\nu+1)t}{g(k/t)} - \int_0^t \frac{\sin(\nu+1)u}{ug(k/u)}du$$
$$- \int_0^t \sin(\nu+1)u \frac{\partial}{\partial u}\left(\frac{1}{g(k/u)}\right)du .$$

Therefore

$$\begin{aligned}\sum &\leqq \frac{1}{g(k/t)}\sum_{n=0}^{\infty}\left|\frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}}k(n,t)\right| \\ &\quad + \sum_{n=0}^{\infty}\left|\frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}}\sum_{\nu=0}^{n}\frac{\lambda_\nu}{(\nu+1)}\int_0^t \frac{\sin(\nu+1)u}{ug(k/u)}du\right| \\ &\quad + \sum_{n=0}^{\infty}\left|\frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}}\sum_{\nu=0}^{n}\frac{\lambda_\nu}{(\nu+1)}\int_0^t \sin(\nu+1)u\frac{\partial}{\partial u}\left(\frac{1}{g(k/u)}\right)du\right| \\ &= \sum_1 + \sum_2 + \sum_3, \text{ say .}\end{aligned}$$

Now, we write, for $T = [t^{-1/(1-\alpha)}]$

$$\sum_1 = \sum_{n=0}^{T-1} + \sum_{n=T}^{\infty} = \sum_{1,1} + \sum_{1,2}, \text{ say .}$$

Since $\sin(\nu+1)t = O(1)$, we have

$$\begin{aligned}\sum_{1,1} &= O\left\{\frac{1}{g(k/t)}\sum_{n=0}^{T-1}\left|\frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}}\sum_{\nu=0}^{n}\frac{\lambda_\nu}{(\nu+1)}\right|\right\} \\ &= O\left\{\frac{1}{g(k/t)}\sum_{\nu=0}^{T-1}\frac{\lambda_\nu}{(\nu+1)}\sum_{n=\nu}^{T-1}\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\right)\right\} \\ &= O\left\{\frac{1}{g(k/t)}\sum_{\nu=0}^{T-1}\frac{1}{\nu+1}\right\} \\ &= O(1) ,\end{aligned}$$

by (2.1), uniformly in $0 < t \leqq \pi$. And, by (3.1),

$$\begin{aligned}\sum_{1,2} &= O\left\{\frac{t^{-1}}{g(k/t)}\sum_{n=T}^{\infty}\left|\frac{\Delta\lambda_n}{(n+1)\lambda_{n+1}}\right|\right\} \\ &= O\left\{\frac{t^{-1}}{g(k/t)}\sum_{n=T}^{\infty}(n+1)^{\alpha-2}\right\} \\ &\qquad \text{(by (2.3))} \\ &= O\left\{\frac{t^{-1}}{g(k/t)}T^{\alpha-1}\right\} \\ &= O(1) ,\end{aligned}$$

uniformly in $0 < t \leqq \pi$. And, by (3.2), we have

$$\sum_2 = O\left\{\sum_{n=0}^{\infty}\left|\frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}}\sum_{\nu=0}^{n}\frac{\lambda_\nu}{(\nu+1)g(\nu+1)}\right|\right\}$$

$$= O\left\{\sum_{\nu=0}^{\infty} \frac{\lambda_\nu}{(\nu+1)g(\nu+1)} \sum_{n=\nu}^{\infty}\left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}}\right)\right\}$$
$$= O\left\{\sum_{\nu=0}^{\infty} \frac{1}{(\nu+1)g(\nu+1)}\right\}$$
$$= O(1) ,$$

by (2.5)(ii), uniformly in $0 < t \leqq \pi$. Also, by using (3.3), we get

$$\sum_3 = O(1) ,$$

uniformly in $0 < t \leqq \pi$.

This terminates the proof of Theorem 1.

6. In this section we give a criterion for the absolute convergence of Lebesgue-Fourier series at a point. First we consider the following corollary of Theorem 1.

COROLLARY. *If* (i) $\phi(t) \in BV(0, \pi)$ *and* (ii) $\Lambda(t)g(k/t) \in BV(0, \pi)$, *then* $\sum_{n=1}^{\infty} A_n(x) \in |R, \exp(n^\alpha), 1|(0 < \alpha < 1)$, *whenever* $g(k/t)$ *stands for any one of the following functions*:

$$\left(\log \frac{k}{t}\right)^{1+c}, \log \frac{k}{t}\left(\log_2 \frac{k}{t}\right)^{1+c}, \cdots, \log \frac{k}{t} \log_2 \frac{k}{t} \cdots \left(\log_p \frac{k}{t}\right)^{1+c}$$

where $\log_p k/t = \log\log_{p-1} k/t$, $\log_1 k/t = \log k/t$, $c > 0$, *and* k *is any suitable positive constant such that* $g(k/\pi) > 0$.

THEOREM 2. *If* (i) $\phi(t) \in BV(0, \pi)$, (ii) $\Lambda(t)g(k/t) \in BV(0, \pi)$ *and* (iii) $\{n^{1-\alpha}A_n(x)\} \in BV$ *for* $0 < \alpha < 1$, *then* $\sum_{n=1}^{\infty} |A_n(x)| < \infty$, *where* $g(k/t)$ *is as defined as in the above corollary.*

Proof of Theorem 2. Mohanty (7) observed that for $\lambda_n = e^{n^\alpha}$ sequences (i) $\{\lambda_n/\lambda_{n+1}\} \in BV$ and (ii) $\{n^{\alpha-1}\lambda_n\backslash(\lambda_n - \lambda_{n-1})\} \in BV$ and hence the conditions (ii) and (iii) of Lemma 2 are satisfied. Thus, in view of the above corollary, the proof follows by Lemma 2.

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