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APOSYNDETIC PROPERTIES OF UNICOHERENT CONTINUA

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In the first part of this paper the structure of n-aposyndetic continua is studied. In particular, those continua which are n-aposyndetic but fail to be $(n+1)$-aposyndetic are investigated. Unicoherence is shown to be a sufficient condition for an n-aposyndetic continuum to be $(n+1)$-aposyndetic. In the final portion of the paper a stronger form of unicoherence is defined. As a point-wise property, aposyndesis and connected im kleinen are shown to be equivalent in continua with this property.

Throughout this paper a *continuum* is a compact connected metric space and M will denote a continuum. If N is a subcontinuum of M, the interior of N in M will be denoted by int N. Suppose $p \in M$ and F is a closed subset of M such that $p \notin F$. *M is aposyndetic at p with respect to F* if there is a subcontinuum N of M such that $p \in \operatorname{int} N \subset N \subset M - F$. Let n be a positive integer. If M is aposyndetic at p with respect to each subset of M consisting of n points, then M is *n-aposyndetic at p*. M is *n-aposyndetic* if it is n-aposyndetic at each point. By convention if M is 1-aposyndetic then M is said to be aposyndetic.

For other terms not defined herein, see [3], [4] and [6].

LEMMA 1. *Suppose M is n-aposyndetic, $p \in M$, F is a subset of $M - \{p\}$ consisting of $n+1$ points, and M is not aposyndetic at p with respect to F. If F_1 and F_2 are disjoint nonempty subsets of F such that $F = F_1 \cup F_2$, there exist subcontinua H and K such that $F_1 \subset H - K$, $F_2 \subset K - H$, $p \in \operatorname{int}(H \cap K)$, and $M = H \cup K$.*

Proof. Suppose F_1 and F_2 are disjoint nonempty subsets of F and $F = F_1 \cup F_2$. For each $x \in F_1$ there is a subcontinuum N_x in $M - (F - \{x\})$ such that $p \in \operatorname{int} N_x$. Clearly $x \in N_x$. Let $A = \bigcup \{N_x : x \in F_1\}$. For each $x \in F_1$ there is a subcontinuum L_x such that $x \in \operatorname{int} L_x$ and $L_x \cap F_2 = \emptyset$. Let $A_1 = A \cup (\cup \{L_x : x \in F_1\})$. Then A_1 is a continuum, $\{p\} \cup F_1 \subset \operatorname{int} A_1$, and $A_1 \cap F_2 = \emptyset$.

Now by interchanging the roles of F_1 and F_2 we obtain a continuum A_2 such that $\{p\} \cup F_2 \subset \operatorname{int} A_2$ and $A_2 \cap F_1 = \emptyset$.

Let $V = (M - A_1) \cap \operatorname{int} A_2$ and $U = (M - A_2) \cap \operatorname{int} A_1$. Let H be the component of $M - V$ which contains A_1 and let K be the component of $M - U$ which contains A_2. Then $F_1 \subset H - K$, $F_2 \subset K - H$,

$p \in \operatorname{int}(H \cap K)$, and $M = H \cup K$.

THEOREM 1. *Suppose M is n-aposyndetic but fails to be $(n+1)$-aposyndetic. Then for each pair of positive integers i and j such that $i + j = n + 1$, there exist subcontinua H and K such that H is not i-aposyndetic, K is not j-aposyndetic, and $M = H \cup K$.*

Proof. Suppose M is not aposyndetic at p with respect to $F = \{x_1, x_2, \cdots, x_n, x_{n+1}\}$. Let i and j be positive integers such that $i + j = n + 1$. Let $F_i = \{x_1, x_2, \cdots, x_i\}$ and $F_j = \{x_{i+1}, x_{i+2}, \cdots, x_{n+1}\}$. By Lemma 1 there are subcontinua H and K such that $F_i \subset H - K$, $F_j \subset K - H$, $p \in \operatorname{int}(H \cap K)$, and $M = H \cup K$.

Now if H is i-aposyndetic, there is a subcontinuum N in $H - F_i$ and a set V open in H such that $p \in V \subset N$. Let $U = (\operatorname{int}(H \cap K)) \cap V$. Then U is open in M and $p \in U \subset N \subset M - F$. Since this is contrary to the supposition, H is not i-aposyndetic.

In a similar manner, it follows that K fails to be j-aposyndetic.

THEOREM 2. *Let n be a positive integer and suppose M is n-aposyndetic. If M is unicoherent, then M is $(n+1)$-aposyndetic.*

Proof. Suppose M fails to be $(n+1)$-aposyndetic. There is a $p \in M$, a set $F = \{x_0, x_1, \cdots, x_n\}$ consisting of $n+1$ points in $M - \{p\}$, and M is not aposyndetic at p with respect to F. By Lemma 1, there are continua H and K such that $\{x_0\} \subset H - K$, $\{x_1, x_2, \cdots, x_n\} \subset K - H$, $p \in \operatorname{int}(H \cap K)$, and $M = H \cup K$. Since $p \in \operatorname{int}(H \cap K) \subset H \cap K \subset M - F$, it follows that $H \cap K$ is not a continuum. Therefore M fails to be unicoherent.

COROLLARY 1. *Suppose M is unicoherent and aposyndetic. Then for each positive integer n, M is n-aposyndetic.*

A continuum M is said to be *k-coherent* (*finitely coherent*) provided that for each pair of proper subcontinua H and K such that $M = H \cup K$, then $H \cap K$ has at most k components (a finite number of components). Thus unicoherence is the same as 1-coherence.

With obvious modifications, Theorem 2 and Corollary 1 also hold for continua which are finitely coherent.

In [5] Vought proves that a planar continuum is locally connected if and only if it is 2-aposyndetic. By combining this result with Corollary 1 we have the following theorem.

THEOREM 3. *Let M be unicoherent planar continuum. Then M is locally connected if and only if M is aposyndetic.*

The following example shows that the theorem does not hold if M fails to be planar.

EXAMPLE 1. Let M be the product of the cone over the Cantor set with the unit interval. Then M is unicoherent and aposyndetic but is not locally connected.

According to [1, Th. 13, p. 100] and [3, Th. 2, p. 437], each planar continuum which fails to separate the plane is unicoherent. Thus the following theorem is an immediate consequence of Theorem 3.

THEOREM 4. (Jones [2]) *Suppose M is a planar continuum which does not separate the plane. Then M is locally connected if and only if M is aposyndetic.*

A *dendrite* is a locally connected continuum which does not contain a simple closed curve. One of many characterizations of a dendrite is that a continuum is a dendrite if and only if it is one-dimensional, unicoherent, and locally connected [3, Cor. 8, p. 442].

Question. If M is a one-dimensional, unicoherent, aposyndetic continuum, does it follow that M is a dendrite?

It is easily seen that if M is hereditarily unicoherent and aposyndetic, then M is locally connected and hence a dendrite. The following results establish a weaker condition under which aposyndesis and locally connectedness are equivalent.

DEFINITION. A decomposable unicoherent continuum M is *strongly unicoherent* provided that for each pair of proper subcontinua H and K such that $M = H \cup K$, both H and K are unicoherent.

EXAMPLE 2. Let M consist of a ray R and a simple closed curve C such that R limits on C. Clearly M is strongly unicoherent, but not hereditarily unicoherent since it contains the non-unicoherent continuum C.

THEOREM 5. *Suppose M is strongly unicoherent and aposyndetic. Then M is hereditarily decomposable.*

Proof. Let N be a proper subcontinuum of M and let x and y be distinct points of N. Since M is aposyndetic, there exist subcontinua H and K such that $x \in H - K$, $y \in K - H$, and $M = H \cup K$ [2]. Now $H \cup N$ and $K \cup N$ are subcontinua of M and $(H \cup N) \cup K = M =$

$H \cup (K \cup N)$. It follows that $H \cap N$ and $K \cap N$ are nonempty continua and $N = (H \cap N) \cup (K \cap N)$. Thus N is decomposable.

COROLLARY 2. *A strongly unicoherent aposyndetic continuum is one-dimensional.*

THEOREM 6. *Suppose M is strongly unicoherent. Then M is aposyndetic at a point p if and only if M is connected im kleinen at p.*

Proof. If M is connected im kleinen at p, it is immediate that M is aposyndetic at p.

Suppose M is aposyndetic at p and is not connected im kleinen at p. There is an open set U containing p such that M is not aposyndetic at p with respect to $M - U$. This property on "p" is inducible. Thus by the Brower Reduction Theorem [6, Th. 11, p. 17], there is an open set V such that $U \subset V$, M is not aposyndetic at p with respect to $M - V$, but for any open set W properly containing V, M is aposyndetic at p with respect to $M - W$.

Let $x \in M - V$. There is a subcontinuum N in $M - \{x\}$ such that $p \in \operatorname{int} N$.

Assertion. There are proper subcontinua H and K such that $M = H \cup K$, $p \in \operatorname{int} H$, and $x \in K - H$. For if N does not separate M, let $H = N$ and $K = \operatorname{Cl}(M - N)$. If N separates M into disjoint open sets S and T, assume $x \in T$; let $H = N \cup S$, and let $K = N \cup T$.

Let $A = (M - V) \cap H$. If $A = \emptyset$, then $M - V \subset K - H$ which implies that M is aposyndetic at p with respect to $M - V$. So assume $A \neq \emptyset$. Since $M - A$ properly contains V, there is a subcontinuum L in $M - A$ such that $p \in \operatorname{int} L$. Now $p \in [(\operatorname{int} H) \cap (\operatorname{int} L)] \subset L \cap H \subset V$ which implies that $L \cap H$ is not a continuum. Since $M = (L \cup H) \cup K$, this contradicts the strong unicoherence of M.

Therefore M is connected im kleinen at p.

COROLLARY 3. *Suppose M is strongly unicoherent. Then M is aposyndetic if and only if M is locally connected.*

Since a strongly unicoherent aposyndetic continuum is one-dimensional (Corollary 2), we have the following characterization of a dendrite.

THEOREM 7. *A continuum M is a dendrite if and only if M is strongly unicoherent and aposyndetic.*

If the answer to the question proposed above is negative, then the following corollary provides some information concerning the structure of such continua.

COROLLARY 4. *Let M be a unicoherent, aposyndetic, one-dimensional continuum. If M is not a dendrite, there exist proper subcontinua H and K such that $M = H \cup K$ and either H or K fails to be unicoherent.*

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