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ON SATURATED REDUCED PRODUCTS

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The first part of this paper characterizes the filters whose reduced products are saturated with respect to quantifier free formulas. It is shown that filters with this property are exactly good filters whose Boolean algebra is compact. In the second part we investigate their set-theoretical properties and prove that such filters exist.

This paper contains a proof of the result announced in [1] as well as a number of results directly connected with this theorem.

The problem is to characterize those filters which have the property that products reduced by these filters are κ-saturated. The special case of the problem in which we talk about ultrafilters only was completely solved by Keisler in [6]. The above problem was first attacked by F. Galvin and also by Ph. Olin and B. Jónsson (see [5]). They showed that the filter of cofinite subsets over ω is ω_1-saturative. Galvin's results were a little more general. In [11], L. Pacholski and C. Ryll-Nardzewski described those atomless filters which were ω_1-saturative. The author of this article then obtained a characterization of (Φ_0, κ)-saturative filters for any κ. S. Shelah after reading a sketch of this article, obtained, using a different method (see [3]), a characterization of κ-saturative filters, thus solving the problem. (For $\kappa = \omega$, it was independently solved also by L. Pacholski.)

The first part of this paper deals with (Φ_0, κ)-saturatedness of reduced products. It is believed that theorems proved here have some applications to algebra. In possible applications the notion (Φ_0, κ)-saturatedness seems, better suited than the full saturatedness ([10]) because e.g., a solution of a system of equations is expressible by quantifier-free formulas. In the second part we deal with products of different kind of filters and we apply these results to get an existence theorem for excellent filters. The third part is devoted to discussion and open problems.

Our notation is standard. λ, κ stand always for cardinals. 0 is the empty set as well as the least element in Boolean algebras. $S_\omega(X)$ is the set of finite subsets of X. If f is a function from X into Y we write it sometimes as $f: X \rightarrow Y$. If $Z \subseteq X$ then $f^*(Z) = \{f(z) \mid z \in Z\}$ and $f \mid Z$ is the function f restricted to Z. A subset X of a Boolean algebra is said to have the finite intersection property if for any $x_1, \cdots, x_n \in X$ $x_1 \cap \cdots \cap x_n \neq 0$. We write it often as $FIP(X)$. D usually stands for a filter over the set $I = \cup D$. The ideal $\{x \subseteq I \mid I - x \in D\}$

is denoted by J. D is called σ-incomplete if there is a sequence $\{X_n \mid n < \omega\} \subseteqq D$ such that $\bigcap_{n>\omega} X_n = 0$. D is κ-good if it is σ-incomplete and if for every $f: S_\omega(\lambda) \to D$, f decreasing (i.e., if $s \subseteqq s'$ then $f(s') \subseteqq f(s)$) there is a $g: S_\omega(\lambda) \to D$, g multiplicative (i.e., $g(s \cup s') = g(s) \cap g(s')$) and $g(s) \subseteqq f(s)$ for each $s \in S_\omega(\lambda)$. If D is a filter over I we can form the Boolean algebra $S(I)/D$. Its elements are the sets $[x]_D = \{y \mid x \dot{-} y \in J\}$. $[x]_D$ is often written as $[x]$. L is a fixed language. If c is a set of constants then $L(c)$ is the language L expanded by c. Structures are denoted by Gothic letters and their universes by correspoding Latin capitals. Thus the universe of $\mathfrak{A}_2$ is A_2. If φ is a formula of L then an instance of φ is any formula obtained from φ by replacing some of its free variables by constants from L. If $\Phi \subseteqq L$ and c is a set of constants then $\Phi(c)$ is the set of all instances of the formulas from Φ in the language $L(c)$. If $\Phi \subseteqq L$ we say that a structure $\mathfrak{A}$ is (Φ, κ)-saturated if for any c and any set of formulas $\Sigma \subseteqq \Phi(c)$ such that $|\Sigma| < \kappa$, if $\varphi \in \Sigma$ then φ has only v as a free variable and Σ is finitely satisfiable in $(\mathfrak{A}, c^{\mathfrak{A}})$ we can find an $a \in A$ which satisfies all the formulas in Σ. Of course (L, κ^+)-saturatedness coincide with the usual κ^+-saturatedness. The reduced product of the sequence $< \mathfrak{A}_i \mid i \in I >$ of structures of the same similarity type (only such products we consider) is denoted by $\Pi \mathfrak{A}_i/D$. Φ_0 is the set of quantifier-free formulas of L.

1. In this section we define some notions and prove a couple of lemmas which will be useful later.

DEFINITION. A Boolean algebra $\mathscr{B}$ is said to be κ-compact if for any $X \subseteqq B$ such that $FIP(x)$ and $|X| < \kappa$ there is a $b \in B$, $b \neq 0$ and $b \leqq x$ for any $x \in X$.

LEMMA 1. *Let $\mathscr{B}$ be a κ-compact Boolean algebra, let $\lambda < \kappa$ and let $\{x_\alpha \mid \alpha < \lambda\} \subseteqq B - \{0\}$ be a set of atomless elements. Then there is $\{y_\alpha \mid \alpha < \lambda\} \subseteqq B - \{0\}$ such that $y_\alpha \leqq x_\alpha$ and $y_\alpha \cap y_\beta = 0$ for any $\alpha, \beta < \lambda$, $\alpha \neq \beta$.*

Proof. Using Zorn's lemma we find sets Z_ξ for $\xi < \delta$ such that:

(i) $Z_\xi \subseteqq \{x_\alpha \mid \alpha < \lambda\}$

(ii) Z_ξ is a maximal family Z such that $FIP(Z)$ and $Z \subseteqq \{x_\alpha \mid \alpha < \lambda\} - \bigcup_{\zeta > \xi} Z_\zeta$

(iii) $\bigcup_{\xi<\delta} Z_\xi = \{x_\alpha \mid \alpha<\lambda\}$.

By κ-compactness, for each $\xi < \delta$ there is a $c_\xi \neq 0$ such that $c_\xi \leqq x$ for each $x \in Z_\xi$. Clearly, (by (ii)) $c_\xi \cap c_{\xi'} = 0$ if $\xi \neq \xi'$. Moreover $c_\xi' s$ are atomless. So it suffices to show that if $x \in B$ is an atomless element, $x \neq 0$ and $\lambda < \kappa$ then there is $\{z_\alpha \mid \alpha < \lambda\} \subseteqq B - \{0\}$ such that

$z_\alpha \leqq x$ and $z_\alpha \cap z_\beta = 0$ for each $\alpha, \beta < \lambda$, $\alpha \neq \beta$. To show this we define, by induction, the sequence z':

Let $z'_0 = x$. Let us assume that z' has been defined for $\xi < \beta$. If $\beta = \beta' + 1$ let z'_β be a nonzero element which is strictly smaller than $z'_{\beta'}$. If β is limit then by κ-compactness of $\mathscr{B}$ and by atomlessness of x we can find z'_β so that $z'_\beta \neq 0$ and $z'_\beta < z'_\xi$ for $\xi < \beta$. Having the sequence z we define z_α to be $z'_\alpha - z'_{\alpha+1}$ for $\alpha < \lambda$. Evidently, the sequence $\{z_\alpha \mid \alpha < \lambda\}$ has the desired properties.

LEMMA 2. *Let $r: X \to S_\omega(X)$ and let R be a function on $S_\omega(X)$ with the properties that $R(s) \neq 0$ and $R(s) \subseteqq \Pi_{x \in s} r(x)$ for every $s \in S_\omega(X)$. If, in addition, R satisfies the condition* (*),

$$(*) \text{ whenever } s \subseteqq s' \text{ and } f \in R(s') \text{ then } f|s \in R(s)$$

then there is an $f \in \Pi_{x \in X} r(x)$ such that $f|s \in R(s)$ for each $s \in S_\omega(X)$.

Proof. We consider $\Pi_{x \in X} r(x)$ as a usual topogical product of discreet spaces $r(x)$. It is a compact space by Tychonov's theorem. Let $F_s = \{f \in \Pi_{x \in X} r(x) \mid f|s \in R(s)\}$. F_s is closed for each $s \in S_\omega(X)$. The set $\{F_s \mid s \in S_\omega\{X\}\}$ has the finite intersection property as follows from (*), so $\bigcap_{s \in S_\omega(x)} F_s \neq 0$. Any $f \in \bigcap_s F_s$ will satisfy the lemma.

DEFINITION. A structure $\mathscr{T} = \langle T, \leqq \rangle$ is called an upper lattice if $\leqq$ is a partial ordering of T and the lowest upper bound of $\{x, y\}$ exists for any $x, y \in T$; we denote it by $x \vee y$.

A filter D is $\mathscr{T}$-good if it is σ-incomplete and for any $f: T \to D$ which is $\leqq$-decreasing (i.e. if $x \leqq y$ then $f(y) \subseteqq f(x)$) there is a $g: T \to D$ such that $g(x \vee y) = g(x) \cap g(y)$ and $g(x) \subseteqq f(x)$ for every $x, y \in T$.

A filter D is said to be κ-separable if for any $\lambda < \kappa$ and for any $\{x_\alpha \mid \alpha < \lambda\} \subseteqq S(I) - J$ such that $x_\alpha \cap x_\beta \in J$ whenever $\alpha \neq \beta$ there is $\{y_\alpha \mid \alpha < \lambda\} \subseteqq S(I) - J$ such that $y_\alpha \subseteqq x_\alpha$ and $y_\alpha \cap y_\beta = 0$ for any $\alpha, \beta < \lambda$, $\alpha \neq \beta$.

REMARKS. (1) A filter is κ-good in the usual meaning if it is $\langle S_\omega(\lambda), \subseteqq \rangle$-good for any $\lambda < \kappa$.

(2) Every filter is ω_1-separable.

The following lemma can be proved analogously to Theorem 3.2 in [7]. Just replace $\subseteqq$ by $\leqq$ in that proof.

LEMMA 3. *If D is κ-good and $|T| \leqq \kappa$ then D is $\langle T, \leqq \rangle$-good.*

LEMMA 4. *If D is κ-good filter then D is κ-separable.*

Proof. Let $\{x_\alpha \mid \alpha < \lambda\} \subseteqq S(I) - J$ has the property $x_\alpha \cap x_\beta \in J$ if

$\alpha \neq \beta$.

Let

$$f(s) = I - \bigcup \{x_\alpha \cap x_\beta \mid \alpha, \beta \in s \text{ and } \alpha \neq \beta\}$$

for $s \in S_\omega(\lambda)$. Then $f: S_\omega(\lambda) \to D$ so, by κ-goodness, there is a $g: S_\omega(\lambda) \to D$ which is multiplicative and $g(s) \subseteqq f(s)$ for $s \in S_\omega(\lambda)$. Let $y_\alpha = x_\alpha \cap g(\{\alpha\})$ for $\alpha < \lambda$. It is clear that $y_\alpha \notin J$ for $\alpha < \lambda$. Now, if $\alpha \neq \beta$ then

$$y_\alpha \cap y_\beta = x_\alpha \cap x_\beta \cap g(\{\alpha\}) \cap g(\{\beta\}) = x_\alpha \cap x_\beta \cap g(\{\alpha, \beta\}) .$$

Since $g(\{\alpha, \beta\}) \subseteqq (\{\alpha, \beta\}) = I - x_\alpha \cap x_\beta$ we see that $y_\alpha \cap y_\beta = 0$.

DEFINITION. A set $K \subseteqq S(X)$ is called κ-compact if for any $Z \subseteqq K$, $|Z| < \kappa$ and $FIP(Z)$ we have $\cap Z \neq 0$. If $K \subseteqq S(X)$ then $Cl(K)$ will be the smallest set $L \subseteqq S(X)$ such that $K \subseteqq L$ and whenever $x, y \in L$ then $x \cap y$ and $x \cap y$ belong to L (i.e., L is closed under $\cup$ and $\cap$). The following lemma is well known so the proof of it is omitted.

LEMMA 5. *Let $K \subseteqq S(X)$. K is κ-compact iff $Cl(K)$ is κ-compact.*

As it was said in the introduction we will be concerned with (Φ, κ)-saturatedness of reduced products. It is easily seen from definitions that a structure $\mathfrak{A}$ is (Φ_0, κ)-saturated iff the set $\mathscr{D}(\mathfrak{A})$ of all subsets of A which are definable (using parameters) by quantifier-free formulas is κ-compact. The set $\mathscr{D}(\mathfrak{A})$ is the closure under $\cup$ and $\cap$ of the set $\mathscr{D}_0(\mathfrak{A})$, where $\mathscr{D}_0(\mathfrak{A})$ are subsets of A definable by atomic formulas or their negations. Thus, in view of Lemma 5, to show that a structure $\mathfrak{A}$ is (Φ_0, κ)-saturated it is enough to show that $\mathscr{D}_0(\mathfrak{A})$ is κ-compact.

If we analyze the definition of reduced products we see that every member of $\mathscr{D}_0(\Pi\mathfrak{A}_i/D)$ can be expressed in one of the following two forms:

(1) $\{f/D \mid \{i \mid f(i) \in B_i\} \notin J\}$ where $B_i \subseteqq A_i$ for $i \in I$

(2) $\{f/D \mid \{i \mid f(i) \in C_i\} \in J\}$ where $C_i \subseteqq A_i$ for $i \in I$.

(Sets of the form (1) are defined by formulas $R(v_0, \cdots)$, sets of the second kind by formulas $\neg R(v_0, \cdots)$). Since sets of the form (1) ((2)) are determined completely by the sequence $B = \{B_i \mid i \in I\}(C = \{C_i \mid i \in I\})$ we will refer to them as $Q_B(R_C)$, i.e., $Q_B = \{f/D \mid \{i \mid f(i) \in B_i\} \notin J\}$.

2. DEFINITION. A filter D over I is called κ-excellent if it is κ-good and if the Boolean algebra $S(I)/D$ is κ-compact.

The question of existence of such filters will be discussed in a latter section. We are now ready to prove the main theorem.

THEOREM 1. *A filter D is κ-excellent iff for every sequence* $<\mathfrak{A}_i | i \in I>$ *of* structures (of the same type) *the reduced product* $\Pi\mathfrak{A}_i/D$ is *(Φ_0, κ)-saturated.*

Proof. Let us assume that D is a κ-excellent filter over I and $<\mathfrak{A}_i | i \in I>$ is a sequence of structures. Let $\lambda < \kappa$ and let $B_i^\alpha, C_i^\alpha \subseteqq A_i$ for every $i \in I$ and $\alpha < \lambda$. Moreover we assume that the set $K = \{Q_{B^\alpha} | \alpha < \lambda\} \cup \{R_{C^\alpha} | \alpha < \lambda\}$ has finite intersection property. It follows from the discussion of the last section that to prove κ-saturatedness of $\Pi\mathfrak{A}_i/D$ it is enough to show that $\cap K \neq 0$ (K has been chosen arbitrarily). Let us define

$$F_\alpha(s) = \{i \in I | B_i^\beta - \bigcup_{\beta \in s} C_i^\beta \neq 0\}$$

for $s \in S_\omega(\lambda)$ and $\alpha < \lambda$. Because $FIP(K)$ we get that $F_\alpha(s) \notin J$ for every $s \in S_\omega(\lambda)$ and $\alpha < \lambda$. Note that F_α is an increasing function for every $\alpha < \lambda$. Let $\mathscr{B} = S(I)/D$ and let Y_α be the set of all atoms of $\mathscr{B}$ which are contained in $[F_\alpha(s)]$ for every $s \in S_\omega(\lambda)$ (if $X \subseteqq I$ then $[X]$ denoted the corresponding element in the Boolean algebra $S(I)/D$). We now define $X_0, X_1, X_2 \subseteqq \lambda$ as follows:
$\alpha \in X_0$ iff $|Y_\alpha| < \omega$ and $[F_\alpha(s)] = \cup Y_\alpha$ for some $s \in S_\omega(\lambda)$
$\alpha \in X_1$ iff $|Y_\alpha| \geqq \kappa$
$\alpha \in X_2$ iff there is an atomless $b \neq 0$ such that $b \leqq [F_\alpha(s)]$ for every $s \in S_\omega(\lambda)$ and $\alpha \notin X_0 \cup X_1$.
We want to show that $X_0 \cup X_1 \cup X_2 = \lambda$. Let $\alpha \in \lambda - (X_0 \cup X_1)$ and let $\{a_\xi | \xi < \nu\}$ be an enumeration of Y_α. If $\nu < \omega$ then $\{[F_\alpha(s)] - \bigcup_{\xi<\nu} a_\xi | s \in S_\omega(\lambda)\}$ has the finite intersection property hence we can find a $b \neq 0, b \leqq [F_\alpha(s)] - \bigcup_{\xi<\nu} a_\xi$ for every $s \in S_\omega(\lambda)$. Since b must be atomless we get that $\alpha \in X_2$. Since $\alpha \notin X_1$ we must have $\omega \leqq \nu < \kappa$. But then $\{[F_\alpha(s)] - a_\xi | \xi < \nu, s \in S_\omega(\lambda)\}$ has the finite intersection property and similarly as above we get $\alpha \in X_2$.

If $\alpha \in X_0$ let $t_\alpha \in S_\omega(\lambda)$ be an element for which $[F_\alpha(t_\alpha)] \leqq [F_\alpha(s)]$ holds for every $s \in S_\omega(\lambda)$, and let $[a_0^\alpha], \cdots, [a_{n_\alpha - 1}^\alpha]$ be the list of atoms included in $[F_\alpha(t_\alpha)]$. We can choose $a_i^\alpha (\subseteqq I)$ in such a way that

$$[a_i^\alpha] = [a_j^\beta] \text{ implies } a_i^\alpha = a_j^\beta \,. \tag{1}$$

Because $|X_1| \leqq \lambda < \kappa$ and $|Y_\alpha| \leqq \kappa$ we can find a sequence $< b_\alpha | \alpha \in X_1 >$ of subsets of I, satisfying:

$$[b_\alpha] \in Y_\alpha \tag{2}$$

$$[b_\alpha] \neq [b_\beta] \text{ if } \alpha \neq \beta \tag{3}$$

$$(4) \qquad [b_\alpha] \text{ is distinct from } [a_i^\beta] \text{ for every } \beta \in X_0 \text{ and } i < n_\beta .$$

If $\alpha \in X_2$ we let $c_\alpha \subseteqq I$ be such that $[c_\alpha]$ is non-zero and atomless and for which $[c_\alpha] \leqq [F_\alpha(s)]$ holds for every $s \in S_\omega(\lambda)$. Using Lemma 1 we can assume that $\{[c_\alpha] \mid \alpha \in X_2\}$ is a set of mutually disjoint (in $\mathscr{B}$) elements. This assumption and (1)—(4) enable us to use Lemma 4, which tells us that we could have chosen a_i^α, b_α, c_α in such a way that the set $X = \{a_i^\alpha \mid \alpha \in X_0 \text{ and } i < n_\alpha\} \cup \{b_\alpha \mid \alpha \in X_1\} \cup \{c_\alpha \mid \alpha \in X_2\}$ is a set of mutually disjoint subsets of I. Let $\{a_\xi \mid \xi < \mu\}$ be an enumeration (without repetitions) of X. Let $Z_k = \{\xi \mid a_\xi \in X_k\}$ for $k < 3$. Clearly $Z_i \cap Z_j = 0$ if $i \neq j$ and $Z_0 \cup Z_1 \cup Z_2 = \mu$.

For any $\alpha \in X_0$ we can find an $r \in S_\omega(Z_0)$ so that $\{F_\alpha(t_\alpha)] = \bigcup_{\xi \in r} [a_\xi]$. We denote such an r by $r(\alpha)$. If $s \in S_\omega(X_0)$ and $t \in S_\omega(\lambda)$ we let $P(s, t)$ to be the set of all $f \in \Pi A_i$ such that:

(i) $\{i \mid f(i) \in B_i^\alpha\} \notin J$ for $\alpha \in s$

(ii) $\{i \mid f(i) \in C_i^\alpha\} \in J$ for $\alpha \in t \cup \bigcup_{\beta \in s} t_\beta$.

Because $FIP(K)$ holds we have $P(s, t) \neq 0$ for $s \in S_\omega(X_0)$, $t \in S_\omega(\lambda)$. If $f \in P(s, t)$ and $\alpha \in s$ let $\bar{f}(\alpha) = \beta$ iff β is the first ξ for which $\{i \mid f(i) \in B_i^\alpha\} \cap a_\xi \notin J$. It follows from the way we defined t_α's that $\bar{f} \in \Pi_{\alpha \in s} r(a)$. Let $R(s, t) = \{\bar{f} \mid f \in P(s, t)\}$. If $t \subseteqq t'$ then $P(s, t') \subseteqq P(s, t)$ so $R(s, t') \subseteqq R(s, t)$. Thus $R(s) = \bigcap_{t \in S_\omega(\lambda)} R(s, t)$ is nonempty since $\Pi_{\alpha \in s} r(\alpha)$ is a finite set. The function R satisfies the condition (*) of Lemma 2 (with X, Y replaced by X_0, Z_0 resp.). Namely, if $s \subseteqq s'$ and $f \in R(s')$ then for any $t \in S_\omega(\lambda)$ there is a $g \in P(s', t)$ such that $\bar{g} = f$; but such a g is a member of $P(s, t)$ as well as $f \mid s = \bar{g} \mid s \in R(s, t)$. Lemma 2 gives us a function $g \in \Pi_{\alpha \in X_0} r(\alpha)$ with the property that $g \mid s \in R(s)$ for $s \in S_\omega(X_0)$. Let $U_\alpha = g^{-1}(\alpha)$ and let h_α be a mapping from λ onto U_α. We may assume without loss of generality that $U_\alpha \neq 0$ for $\alpha \in Z_0$. For $s, t \in S_\omega(\lambda)$ and $\alpha \in Z_0$ we define:

$$(5) \qquad H_\alpha(s, t) = a_\alpha \cap \{i \mid \bigcap_{\xi \in s} B_i^{h_\alpha(\xi)} - \bigcup_{\beta \in t} C_i^\beta \neq 0\} .$$

We want to show that $H_\alpha(s, t) \notin J$. Firstly, if $u = \{h_\alpha(\xi) \mid \xi \in U_\alpha\}$ then $g^*(u) = \{\alpha\}$. Since $g \mid u \in R(u) \subseteqq R(u, t \cup \bigcup_{\alpha \in u} t_\alpha)$ there is an $f \in P(u, t \cup \bigcup_{\alpha \in u} t_\alpha)$ such that $\bar{f} = g \mid u$. Hence $\{i \mid f(i) \in B_i^{h_\alpha(\xi)}\} \cap a_\alpha \notin J$ for $\xi \in s$ and $\{i \mid f(i) \in C_i^\beta\} \in J$ for each $\beta \in t \cup \bigcup_{\alpha \in u} t_\alpha$. The fact that $[a_\alpha]$ is an atom in $\mathscr{B}$ implies that $\{i \mid f(i) \in \bigcap_{\xi \in s} B_i^{h_\alpha(\xi)}\} \cap a_\alpha \notin J$. It is now obvious that $H_\alpha(s, t) \notin J$.

If $\alpha \in Z_1 \cup Z_2$ (we let U_α be $\{\alpha\}$ and h_α be the mapping from λ onto $\{\alpha\}$. $H_\alpha(s, t)$ for $\alpha \in Z_1 \cup Z_2$ is defined then also by formula (5). Because $h_\alpha(\xi) = \alpha$ for $\alpha \in Z_1 \cup Z_2$ we get $H_\alpha(s, t) \notin J$ immediately.

Let $a = I - \bigcup_{\alpha < \mu} a_\alpha$ and let $\{X_n \mid n < \omega\} \subseteqq D$ be such that $\bigcap_{n < \omega} X_n = 0$. Let

$$H(r, s, t) = (a \cup \bigcup_{\beta \in \mu - r} a_\beta \cup \bigcup_{\beta \in r} H_\beta(s, t)) \cap X_{|r \cup s \cup t|}$$

for $< r, s, t > \in T = S_\omega(\nu) \times S_\omega(\lambda) \times S_\omega(\lambda)$. Then $< T, \leqq >$, where $\leqq$ is the direct product of the three inclusions, is an upper lattice, so, by Lemma 3, D is $< T, \leqq >$-good. Because H maps T into D and H is $\leqq$-decreasing there is a $G: T \to D$ such that $G(r, s, t) \subseteqq H(r, s, t)$ and $G(r \cup r', s \cup s', t \cup t') = G(r, s, t) \cap G(r', s', t')$. Let

$$d_\alpha = a_\alpha \cap G(\{\alpha\}, \{\alpha\}, \{\alpha\}).$$

Then

$$\begin{aligned} G(\{\alpha\}, s, t) \cap d_\alpha &= a_\alpha \cap G(\{\alpha\}, s \cup \{\alpha\}, t \cup \{\alpha\}) \\ &\subseteqq a_\alpha \cap H(\{\alpha\}, s \cup \{\alpha\}, t \cup \{\alpha\}) \\ &= a_\alpha \cap (a \cup \bigcup_{\beta \in \mu - \{\alpha\}} a_\beta \cup H_\alpha(s \cup \{\alpha\}, t \cup \{\alpha\}) \\ &= G_\alpha(s \cup \{\alpha\}, t \cup \{\alpha\}) \subseteqq H_\alpha(s, t) \, . \end{aligned}$$

Thus

$$G(\{\alpha\}, s, t) \cap d_\alpha \subseteqq H_\alpha(s, t) \, . \tag{6}$$

We have that $d_\alpha \cap d_\beta = 0$ if $\alpha \neq \beta$ so if $i \in \bigcup_{\alpha < \mu} d_\alpha$ we let α_i to be that α for which $i \in d_\alpha$. Let $T(i) = \{< \alpha, \beta > \in \lambda \times \lambda \mid i \in G(\{\alpha_i\}, \{\alpha\}, \{\beta\}\}$ for $i \in \bigcup_\alpha d_\alpha$. $|T(i)| < \omega$ because $G(r, s, t) \subseteqq X_{|r \cup s \cup t|}$ and $i \in G(\{\alpha_i\}, V_i, W_i)$ where $V_i = \text{Dom}\, T(i)$ and $W_i = RgT(i)$ (this is due to multiplicativity of G). The last statement is true for every $i \in \bigcup_\alpha d_\alpha$. It follows from (6) that $i \in H_{\alpha_i}(V_i, W_i)$. A function f' then can be defined on $\bigcup_\alpha d_\alpha$ satisfying

$$f'(i) \in \bigcap_{\xi \in V_i} B_i^{h_{\alpha_i}(\xi)} - \bigcup_{\beta \in W_i} C_i^\beta \, .$$

f' is a piece of a function f which will belong to $\bigcap_\alpha Q_{\beta\alpha} \cap \bigcap_\alpha R_{C\alpha}$. We check that this piece has the right properties. Let $\gamma < \lambda$. Hence there are α and β such that $r = h_\alpha(\beta)$. Then

$$\{i \mid (\exists t)[i \in G(\{\alpha\}, \{\beta\}, t)]\} \cap d_\alpha \subseteqq \{i \mid f'(i) \in B_i^{h_\alpha(\beta)}\} \tag{7}$$

and

$$\{i \mid f'(i) \in C_i^\beta\} \cap \bigcup_\alpha d_\alpha \subseteqq I - \bigcup_s \bigcup_{\alpha < \mu} G(\{\alpha\}, s, \{\beta\}) \, . \tag{8}$$

Because the left hand side of (7) does not belong to J it is clear that no matter how we extend f' to a function from ΠA_i the result will belong to $\bigcap_\alpha Q_{\beta\alpha}$. The right side of (8) belongs to J. Hence if we extend f' to f in such a way that $\{i \in I - \bigcup_\alpha d_\alpha \mid f(i) \in C_i^\beta\} \in J$ for $\beta < \lambda$ we are done (J is closed under $\cup$). To do this we define

$$F(s) = (I - \{i \mid \bigcup_{\alpha \in s} C_i^\alpha = A_i\}) \cap X_{|s|}$$

for $s \in S_\omega(\lambda)$. Because $FIP(K)$ we have that $F: S_\omega(\lambda) \to D$. By κ-goodness of D there is a multiplicative function $G: S_\omega(\lambda) \to D$ such that $G(s) \subseteqq F(s)$ for each $s \in S_\omega(\lambda)$. Let $t(i) = \{\alpha \mid i \in (\{\alpha\})\}$. Then $i \in G(t(i)) \subseteqq F(t(i))$ hence we can define f'' on $I - \bigcup_\alpha d_\alpha$ such that $f''(i) \in A_i - \bigcup_{\alpha \in t(i)} C_i^\alpha$. Consequently, $\{i \notin \bigcup_\alpha d_\alpha \mid f''(i) \in C_i^\alpha\} \subseteqq I - G(\{\alpha\}) \in J$ for every $\alpha < \lambda$. As we have explained before $f = f' \cup f''$ is a function beloging to $\bigcap_\alpha Q_{B^\alpha} \cap \bigcap_\alpha R_{C^\alpha}$.

For the other direction we assume that every direct product reduced by D is (Φ_0, κ)-saturated. In particular $2^I/D$ (2 being the 2-element Boolean algebra) is (Φ_0, κ)-saturated. Realizing that $2^I/D = S(I)/D$ and that (Φ_0, κ)-saturatedness of a Boolean algebra is a stronger property than κ-compactness we get that $S(I)/D$ is κ-compact.

To prove that D must be κ-good we take a $\lambda < \kappa$ and a function $f: S_\omega(\lambda) \to D$, f decreasing. Let $A_i = \{s \mid i \in f(s)\}$ and let $\mathfrak{A}_i = \langle A_i, \subseteqq \rangle$. If $s \in S_\omega(\lambda)$ then $f(s) \in D$ so there is a function $\bar{s} \in \Pi A_i$ such that $\{i \mid \bar{s}(i) = s\} \in D$. Let $\Sigma = \{c_s \subseteqq v_0 \mid s \in S_\omega(\lambda)\}$. This is a set of quantifier free formulas of the language $L(c)$. Interpreting the constant c_s in $\Pi\mathfrak{A}_i/D$ by $\bar{s}/D$ we get that Σ is finitely satisfiable in $\Pi\mathfrak{A}_i/D$; using our assumption we can find an $h \in \Pi A_i$ which satisfies Σ, i.e., $\{i \mid \bar{s}(i) \subseteqq h(i)\} \in D$ for each $s \in S_\omega(\lambda)$. Let $g(s) = \{i \mid s \subseteqq h(i)\}$. It is easily seen that $g: S_\omega(\lambda) \to D$ $g(s) \subseteqq f(s)$ for $s \in S_\omega(\lambda)$ and g is multiplicative.

To prove σ-incompleteness of D we let $P_n = \{m < \omega \mid m \geqq n\}$ for, $n < \omega$ and $\mathfrak{A}_\xi = \langle \omega, P_n \rangle_{n<\omega}$. Then $\Sigma = \{P_n(v_0) \mid n < \omega\} \subseteqq \Phi_0$ is finitely satisfiable in $\mathfrak{A}^I/D$, hence there is an $f: I \to \omega$ such that $\{i \mid f(i) \in P_n\} \in D$ for every $n < \omega$. If $X_n = \{i \mid f(i) \in P_n\}$ we have $\{X_n \mid n < \omega\} \subseteqq D$ and $\bigcap_{n<\omega} X_n = 0$. The proof of the theorem is thus completed.

3. Definition. A structure $\mathfrak{A}$ is called rich if for any formula φ of $L(\mathfrak{A})$ there is a relational symbol R of $L(\mathfrak{A})$ such that $\mathfrak{A} \models \varphi \leftrightarrow R$. A theory T in a language L is called open if for any formula φ of L there is a quantifier-free formula ψ such that $T \vdash \varphi \leftrightarrow \psi$.

The proof of the following result can be found in [11].

Theorem 2. *If $\{\mathfrak{A}_i \mid i \in I\}$ is a set of rich structures, D is a filter over I and $S(I)/D$ is atomless then the theory of $\Pi\mathfrak{A}_i/D$ is open.*

Using this result we get the following corollary to Theorem 1.

Corollary 1. *If D is κ-excellent and $\mathscr{B} = S(I)/D$ is atomless then $\Pi\mathfrak{A}_i/D$ is κ-saturated for any sequence $\langle \mathfrak{A}_i \mid i \in I \rangle$.*

Proof. We expand $\mathfrak{A}_i$ to a rich structure $\mathfrak{A}_i'$. By Theorem 2 the theory of $\Pi\mathfrak{A}_i'/D$ is open. By Theorem 1 $\Pi\mathfrak{A}_i'/D$ is (Φ_0, κ)-saturated hence $\Pi\mathfrak{A}_i'/D$ is κ-saturated and so is its reduction $\Pi\mathfrak{A}_i/D$.

We can get a better result than Corollary 1 but to that we need a few more facts.

LEMMA 6 (i). *If* $a \subseteqq I$, $a \notin J$ *and* D *is a* κ*-good filter over* I *then* $D(a)$*—the filter generated by* D *and* $\{a\}$*—is* κ*-good.*

(ii) *If* $a \subseteqq I$ *is such that* $[a]$ *is an atom in* $S(I)/D$ *then* $D(a)$ *is an ultrafilter.*

(iii) *If* $a \in S(I) - J$ *and* $S(I)/D$ *is* κ*-compact then* $S(I)/D(a)$ *is* κ*-compact.*

(iv) *If* $a \in S(I) - J$ *and* $[a]$ *is an atomless element in* $S(I)/D$ *then* $S(I)/D(a)$ *is atomless.*

Proof. (i) is almost obvious. (ii), (iii) and (iv) follows from the fact that $S(I)/D(a) \cong \{[x] \mid [x] \leqq [a]\}$, where the last set is viewed as a subalgebra of $S(I)/D$. The isomorphism is given by $[b]_{D(a)} \to [b \cap a]_D$.

The proof of the next theorem involves different method than those discussed here. For a reference, see e.g., [3].

THEOREM 3. *If* $n < \omega$ *and for every* $j < n$ *the structure* $\mathfrak{A}_j$ *is* κ*-saturated then* $\prod_{j<n} \mathfrak{A}_i$ *is* κ*-saturated.*

COROLLARY 2. *If* D *is* κ*-excellent and* $S(I)/D$ *has finitely many atoms then* $\Pi \mathfrak{A}_i/D$ *is* κ*-saturated.*

Proof. Let $a_0, \cdots, a_{n-1} \subseteqq I$ be mutually disjoint and such that $\{[a_i] \mid i < n\}$ is the set of all atoms of $S(I)/D$. Let $a_n = I - \bigcup_{i<n} a_i$ and let $D_i = D(a_i)$ for $i \leqq n$. If follows from Lemma 6(i) and (ii) that for $i < n$ D_i is a κ-good ultrafilter. Lemma 6(i), (iii) and (iv) imply that D_n is κ-excellent and that the algebra $S(I)/D_n$ is atomless. By Theorem 2.1 of [6] we get that $\Pi \mathfrak{A}_i/D_j$ is κ-saturated for $j < n$. That $\Pi \mathfrak{A}_i/D_n$ is κ-saturated we get using Corollary 1. Note that $D = \bigcap_{j \leqq n} D_j$. Using Theorem 1.2 in [4] we get

$$(\cdot) \Pi \mathfrak{A}_i/D \cong \prod_{j \leqq n} (\Pi \mathfrak{A}_i/D_j) .$$

Theorem 3 yields that the right hand side of $(\cdot)$ is κ-saturated and so is $\Pi \mathfrak{A}_i/D$.

Our next corollary deals with Boolean algebras:

COROLLARY 3. *If a Boolean algebra* $\mathscr{B}$ *has at most finitely many atoms and it is of the form* $S(I)/D$ *where* D *is a* κ*-good filter then* $\mathscr{B}$ *is* κ*-saturated iff* $\mathscr{B}$ *is* κ*-compact.*

Proof. It follows from Corollary 2.

4. Let us pass now to a brief investigation of κ-excellent filters. We will be primarily interested with products of filters but we will also touch upon questions which will naturally arise.

Throughout this section D_k will be a filter over I_k, where $k \leqq 1$. As before we define $J_k = \{x \subseteqq I_k \mid I_k - x \in D_k\}$. If $x \subseteqq I_0 \times I_1$ and $i \in I$, then $x^i = \{j \in I_0 \mid < j, i >\} \in x$. The product $D_0 \times D_1$ of D_0 and D_1 is the following filter over $I_0 \times I_1$:

$$\{x \subseteqq I_0 \times I_1 \mid \{i \mid x^i \in D_1\} \in D_2\} .$$

The following proposition is obvious.

PROPOSITION 1. $D_0 \times D_1$ *is* σ*-incomplete iff* D_0 *or* D_1 *is* σ*-incomplete.*

The next result is known as well and is proved in [8].

PROPOSITION 2. $D_0 \times D_1$ *is* κ*-good iff* D_1 *is.*

DEFINITION. A filter D is called κ-compact if the algebra $S(I)/D$ is κ-compact.

PROPOSITION 3. *If* $D_0 \times D_1$ *is* κ*-compact then* D_1 *is* κ*-compact.*

Proof. Let $\{x_\alpha \mid \alpha < \lambda\} \subseteqq S(I_1) - J_1$ be such that $\{[x_\alpha] \mid \alpha < \lambda\}$ has the finite interesection property. Let $y_\alpha = I_0 \times x_\alpha$. Then $y_\alpha \notin J_0 \times J_1$ and $\{[y_\alpha] \mid \alpha < \lambda\}$ has f.i.p. in $S(I_0 \times I_1)/D_0 \times D_1$. Let $y \subseteqq I_0 \times I_1$ be a set satisfying $y \notin J_0 \times J_1$ and $[y] \leqq [y_\alpha]$ for every $\alpha < \lambda$. Then $x = \{i \in I_0 \mid y^i \notin J_0\} \notin J_1$ hence $[x] \neq 0$ and it is clear that $[x] \leqq [x_\alpha]$ for every $\alpha < \lambda$.

The other implication is false. Take e.g., $I_0 = \omega$, $D_0 = \{\omega\}$, $I_1 = 2$ and $D_1 = \{1\}$. Then $S(2)/D_1$ is finite hence saturated, but $S(I_0 \times I_1)/D_0 \times D_1 = S(I_0 \times I_1)$ is not ω_1-compact. Nevertheless we have a partial reverse.

PROPOSITION 4. *If* D_0 *is an intersection of finitely many ultrafilters over* I_0 *and* D_1 *is* κ*-compact then* $D_0 \times D_1$ *is* κ*-compact.*

Proof. Let D_0 be an intersect on of n distinct ultrafilters. By Theorem 1.2 in [4] we get

$$2^{I_0 \times I_1}/D_0 \times D_1 \cong (2^n)^{I_1}/D_1 .$$

(2^n is the direct product of n copies of the algebra 2).

The referee has suggested the following continuation of the proof which is simpler than our original one. Because $(2^n)^{I_1}/D_1 \cong (2^{I_1}/D_1)^n$ it suffices to show that if $\mathscr{B}_0, \cdots \mathscr{B}_{n-1}$ are κ-compact Boolean algebras then $\mathscr{B} = \mathscr{B}_0 \times \cdots \times \mathscr{B}_{n-1}$ is also κ-compact. Indeed, if $\{b_\alpha \mid \alpha < \lambda\}$ $(\lambda < \kappa)$ has f.i.p. in $\mathscr{B}$ and $b_\alpha = \langle b_{\alpha 0}, \cdots, b_{\alpha n-1}\rangle$ then for some $m < n$ $\{b_{\alpha m} \mid \alpha < \lambda\}$ has f.i.p. in $\mathscr{B}_m$. Let c_m be such that $0 < c_m \leqq b_{\alpha m}$ for $\alpha < \lambda$ and let c_k be 0 for $k \neq m$. Then $0 < \langle c_0, \cdots, c_{n-1}\rangle \leqq b_\alpha$ for every $\alpha < \lambda$.

In order to avoid rigid expressions we adopt the following convention:

CONVENTION. We say that a set $\{z_\alpha \mid \cdots \alpha \cdots\} \subseteq S(X)$ has f.i.p. in $S(X)/Y$ if $[z_\alpha]_Y \neq 0$ and $\{[z_\alpha]_Y \mid \cdots \alpha \cdots\}$ has f.i.p. in $S(X)/Y$ where Y is a filter over X.

We may now ask: What happens if D_0 is not an intersection of finitely many ultrafilters and $D_0 \times D_1$ is κ-compact? We naturally expect that D_1 has some additional property. This property, called fineness is captured in the following definition.

DEFINITION. A filter D over I is called κ-fine if it is σ-incomplete and for every $\lambda < \kappa$ and every function $f\colon S_\omega(\lambda) \to S(I) - J$ which is $\subseteq$-decreasing there is a $g\colon S_\omega(\lambda) \to S(I) - J$ which is multiplicative and $g(s) \subseteq f(s)$ for every $s \in S_\omega(\lambda)$.

REMARK. It follows from the proof of existence of good ultrafilters (see e.g., [7] or [9]) that any filter generated by at most κ sets is κ^+-fine (and it is not κ^+-good). For ultrafilters these notions (good and fine) coincide.

PROPOSITION 5. *If D_0 is not intersection of finitely many ultrafilters, D_1 is λ-regular for every $\lambda < \kappa$ and $D_0 \times D_1$ is κ-compact then D_1 is κ-fine.*

Proof. Since D_0 is not an intersection of finitely many ultrafilters we can find $\{a_i \mid i < \omega\} \subseteq S(I_0) - J_0$ such that $a_i \cap a_j = 0$ if $i \neq j$. Let $\lambda < \kappa$ and let $f\colon S_\omega(\lambda) \to S(I_1) - J_1$ be a decreasing function. Because D_1 is λ-regular we can assume without loss of generality that $\bigcap_{\alpha \in X} f(\{\alpha\}) = 0$ for any infinite $X \subseteq \lambda$. Let $T(i) = \{s \in S_\omega(\lambda) \mid i \in f(s)\}$, where $i \in I_1$. Then for every $i \in I_1$, $|T(i)| < \omega$, so let $T(i)$ be $\{s^i_k \mid k < n(i)\}$. For $\alpha < \lambda$ we define $x_\alpha \subseteq I_0 \times I_1$ by

$$x^i_\alpha = \cup\{a_k \mid \alpha \in s^i_k\},\ i \in I_1 .$$

Now $x_\alpha \notin J_0 \times J_1$ because $\{i \mid \exists s \in T(i))[\alpha \in s]\} \notin J_1$ for any $\alpha < \lambda$. We want to show that

$$(1) \qquad \{i \in I_1 \mid \bigcap_{\alpha \in s} x^i_\alpha \notin J_0\} = f(s) .$$

That the right hand side of (1) is included in the left hand side is obvious. For the other inclusion, if $\bigcap_{\alpha \in s} x^i_\alpha \notin J_0$ there must be a $k < \omega$ such that $a_k \subseteqq \bigcap_{\alpha \in s} x^i_\alpha$. Then $\alpha \in s^i_k$ for every $\alpha \in s$, hence $s \subseteqq s^i_k \in T(i)$, so $i \in f(s)$ as follows from monotonicity of f.

Consequently the set $\{x_\alpha \mid \alpha < \lambda\}$ has f.i.p. in $S(I_0 \times I_1)/D_0 \times D_1$ (see the convention in the previous proof) and by κ-compactness of $D_0 \times D_1$ there is an $x \in S(I_0 \times I_1) - J_0 \times J_1$ such that $[x] \leqq [x_\alpha]$ for each $\alpha < \lambda$. Let $d = \{i \in I_1 \mid x^i \notin J_0\}$ and let $g(\{\alpha\})$ be $\{i \in d \mid x^i - x^i_\alpha \in J_0\}$. We extend g to a function on $S_\omega(\lambda)$ $g(s) = \bigcap_{\alpha \in s} g(\{\alpha\})$. The g is a multiplicative function from into $S(I_1) - J_1$. We have to show now that $g(s) \subseteqq f(s)$ for $s \in S_\omega(\lambda)$. Let $i \in g(s)$. Then $x^i - x^i_\alpha \in J_0$ for each $\alpha \in s$, so $x^i - \bigcap_{\alpha \in s} x^i_\alpha \in J_0$ Since $i \in d$, $x^i \notin J_0$ hence there must be a $k < \omega$ such that $a_k \subseteqq \bigcap_{\alpha \in s} x^i_\alpha$. That means $\alpha \in s^i_k$ for every $\alpha \in s$ hence $s \subseteqq s^i$ which gives $i \in f(s)$.

The result has the following interesting corollary.

COROLLARY. *If D_0 is not an intersection of finitely many ultrafilters, D_1 is λ-regular ultrafilter for every $\lambda < \kappa$ and $D_0 \times D_1$ is κ-compact then D_1 is κ-good.*

As we have seen before compactness of D_1 is not enough to assure compactness of $D_0 \times D_1$. Proposition 5 suggests that an additional property should be assumed about D_1; this suggestion is shown to be true in the next result.

PROPOSITION 6. *If D_1 is κ-compact and κ-fine then $D_0 \times D_1$ is κ-compact and κ-fine.*

Proof. Let $\{x_\alpha \mid \alpha < \lambda\} \subseteqq S(I_0 \times I_1) - J_0 \times J_1$ has f.i.p. in $S(I_0 \times s_1)/D_0 \times D_1$. Let

$$f(s) = \{i \in I_1 \mid \bigcap_{\alpha \in s} x^i_\alpha \notin J_0\}$$

for $s \in S_\omega(\lambda)$. f is a $\subseteqq$-decreasing function into $S(I_1) - J_1$ so by our assumption there is a $g: S_\omega(\lambda) \to S(I_1) - J_1$ which is multiplicative and $g(s) \subseteqq f(s)$ for every $s \in S_\omega(\lambda)$. By κ-compactness of D_1 there is a $y \in S(I_1) - J_1$ such that $y - g(s) \in J_1$ (i.e., $[y] \leqq [g(s)]$) for each $s \in S_\omega(\lambda)$. Put $t(i)$ equal to $\{\alpha \mid i \in g(\{\alpha\})\}$. Because D_1 is σ-incomplete we can assume that $t(i)$ is finite. Let x be that subset of $I_0 \times I_1$ for which the following holds: if $i \in y$ then $x^i = \bigcap_{\alpha \in t(i)} x^i_\alpha$ and if $i \in I_1 - y$ then $x^i = 0$. It is easy to check that $x \notin J_0 \times J_1$ and $[x] \leqq [x_\alpha]$ for each $\alpha < \lambda$. The fact that $D_0 \times D_1$ is κ-fine follows just from the assumption that D_1 is κ-fine. The proof is similar to the proof of Proposition

2 so we leave it to the reader.

PROPOSITION 7. *If D is κ-compact and κ-good then D is κ-fine.*

Proof. Let $f: S_\omega(\lambda) \to S(I) - J$ be $\subseteqq$-decreasing. Thus $\{f(s) \mid s \in S_\omega(\lambda)\}$ has f.i.p. in $S(I)/D$, therefore there is a $Y \subseteqq I$ such that $Y - f(s) \in J$ for every $s \in S_\omega(\lambda)$ and $Y \notin J$. Let $F(s) = I - (Y - f(s))$. F is a function from $S_\omega(\lambda)$ into D, so a function $G: S_\omega(\lambda) \to D$ exists which is multiplicative and $G(s) \subseteqq F(s)$. Let $g(s) = G(s) \cap Y$ for $s \in S_\omega(\lambda)$. Obviously $g: S_\omega(\lambda) \to S(I) - J$ and g is multiplicative. Moreover, for every $s \in S_\omega(\lambda) g(s) = G(s) \cap Y \subseteqq F(s) \cap Y = (I - (Y - f(s))) \cap Y \subseteqq f(s)$. So D is κ-fine.

Using the last result and Propositions 2, 3 and 6 we immediately get

PROPOSITION 8. *$D_0 \times D_1$ is κ-good and κ-compact iff D_1 is. If D_1 is σ-incomplete then we also have that $D_0 \times D_1$ is κ-excellent iff D_1 is κ-excellent.*

The last result yields a simple proof of existence of κ-excellent filters. In fact, for any Boolean algebra $\mathscr{B}$ and any κ there is a filter D which is κ^+-excellent and $\mathscr{B} \equiv S(I)/D$. The argument, which is due to Keisler, runs as follows: using Ershov's theorem (see [2]) we find a filter F over ω such that $\mathscr{B} \equiv S(\omega)/F$. Then we get a κ^+-good σ-incomplete ultrafilter over I' (see [11]) G and we form the filter $D = F \times G$. Now G being an ultrafilter is λ-compact for any λ. Hence by Proposition 8 D is κ^+-excellent. Because $S(I)D \cong 2^I/D \cong (2^\omega/F)^{I'}/G \equiv 2^\omega/F$ we get $S(I)/D \equiv \mathscr{B}$.

5. This section contains a number of remarks relevant to the topics discussed before as well as some open problems.

(a) We have been investigating various properties of filters. It may be interesting to note that these properties can be put into one of the following three groups: (i) properties which have something to do only with elements of the filter: (ii) properties which have something to do with elements of $S(I) - J$; (iii) properties of the Boolean algebra $S(I)/D$. E.g., goodness, regularity and incompleteness are properties of the first kind. To be fine or to be separable are properties of the second kind while compactness is a property of type (iii). Herewith a set of natural questions is connected. Given a class of filters (e.g., those which make reduced products saturated) we can ask whether the class can be characterized by properties listed under, say, (iii). Or we can ask whether "to be κ-excellent" cannot be reduced to a property of type (i) (we believe it can't).

(b) Let Φ be the set of formulas which has the property: if $\varphi \in \Phi$ and D is a filter then $\Pi\mathfrak{A}_i/D \models \varphi(f_0/D, \cdots, f_n/D)$ iff $\{i \mid \mathfrak{A}_i \models \varphi(f_0(i)\cdots)\} \notin J$. If R is a relational symbol then $\neg R \in \Phi$, Φ is closed under v and if $\varphi \in \Phi$ then $(\exists v)\varphi \in \Phi$. Similarly as in [1] we can prove that if D is a κ-fine filter then $\Pi\mathfrak{A}_i/D$ is (Φ, κ)-saturated. What other model-theoretic properties products reduced by fine filters have? The same question can be asked for separable filters.

(c) Are there any natural examples of κ-compact filters? It is proved in [11] that on ω every countably generated σ-incomplete filter is κ-compact. Is the situation similar on larger cardinals? Specifically, if F_κ is the filter over $S_\omega(\lambda)$ generated by the set $\{\{s \mid \alpha \in s\} \mid \alpha < \kappa\}$ is F_κ κ^+-compact?

(d) While ω_1-complete ultrafilters are rather exceptions there are natural and "small" ω-complete filters. Even though they play an important role in set theory their model-theoretical properties have not been studied in detail. It is conceivable that many results about first order properties of reduced products can be extended to $L_{\omega_1\omega_1}$-properties of products reduced by ω_1-complete filters.

(e) Another notion which could be of interest is (κ, λ)-saturatedness. A structure is (κ, λ)-saturated if every set of formulas of power κ is satisfiable in it provided the set has the property that every subset of power $< \lambda$ is satisfiable. Almost every result about saturated structures can be reformulated so a question about (κ, λ)-saturated structures. In connection with this one could ask whether there is a good filter D such that $S(I)/D$ is (κ, ω_1)-compact but not (κ, ω)-compact (i.e., not κ-compact). The referee has pointed out that no ω_1-complete filter makes all structures (ω_1, ω_1)-saturated.

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