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# ON FREDHOLM TRANSFORMATIONS IN YEH-WIENER SPACE

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**Let $C_Y$ denote the Yeh-Wiener space, i.e., the space of all real-valued continuous functions $f(x, y)$ on $I^2 \equiv [0, 1] \times [0, 1]$ such that $f(0, y) = f(x, 0) \equiv 0$. Yeh has defined a Gaussian probability measure on $C_Y$ such that the mean of the process**

$$m(x, y) \equiv \int_{C_Y} f(x, y) d_Y f = 0$$

**and the convariance**

$$R(s, t, x, y) \equiv \int_{C_Y} f(s, t) f(x, y) d_Y f = (1/2) \min(s, x) \min(t, y) .$$

**Consider now a linear transformation of $C_Y$ onto $C_Y$ of the form**

$$\begin{aligned} T: f(x, y) &\to g(x, y) \\ &= f(x, y) + \int_{I^2} K(x, y, s, t) f(s, t) ds dt , \end{aligned} \tag{1.1}$$

**which is often called a Fredholm transformation. The main purpose of this paper is to find the corresponding Radon-Nikodym derivative thus showing how the Yeh-Wiener integrals transform under the transformation.**

The transformations considered here contain the Volterra transformation

$$T_1[f(x, y)] = f(x, y) + \int_0^y \int_0^x K(x, y, s, t) f(s, t) ds dt$$

as a special case.

Such transformations in Wiener space have been studied a great deal by Cameron and Martin [1], Woodward [9], Segal [5], [6], and Shepp [7], and the results have proved very useful in the evaluation of various Wiener integrals.

The transformation theorems in this paper are based on stochastic integrals called the generalized Paley-Wiener-Zygmud (P.W.Z.) integrals given in [3] and [4]. For a function $h(x, y) \in L^2(I^2)$ and $f(x, y) \in C_Y$, the generalized P.W.Z. integral is defined to be

$$\int_{I^2} h f d^* f \equiv \lim_{n \to \infty} \int_{I^2} (hf)_n df , \tag{1.2}$$

where $(hf)_n$ is the $n^{\text{th}}$ partial sum of the Fourier expansion of $h(x, y)f(x, y)$ with respect to a C.O.N. set belonging to a class of C.O.N. systems $\{\alpha_k(x, y)\}$ with each $\alpha_k(x, y)$ of B.V. satisfying the condition

$$\underset{n\to\infty}{\text{l.i.m.}} \sum_{j=1}^{n} \alpha_j(x, y) \int_0^y \int_0^x \alpha_j(s, t) ds dt = \frac{1}{4} ,$$

and the limit and on the right of (1.2) is an ordinary Riemann-Stieltjes (R-S) integral. It it known that the limit in (1.2) exists for almost all $f$ in $C_Y$ and it is essentially independent of the particular choice of the C.O.N. set in the class. (For details see [3] and [4].)

The Fredholm determinant $D(K)$ of $K(x, y, s, t) \in L^2(I^4)$ for $\lambda = -1$ is defined by

$$D(K) \equiv 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{I^{2k}} \begin{vmatrix} K(x_1, y_1, x_1, y_1) \cdots K(x_1, y_1, x_k, y_k) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ K(x_k, y_k, x_1, y_1) \cdots K(x_k, y_k, x_k, y_k) \end{vmatrix} dx_1 dy_1 \cdots dx_k dy_k .$$

## 2. Statement of main results.

THEOREM I. *Suppose that each $K_i(x, y, s, t)$, $i = 1, 2, 3, 4$, is continuous on $I^4$ and absolutely continuous in $x, y$ for each $(s, t) \in I^2$ and $K_1(0, y, s, t) = K_1(x, 0, s, t) = K_3(x, 0, s, t) = K_4(0, y, s, t) \equiv 0$. Let $K(x, y, s, t)$ be defined on $I^4$ by*

$$K(x, y, s, t) = \begin{cases} K_1(x, y, s, t) & \text{if } x < s, y < t \\ K_2(x, y, s, t) & \text{if } x > s, y > t \\ K_3(x, y, s, t) & \text{if } x > s, y < t \\ K_4(x, y, s, t) & \text{if } x < s, y > t \\ 2^{-1}(K_1 + K_3)(x, y, x, t) & \text{if } x = s, y < t \\ 2^{-1}(K_2 + K_4)(x, y, x, t) & \text{if } x = s, y > t \\ 2^{-1}(K_1 + K_4)(x, y, s, y) & \text{if } x < s, y = t \\ 2^{-1}(K_2 + K_3)(x, y, s, y) & \text{if } x > s, y = t \\ 4^{-1}(K_1 + K_2 + K_3 + K_4)(x, y, x, y) & \text{if } x = s, y = t , \end{cases} \tag{2.1}$$

*where $(K_1 + K_3)(x, y, x, t) = K_1(x, y, x, t) + K_3(x, y, x, t)$, etc., and let*

$$\begin{aligned} \phi(x, s, t) &\equiv K(x, t^+, s, t) - K(x, t^-, s, t) \\ \psi(y, s, t) &\equiv K(s^+, y, s, t) - K(s^-, y, s, t) . \end{aligned} \tag{2.2}$$

To be definite at each jump discontinuity, let us agree that the partials take left-hand limit whenever it fails to exist at a point, i.e.,

$$\frac{\partial^2}{\partial y \partial x} K(x, y, s, t) \equiv H(x, y, s, t) = \lim_{(u,v)\to(x^-,y^-)} \frac{\partial^2}{\partial v \partial u} K(u, v, s, t)$$

$$\frac{\partial}{\partial x}\phi(x, s, t) \equiv A(x, s, t) = \lim_{u\to x^-} \frac{\partial}{\partial u}\phi(u, s, t) , \tag{2.3}$$

$$\frac{\partial}{\partial y}\psi(y, s, t) \equiv B(y, s, t) = \lim_{v\to y^-} \frac{\partial}{\partial v}\psi(v, s, t) ,$$

and assume that there exists an integrable function $M(s, t)$ such that for all $(s, t) \in I^2$

$$\left.\begin{array}{l} \sup_{(x\ y)\in I^2} |H(x, y, s, t)| \\ \mathrm{var}_{(x\ y)\in I^2} H(x, y, s, t) \\ \mathrm{var}_{x\in I} H(x, 1, s, t) \\ \mathrm{var}_{y\in I} H(1, y, s, t) \\ \mathrm{var}_{x\in I} A(x, s, t) \\ \mathrm{var}_{y\in I} B(y, s, t) \end{array}\right\} \leqq M(s, t) . \tag{2.4}$$

Also assume that

$$|A(x, s, t)|, |B(y, s, t)| \leqq \beta, \text{ a constant, and} \tag{2.5}$$

$$D(K) \neq 0 . \tag{2.6}$$

Then for any Yeh-Wiener measurable functional $F(f)$, we have under the Fredholm transformation (1.1)

$$\int_{C_Y} F(g) d_Y g \stackrel{*}{=} |D(K)| \int_{C_Y} F(Tf)\cdot\exp\{-\Phi(f)\} d_Y f , \tag{2.7}$$

where

$$\begin{aligned} \Phi(f) = & \int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}\int_{I^2} K(x, y, s, t) f(s, t) ds\, dt\right]^2 dx dy \\ & + 2\int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}\int_{I^2} K(x, y, s, t) f(s, t) ds\, dt\right] d^* f(x, y) , \end{aligned} \tag{2.8}$$

and " $\stackrel{*}{=}$ " indicates the existence of one side implies that of the other and the equality.

THEOREM II. *Let $h(x, y) \in L^2$ on $I^2$, $K(x, y, s, t)$ and $F(f)$ as in Theorem I. Then under the (nonlinear) transformation*

$$\begin{aligned} L: f(x, y) \to g(x, y) = & f(x, y) + f_0(x, y) \\ & + \int_{I^2} K(x, y, x, s, t) f(s, t) ds\, dt , \end{aligned} \tag{2.9}$$

*where* $f_0(x, y) = \int_0^y \int_0^x h(s, t) ds\, dt$, *we have*

$$\int_{C_Y} F(g) d_Y g \overset{*}{=} |D(K)| \int_{C_Y} F(Lf) \cdot \exp\{-\Psi(f)\} d_Y f \ ,$$

*where*

$$\Psi(f) = \int_{I^2} \left[ h(x, y) + \frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds\, dt \right]^2 dx\, dy$$
$$+ 2 \int_{I^2} \left[ h(x, y) + \frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) ds\, dt \right] d^* f(x, y) \ .$$

3. **Definitions.** Corresponding to each continuous function $f(x, y)$ on $I^2$, the ($n^{\text{th}}$) quasi-polyhedric function $f_{(n)}(x, y)$ of $f(x, y)$ is defined to be

$$f_{(n)}(x, y) = a_{ij} xy + b_{ij} x + c_{ij} y + d_{ij} \tag{3.1}$$

on each square $Q_{ij} \equiv [(i-1)/n, i/n] \times [(j-1)/n, j/n]$, $i, j = 1, 2, \cdots, n$, where $a_{ij}$, $b_{ij}$, $c_{ij}$, and $d_{ij}$ are so chosen that $f_{(n)}(x, y)$ and $f(x, y)$ agree at the vertices $(i/n, j/n)$, $(i/n, (j-1)/n)$, $((i-1)/n, j/n)$ and $((i-1)/n, (j-1)/n)$.

REMARKS. (i). Since the function $f_{(n)}(x, y)$ is linear horizontally and vertically in each square $Q_{ij}$, we see that $f_{(n)}(x, y)$ is continuous on $I^2$. Furthermore the sequence $\{f_{(n)}(x, y)\}$ converges to $f(x, y)$ uniformly on $I^2$ as $n \to \infty$. Evaluating $a_{ij}$, $b_{ij}$, $c_{ij}$, and $d_{ij}$ explicitly, and then combining terms, we have on each square $Q_{ij}$,

$$\begin{aligned} f_{(n)}(x, y) &= f\left(\frac{i}{n}, \frac{j}{n}\right)[n^2 xy - n(j-1)x - n(i-1)y + (i-1)(j-1)] \\ &+ f\left(\frac{i-1}{n}, \frac{j}{n}\right)[-n^2 xy + n(j-1)x + niy - i(j-1)] \\ &+ f\left(\frac{i}{n}, \frac{j-1}{n}\right)[-n^2 xy + njx + n(i-1)y - j(i-1)] \\ &+ f\left(\frac{i-1}{n}, \frac{j-1}{n}\right)[n^2 xy - njx - niy + ij] \ . \end{aligned} \tag{3.2}$$

(ii) If $K(x, y, s, t)$ is continuous on $I^4$, then for each $(s, t) \in I^2$ we can think of $K(x, y, s, t)$ as a function in $x, y$ and so we have the ($n^{\text{th}}$) quasi-polyhedric function $K_{(n)}(x, y, s, t)$ in $x, y$, namely for $(x, y) \in Q_{ij}$; $i, j = 1, 2, \cdots, n$, we have

$$
\begin{aligned}
&K_{(n)}(x, y, s, t) \\
&\quad = K\Big(\frac{i}{n}, \frac{j}{n}, s, t\Big)[n^2xy - n(j-1)x - n(i-1)y + (i-1)(j-1)] \\
&\qquad + K\Big(\frac{i-1}{n}, \frac{j}{n}, s, t\Big)[-n^2xy + n(j-1)x + niy - i(j-1)] \\
&\qquad + K\Big(\frac{i}{n}, \frac{j-1}{n}, s, t\Big)[-n^2xy + njx + n(i-1)y - j(i-1)] \\
&\qquad + K\Big(\frac{i-1}{n}, \frac{j-1}{n}, s, t\Big)[n^2xy - njx - niy + ij]\,.
\end{aligned}
\tag{3.3}
$$

We also see that $K_{(n)}(x, y, s, t) \rightrightarrows K(x, y, s, t)$ on $I^4$ as $n \to \infty$, provided that $K(x, y, s, t)$ is continuous on $I^4$. Here "$\rightrightarrows$" indicates uniform convergence.

For each $\varepsilon > 0$ let

$$
\begin{aligned}
\Omega_\varepsilon(s) &= \operatorname{sgn} s - s/\varepsilon \quad \text{if} \quad |s| \leqq \varepsilon \\
&= 0 \qquad\qquad\quad\ \ \text{if} \quad |s| > \varepsilon\,.
\end{aligned}
\tag{3.4}
$$

Let $K(x, y, s, t)$ and $\psi(y, s, t)$ be as in Theorem I. Then the function defined by

$$
L_\varepsilon(x, y, s, t) \equiv K(x, y, s, t) + 2^{-1}[\Omega_\varepsilon(s - x) - \Omega_\varepsilon(s^+)]\psi(y, s, t)
\tag{3.5}
$$

is continuous in $x, s$. Now define

$$
J(x, s, t) \equiv L_\varepsilon(x, t^+, s, t) - L_\varepsilon(x, t^-, s, t)\ ,
\tag{3.6}
$$

and

$$
\begin{aligned}
K_\varepsilon(x, y, s, t) &\equiv L_\varepsilon(x, y, s, t) + 2^{-1}[\Omega_\varepsilon(t - y) - \Omega_\varepsilon(t^+)]J(x, s, t) \\
&= K(x, y, s, t) + 2^{-1}[\Omega_\varepsilon(s - x)\Omega_\varepsilon(s^+)]\psi(y, s, t) \\
&\quad + 2^{-1}[\Omega_\varepsilon(t - y) - \Omega_\varepsilon(t^+)]\phi(x, s, t) \\
&\quad + 4^{-1}[\Omega_\varepsilon(s - x) - \Omega_\varepsilon(s^+)][\Omega_\varepsilon(t - y) - \Omega_\varepsilon(t^+)](K_1 + K_2 \\
&\quad - K_3 - K_4)(s, t, s, t)\,.
\end{aligned}
\tag{3.7}
$$

Then $K_\varepsilon(x, y, s, t)$ is continuous on $I^4$, and $K_\varepsilon(0, y, s, t) = K_\varepsilon(x, 0, s, t) \equiv 0$. Furthermore $K_\varepsilon(x, y, s, t)$ is uniformly bounded in $\varepsilon, x, y, s, t$, and $\lim_{\varepsilon \to 0+} K_\varepsilon(x, y, s, t) = K(x, y, s, t)$. Now, define

$$
\begin{aligned}
C_\varepsilon(x, s) &= \varepsilon/2 \quad \text{if} \quad -\varepsilon < s - x \leqq \varepsilon \\
&= 0 \qquad \text{otherwise}\,.
\end{aligned}
\tag{3.8}
$$

Then from (3.7), (3.4), (3.8), and (2.3) it follows

$$
(3.9)\quad
\begin{aligned}
H_\varepsilon(x, y, s, t) &\equiv \frac{\partial^2}{\partial y\partial x} K_\varepsilon(x, y, s, t) \\
&= H(x, y, s, t) + C_\varepsilon(x, s)B(y, s, t) + C_\varepsilon(y, t)A(x, s, t) \\
&\quad + C_\varepsilon(x, s)\cdot C_\varepsilon(y, t)\cdot(K_1 + K_2 - K_3 - K_4)(s, t, s, t) ,
\end{aligned}
$$

with the understanding that whenever the partial derivatives are not defined at a point, then the value at the point to be the left-hand limit with respect to $x$ and $y$ (for the uniqueness sake) as in (2.3). Thus by using the mean-value property and the dominated convergence, we see that for almost all $(x, y)$ in $I^2$

$$
(3.10)\quad
\begin{aligned}
&\lim_{\varepsilon\to 0+}\int_{I^2} H_\varepsilon(x, y, s, t)f(s, t)ds\, dt \\
&= \int_{I^2} H(x, y, s, t)f(s, t)ds\, dt + \int_0^1 B(y, x, t)f(x, t)dt \\
&\quad + \int_0^1 A(x, s, y)f(s, y)ds + (K_1+K_2-K_3-K_4)(x, y, x, y)\cdot f(x, y) .
\end{aligned}
$$

But the right hand side of (3.10) is exactly equal to

$$(\partial^2/\partial y\partial x)\int_{I^2} K(x, y, s, t)f(s, t)ds\, dt .$$

Therefore for a.a. $(x, y)$ in $I^2$

$$
(3.11)\quad
\begin{aligned}
&\lim_{\varepsilon\to 0+}\int_{I^2}\frac{\partial^2}{\partial y\partial x} K_\varepsilon(x, y, s, t)f(s, t)ds\, dt \\
&= \frac{\partial^2}{\partial y\partial x}\int_{I^2} K(x, y, s, t)f(s, t)ds\, dt .
\end{aligned}
$$

## 4. Some Preliminary Lemmas.

LEMMA 1. *Let $K_{(n)}(x, y, s, t)$ be the $n$-th polyhedric function in $(x, y)$ with the understanding that $K_{(n)}(x, y, s, t) = 0$ for all $(x, y, s, t) \notin I^4$. For $i, j, p, q = 1, 2, \cdots, n$ let*

$$
(4.1)\quad A_{ijpq}^{(n)} \equiv \int_{(q-1)/n}^{(q+1)/n}\int_{(p-1)/n}^{(p+1)/n} K_{(n)}\Big(\frac{i}{n}, \frac{j}{n}, s, t\Big)(1-|ns-p|)(1-|nt-q|)ds\,dt .
$$

*Then for any $f \in C_Y$ we obtain:*

$$
(\mathrm{i})\quad \int_{I^2} K_{(n)}\Big(\frac{i}{n}, \frac{j}{n}, s, t\Big)f_{(n)}(s, t)ds\, dt = \sum_{p,q=1}^{n} A_{ijpq}^{(n)} f\Big(\frac{p}{n}, \frac{q}{n}\Big) ,
$$

$$
(\mathrm{ii})\quad \int_{I^2}\frac{\partial^2}{\partial y\partial x} K_{(n)}(x, y, s, t)f_{(n)}(s, t)ds\, dt = n^2 \sum_{p,q=1}^{n} B_{ijpq}^{(n)} f\Big(\frac{p}{n}, \frac{q}{n}\Big)
$$

*for* $(x, y) \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \times \left(\frac{j-1}{n}, \frac{j}{n}\right)$, *where*

$$B^{(n)}_{ijpq} \equiv A^{(n)}_{ijpq} - A^{(n)}_{i-1,jpq} - A^{(n)}_{i,j-1,pq} + A^{(n)}_{i-1,j-1pq} ,$$

(iii) $$\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_{(n)}(x, y, s, t)f_{(n)}(s, t)ds\,dt\right]^2 dx\,dy$$
$$= n^2\sum_{i,j=1}^{n}\left[\sum_{p,q=1}^{n} B^{(n)}_{ijpq}f\left(\frac{p}{n}, \frac{q}{n}\right)\right]^2 ,$$

(iv) $$\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_{(n)}(x, y, s, t)f_{(n)}(s, t)ds\,dt\right]d f_{(n)}(x, y)$$
$$= n^2\sum_{i,j=1}^{n}\left[\sum_{p,q=1}^{n} B^{(n)}_{ijpq}f\left(\frac{p}{n}, \frac{q}{n}\right)\right]\cdot \Delta_{ij}f_{(n)} ,$$

*where*

$$\Delta_{ij}f_{(n)} \equiv f\left(\frac{i}{n}, \frac{j}{n}\right) - f\left(\frac{i-1}{n}, \frac{j}{n}\right) - f\left(\frac{i}{n}, \frac{j-1}{n}\right)$$
$$+ f\left(\frac{i-1}{n}, \frac{j-1}{n}\right) .$$

The proof of this lemma is similar to that of corresponding results in [1]. Next, we consider a transformation of $C_Y$ to $C_Y$:

$$T: g(x, y) = f_{(n)}(x, y) + \int_{I^2}K_{(n)}(x, y, s, t)f_{(n)}(s, t)ds\,dt .$$

Then by (i) of Lemma 1 and the fact that $f_{(n)}(i/n, j/n) = f(i/n, j/n)$ at each $i$ and $j$, we have

$$\text{(4.2)}\quad T: g\left(\frac{i}{n}, \frac{j}{n}\right) = f\left(\frac{i}{n}, \frac{j}{n}\right) + \sum_{p,q=1}^{n} A^{(n)}_{ijpq}f\left(\frac{p}{n}, \frac{q}{n}\right); \quad i,j=1,2,\cdots,n .$$

The determinant $\Delta(K_{(n)})$ of this transformation is given by

$$\text{(4.3)}\qquad \Delta(K_{(n)}) \equiv \det\,(A^*_{IP} + \delta_{IP})_{I,P} = 1, 2, \cdots, n^2$$

where $A^*_{IP} = A^{(n)}_{ijpq}$ with $I \equiv (i-1)n+j$, $P \equiv (p-1)n+q$, $1 \leqq i, j, p, q \leqq n$.

LEMMA 2. *Let $F(f)$ be a Yeh-Wiener measurable functional which depends only on the $n^2$ values of $f(x, y)$ at $(x, y) = (i/n, j/n)$; $i, j = 1, 2, \cdots, n$. Let the $n^{\text{th}}$ quasi-polyhedric function in $x, y$, $K_{(n)}(x, y, s, t)$, satisfy that $K_{(n)}(x, y, s, t) = 0$ if $x = 0$ or $y = 0$, and that $\Delta(K_{(n)}) \neq 0$. Then*

$$\int_{C_Y} F(g)d_Y g \overset{*}{=} |\Delta(K_{(n)})| \int_{C_Y} F[f_{(n)} + \int_{I^2} K_{(n)}(\cdot, \cdot, s, t) f_{(n)}(s, t) ds\, dt]$$
$$\exp\left\{- \int_{I^2}\left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds\, dt\right]^2 dx\, dy\right.$$
$$\left. -2\int_{I^2}\left[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_{(n)}(x, y, s, t) f_{(n)}(s, t) ds\, dt\right] df_{(n)}(x, y)\right\} d_Y f .$$

*In view of Lemma* 1 *and a Yeh-Wiener integral formula* (see Theorem I, [11]) *the proof is word for word identical to that of the corresponding lemma in* [1].

LEMMA 3. *If $K(x, y, s, t)$ is continuous on $I^4$, and if the Fredholm determinant $D(K) \neq 0$, then*

$$\lim_{n\to\infty} \Delta(K_{(n)}) = D(K) .$$

*Proof.* Using (4.3), we may expand $\Delta(K_{(n)})$ as:

$$\begin{aligned} \Delta(K_{(n)}) &= \det (A^*_{IP} + \delta_{IP})_{I,P=1,2,\cdots,n^2} \\ &= 1 + \sum_{P=1}^{n^2} A^*_{PP} + \frac{1}{2!} \sum_{P_1,P_2=1}^{n^2} \det (A^*_{P_iP_j})_{i,j=1,2} \\ &\quad + \cdots + \frac{1}{(n^2)!} \sum_{P_1,\cdots,P_{n^2}=1}^{n^2} \det (A^*_{P_iP_j})_{i,j=1,\cdots, n^2} \\ &= 1 + \sum_{p,q=1}^{n} A^{(n)}_{pqpq} + \frac{1}{2!} \sum_{p_1,q_1,p_2,q_2=1}^{n} \det (A^{(n)}_{p_iq_ip_jq_j})_{i, j=1, 2} \\ &\quad + \cdots + \frac{1}{(n^2)!} \sum_{p_1,q_1,p_2,q_2,\cdots,p_{n^2},q_{n^2}=1}^{n} \det (A^{(n)}_{p_iq_ip_jq_j})_{i, j=1, 2, \cdots, n^2} . \end{aligned} \tag{4.4}$$

Let $M$ be the bound for $K$ on $I^4$. Then by (4.1) we have $|n^2 A^{(n)}_{ijpq}| \leqq M$ uniformly in $n, i, j, p, q$. Thus by Hadamard's inequality it follows:

$$|\det (A^{(n)}_{p_iq_ip_jq_j})_{i,j=1,\cdots,N}| \leqq (M/n^2)^N N^{N/2} . \tag{4.5}$$

Upon using (4.5) in (4.4), we conclude that $|\Delta(K_{(n)})|$ is uniformly bounded by the convergent series $1 + \sum_{N=1}^{\infty} M^N N^{N/2}/N!$, and for each $N$

$$\lim_{n\to\infty} \sum_{p_1,q_1,\cdots,p_N,q_N=1}^{n} \det (A^{(n)}_{p_iq_ip_jq_j})_{i, j=1,\cdots,N}$$
$$= \int_{I^{2N}} \begin{vmatrix} K(s_1, t_1, s_1, t_1)\cdots K(s_1, t_1, s_N, t_N) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K(s_N, t_N, s_1, t_1) \quad K(s_N, t_N, s_N, t_N) \end{vmatrix} ds_1 dt_1 \cdots ds_N dt_N .$$

Hence the conclusion follows.

Similarly it follows:

LEMMA 4. *Let $K(x, y, s, t)$ be a bounded integrable function with $D(K) \neq 0$ and let $\{K_\lambda(x, y, s, t)\}$ be a set of Borel measurable functions which are uniformly bounded in $\lambda$, $x$, $y$, $s$, $t$, and let $\lim_{\lambda\to 0^+} K_\lambda(x, y, s, t) = K(x, y, s, t)$. Then $\lim_{\lambda\to 0^+} D(K_\lambda) = D(K)$.*

LEMMA 5. *For each $(s, t) \in I^2$ let $H(x, y, s, t)$ satisfy that*

$$\sup_{(x,y)\in I^2} |H(x, y, s, t)|, \quad \operatorname*{var}_{(x,y)\in I^2} H(x, y, s, t), \operatorname*{var}_{x\in I} H(x, 1, s, t), \operatorname*{var}_{y\in I} H(1, y, s, t)$$

*are all dominated by an integrable function $M(s, t)$. Then*

$$\begin{aligned}\int_{I^2}\Big[\int_{I^2} H(x, y, s, t) f(s, t) ds\, dt\Big] df(x, y) \\ = \int_{I^2} f(s, t)\Big[\int_{I^2} H(x, y, s, t) df(x, y) ds\, dt\,.\end{aligned} \tag{4.6}$$

*Proof.* Let "$\|\cdot\|$" denote the supremum of the absolute value. Then from the fact that

$$\left.\begin{aligned}&\operatorname{var}_{(x,y)\in I^2}\int_{I^2} H(x, y, s, t) f(s, t) ds\, dt \\ &\operatorname{var}_{y\in I}\int_{I^2} H(x, 1, s, t, ) f(s, t) ds\, dt \\ &\operatorname{var}_{y\in I}\int_{I^2} H(1, y, s, t) f(s, t) ds\, dt\end{aligned}\right\} \leqq \|f\|\int_{I^2} M(s, t) ds\, dt < \infty\,, \tag{4.7}$$

the left member of (4.6) exists as an ordinary $R-S$ integral. Hence for the net: $x_i = i/n$, $y_j = j/n$; $i, j = 1, 2, \cdots, n$, and $x_{i-1} \leqq x_i^* \leqq x_i$, $y_{j-1} \leqq y_j^* \leqq y_j$, we have with $\Delta_{ij} f(x, y) = f(x_i, y_j) - f(x_{i-1}, y_j) - f(x_i, y_{j-1}) + f(x_{i-1}, y_{j-1})$,

$$\begin{aligned}&\int_{I^2}\Big[\int_{I^2} H(x, y, s, t) f(s, t) ds\, dt\Big] df(x, y) \\ &= \lim_{n\to\infty}\sum_{i,j=1}^{n}\Big[\int_I H(x_i^*, y_j^*, s, t) f(s, t) ds\, dt\Big]\Delta_{ij} f(x, y) \\ &= \lim_{n\to\infty}\int_2 f(s, t)\Big[\sum_{i,j=1}^{n} H(x_i^*, y_j^*, s, t)\Delta_{ij} f(x, y)\Big] ds\, dt\,.\end{aligned} \tag{4.8}$$

But

$$\begin{aligned}&\Big|\sum_{i,j=1}^{n} H(x_i^*, y_j^*, s, t)\Delta_{ij} f(x, y)\Big| \\ &\leqq \|f\|\big[\operatorname*{var}_{(x,y)\in I^2} H(x, y, s, t) + \operatorname*{var}_{x\in I} H(x, 1, s, t) + \operatorname*{var}_{y\in I} H(1, y, s, t) \\ &\quad + |H(1, 1, s, t)|\big] \\ &\leqq 4\|f\| M(s, t)\,,\end{aligned} \tag{4.9}$$

and since $M(s, t)$ is finite a.e. we see that for a.a.$(s, t) \in I^2$

$$\lim_{n\to\infty}\sum_{i,j=1}^{n} H(x_i^*, y_j^*, s, t)\Delta_{ij}f(x, y) = \int_{I^2} H(x, y, s, t)df(x, y) .$$

Thus (4.6) follows from this, (4.8) and (4.9) by dominated convergence.

LEMMA 6. *Let $H(x, y, s, t)$ be as in Lemma 5, and let*

$$H^n(x, y, s, t) \equiv n^2\int_{(q-1)/n}^{q/n}\int_{(p-1)/n}^{p/n} H(u, v, s, t)du\, dv \tag{4.10}$$

$$\text{for } (x, y)\in\left(\frac{p-1}{n}, \frac{p}{n}\right]\times\left(\frac{q-1}{n}, \frac{q}{n}\right], p, q = 1, 2, \cdots, n .$$

*Then for every $f(x, y)\in C_Y$*

$$\begin{aligned}\lim_{n\to\infty}\int_{I^2}\left[\int_{I^2} H(x, y, s, t)f_{(n)}(s, t)ds\, dt\right]^2 dx\, dy \\ = \int_{I^2}\left[\int_{I^2} H(x, y, s, t)f(s, t)ds\, dt\right]^2 dx\, dy ,\end{aligned} \tag{4.11}$$

$$\left|\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f_{(n)}(s, t)ds\, dt\right]^2 dx\, dy\right| \leqq \left(\|f\|\int_{I^2} M(s, t)ds\, dt\right)^2 , \tag{4.12}$$

$$\begin{aligned}\lim_{n\to\infty}\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f_{(n)}(s, t)ds\, dt\right]df_{(n)}(x, y) \\ = \int_{I^2}\left[\int_{I^2} H(x, y, s, t)f(s, t)ds\, dt\right]df(x, y) .\end{aligned} \tag{4.13}$$

*Proof.* (4.11) and (4.12) are immediate. By (4.7) the function $\int_{I^2} H(x, y, s, t)f(s, t)ds\, dt$ is of B.V. Now

$$\begin{aligned}&\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f(s, t)ds\, dt\right]df(x, y) \\ &= \sum_{p,q=1}^{n}\int_{I^2}\left[n^2\int_{(p-1)/n}^{q/n}\int_{(q-1)/n}^{p/n} H(u, v, s, t)du, dv\right]f(s, t)ds\, dt\cdot\Delta_{p,q}f ,\end{aligned}$$

where $\Delta_{p,q}f = f\left(\frac{p}{n}, \frac{q}{n}\right) - f\left(\frac{p-1}{n}, \frac{q}{n}\right) - f\left(\frac{p}{n}, \frac{q-1}{n}\right) + f\left(\frac{p-1}{n}, \frac{q-1}{n}\right)$.

Since

$$\int_{2}\left[n^2\int_{(q-1)/n}^{q/n}\int_{(p-1)/n}^{p/n} H(u, v, s, t)du\, dv\right]f(s, t)ds\, dt$$

is the average value of $\int_{I^2} H(x, y, s, t)f(s, t)ds\, dt$ in the square $((q-1)/n, q/n]\times((q-1)/n, p/n]$, the existence of the $R-S$ integral implies

$$\tag{4.14}\begin{aligned}&\lim_{n\to\infty}\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f(s, t)ds\,dt\right]df(x, y)\\&\quad=\int_{I^2}\left[\int_{I^2} H(x, y, s, t)f(s, t)ds\,dt\right]df(x, y)\,.\end{aligned}$$

Now,

$$\tag{4.15}\begin{aligned}&\operatorname*{var}_{(x,y)\in I^2} H^n(x, y, s, t) \leqq \operatorname*{var}_{(x,y)\in I^2} H(x, y, s, t) \leqq M(s, t)\\&\operatorname*{var}_{x\in I} H^n(x, 1, s, t) \leqq \operatorname*{var}_{x\in I} H(x, 1, s, t) \leqq M(s, t)\\&\operatorname*{var}_{y\in I} H^n(1, y, s, t) \leqq \operatorname*{var}_{y\in I} H(1, y, s, t) \leqq M(s, t)\\&|H^n(1, 1, s, t)| \leqq \sup_{(x,y)\in I^2}|H(x, y, s, t)| \leqq M(s, t)\,.\end{aligned}$$

Seeing the fact that $H^n(x, y, s, t)$ is constant in $x$, $y$ in each square $\left(\frac{p-1}{n}, \frac{p}{n}\right]\left(\frac{q-1}{n}, \frac{q}{n}\right]$ and that $f_{(n)}(x, y)$ and $f(x, y)$ agree on the vertices of this square, we can write

$$\begin{aligned}&\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f_{(n)}(s, t)ds\,dt\right]df_{(n)}(x, y)\\&\quad=\sum_{p,q=1}^{n}\int_{I^2} H^n\left(\frac{p}{n}, \frac{q}{n}, s, t\right)f_{(n)}(s, t)ds\,dt\cdot\Delta_{pq}f\\&\quad=\int_{I^2} f_{(n)}(s, t)\left[\sum_{p,q=1}^{n} H^n\left(\frac{p}{n}, \frac{q}{n}, s, t\right)\Delta_{pq}f\right]ds\,dt\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\left|\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f(s, t)ds\,dt\right]df(x, y)\right.\\&\qquad\left.-\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f_{(n)}(s, t)ds\,dt\right]df_{(n)}(x, y)\right|\\&\quad=\left|\int_{I^2}[f(s, t)-f_{(n)}(s, t)]\left[\sum_{p,q=1}^{n} H^n\left(\frac{p}{n}, \frac{q}{n}, s, t\right)\cdot\Delta_{pq}f\right]ds\,dt\right|\\&\quad\leqq 4\|f-f_n\|\,\|f\|\int_{I^2} M(s, t)ds\,dt\,,\end{aligned}$$

thus obtaining

$$\tag{4.16}\begin{aligned}&\lim_{n\to\infty}\int_{2}\left[\int_{I^2} H^n(x, y, s, t)f_{(n)}(s, t)ds\,dt\right]df_{(n)}(x, y)\\&\quad=\lim_{n\to\infty}\int_{I^2}\left[\int_{I^2} H^n(x, y, s, t)f(s, t)ds\,dt\right]df(x, y)\,.\end{aligned}$$

Hence (4.13) follows from (4.14) and (4.16).

COROLLARY. *Let $K(x, y, s, t)$ be as in Theorem I, $K_\varepsilon(x, y, s, t)$ the* *corresponding function defined by* (3.7), *and* $K_{\varepsilon,(n)}(x, y, s, t)$ *the* $n^{\text{th}}$ *quasi-polyhedric function of* $K_\varepsilon$ *in* $x$, $y$. *Then*

$$\begin{aligned}(4.17)\qquad &\lim_{n\to\infty}\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_{\varepsilon,(n)}(x, y, s, t)f_{(n)}(s, t)ds\,dt\right]^2 dx\,dy \\ &= \int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)ds\,dt\right]^2 dx\,dy ,\end{aligned}$$

$$\begin{aligned}(4.18)\qquad &\lim_{n\to\infty}\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_{\varepsilon,(n)}(x, y, s, t)f_{(n)}(s, t)ds\,dt\right]df_{(n)}(x, y) \\ &= \int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)ds\,dt\right]df(x, y) .\end{aligned}$$

*Proof.* In view of (3.9), (3.8), and (2.4) we obtain

$$\begin{aligned}&\sup_{(x,y)\in I^2}|H_\varepsilon(x, y, s, t)| \leqq M(s, t) + \frac{\beta}{\varepsilon} + \frac{1}{4\varepsilon^2}|(K_1 + K_2 - K_3 \\ &\qquad - K_4)(s, t, s, t)| , \\ &\operatorname*{var}_{(x,y)\in I^2} H_\varepsilon(x, y, s, t) \leqq M(s, t) + \frac{2}{\varepsilon}M(s, t) + \frac{1}{\varepsilon^2}|(K_1 + K_2 - K_3 \\ (4.19)\qquad &\qquad - K_4)(s, t, s, t)| , \\ &\operatorname*{var}_{x\in I} H_\varepsilon(x, 1, s, t) \leqq M(s, t) + \frac{1}{2\varepsilon}[2\beta + M(s, t)] + \frac{1}{2\varepsilon^2}|(K_1+K_2 \\ &\qquad - K_3 - K_4)(s, t, s, t)| , \\ &\operatorname*{var}_{y\in I} H_\varepsilon(1, y, s, t) \leqq M(s, t) + \frac{1}{2\varepsilon}[2\beta + M(s, t)] + \frac{1}{2\varepsilon^2}|(K_1+K_2 \\ &\qquad - K_3 - K_4)(s, t, s, t)| .\end{aligned}$$

Let $M_\varepsilon(s, t)$ be the sum of the four right members of (4.19). From (3.3) it is obvious that

$$(4.20)\qquad \frac{\partial^2}{\partial x\partial y}K_{\varepsilon\,(n)}(x, y, s, t) = n^2\int_{(q-1)/n}^{q/n}\int_{(p-1)/n}^{p/n}\frac{\partial^2}{\partial v\partial u}K_\varepsilon(u, v, s, t)du\,dv$$

for $(p - 1)/n < x \leqq p/n$, $(q - 1)/n < y \leqq q/n$; $p, q = 1, \cdots, n$. Hence from (4.10), (4.20), and the fact that $(\partial^2/\partial v\partial u)K_\varepsilon(u, v, s, t)\equiv H_\varepsilon(u, v, s, t)$, we see that

$$(4.21)\qquad \frac{\partial^2}{\partial y\partial x}H_{\varepsilon\cdot(n)}(x, y, s, t) \equiv H_\varepsilon^n(x, y, s, t) .$$

Thus the conclusions follow directly from Lemma 6.

**5. Some discussions on resolvent kernels.** Let us denote the resolvent kernel of $K(x, y, s, t)$ in the Fredholm transformation (1.1) by $K^*(x, y, s, t)$, i.e., if

$$g(x, y) = f(x, y) + \int_{I^2} K(x, y, s, t) f(s, t) ds\, dt , \tag{5.1}$$

then

$$f(x, y) = g(x, y) + \int_{I^2} K^*(x, y, s, t) g(s, t) ds\, dt . \tag{5.2}$$

Let $K(x, y, s, t)$ be a bounded $L^2$-kernel on $I^4$ with $D(K) \neq 0$, and set

$$\mathscr{D}(x, y, s, t) \equiv K^*(x, y, s, t) \cdot D(K) . \tag{5.3}$$

Then, using the familiar results for resolvent kernels for Fredholm integral equations with $\lambda = -1$ (see [8], pp. 66–75), we can establish

$$K(x, y, s, t) + K^*(x, y, s, t) = -\int_{I^2} K(x, y, u, v) \cdot K^*(u, v, s, t) du dv , \tag{5.4}$$

$$\mathscr{D}(x, y, s, t) = \sum_{n=0}^{\infty} C_n(x, y, s, t)/n! \tag{5.5}$$

where $C_0(x, y, s, t) = -K(x, y, s, t)$, and other $C_n's$ are found successively by

$$\begin{aligned} C_n(x, y, s, t) &= J_n(K) \cdot K(x, y, s, t) \\ &\quad - n \int_{I^2} K(x, y, u, v) C_{n-1}(u, v, s, t) du dv , \\ J_n(K) &= -\int_{I^2} \overset{(n)}{\cdots} \int_{I^2} \begin{vmatrix} K(x_1, y_1, x_1, y_1) \cdots K(x_1, y_1, x_n, y_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K(x_n, y_n, x_1, y_1) \cdots K(x_n, y_n, x_n, y_n) \end{vmatrix} \\ &\qquad dx_1 dy_1 \cdots dx_n dy_n . \end{aligned} \tag{5.6}$$

By Hadamard's inequality it follows from (5.6) that

$$|C_n(x, y, s, t)| \leqq (n + 1)^{(n+1)/2} ||K||^{n+1} .$$

Therefore the series in (5.5) is absolutely and uniformly convergent. If $K(x, y, s, t)$ satisfies the assumptions in Theorem I, then

$$\int_{I^2} K(x, y, u, v) C_n(u, v, s, t) du dv$$

is continuous on $I^4$ for $n = 0, 1, 2, \cdots$, and hence from (5.6) we see that the jumps for $C_n(u, v, s, t)$ coincides with those for $K(x, y, s, t)$, and thus it takes average value at each jump, and so does $\sum_{n=0}^{\infty} C_n(x, y, s, t)/n!$ by uniform convergence. By absolute convergence we may rearrange the terms in the series (5.5), and then use the uniform convergence to obtain

$$
\begin{aligned}
\mathscr{D}(x, y, s, t) &= \sum_{n=0}^{\infty} \frac{1}{n!}\Big[J_n . K(x, y, s, t) \\
&\qquad - n\int_{I^2} K(x, y, u, v) . C_{n-1}(u, v, s, t) du dv\Big] \\
&= C \cdot K(x, y, s, t) - \int_{I^2} K(x, y, u, v) \cdot \mathscr{D}(u, v, st) du dv \, ,
\end{aligned}
\tag{5.7}
$$

where $C = \sum_{n=0}^{\infty} J_n(K)/n!$. Corresponding to each $K \in (x, y, s, t)$, define $K^*_{(\varepsilon)}(x, y, s, t)$ by

$$
\begin{aligned}
K^*_{(\varepsilon)}(x, y, s, t) &= D(K)\mathscr{D}_{(\varepsilon)}(x, y, s, t) \\
&= D(K)\Big[C \cdot K_\varepsilon(x, y, s, t) \\
&\qquad - \int_{I^2} K_\varepsilon(x, y, u, v) \cdot \mathscr{D}(u, v, s, t) du dv\Big] \, .
\end{aligned}
\tag{5.8}
$$

Then $K^*_{(\varepsilon)}(x, y, s, t)$ is continuous on $I^4$, and since $\mathscr{D}(u, v, s, t)$ is bounded, we have that $K^*_{(\varepsilon)}(x, y, s, t)$ is uniformly bounded in $\varepsilon, x, y, s, t$, and from (3.9), (4.19), and (5.8) it follows that

$$
\begin{aligned}
H^*_{(\varepsilon)}(x, y, s, t) &\equiv \frac{\partial^2}{\partial y \partial x} K^*_{(\varepsilon)}(x, y, s, t) \\
&= D(K)\Big[C \cdot H_\varepsilon(x, y, s, t) \\
&\qquad - \int_{I^2} H_\varepsilon(x, y, u, v) \cdot \mathscr{D}(u, v, s, t) du dv\Big] \, ,
\end{aligned}
\tag{5.9}
$$

and $\sup_{(x, y) \in I^2} |H^*_{(\varepsilon)}(x, y, s, t)|$, $\operatorname{var}_{(x,y) \in I^2} H^*_{(\varepsilon)}(x, y, s, t)$, $\operatorname{var}_{x \in I} H^*_{(\varepsilon)}(x, 1, s, t)$, and $\operatorname{var}_{y \in I} H^*_{(\varepsilon)}(1, y, s, t)$ are all dominated by $|D(K)|(|C| + \|\mathscr{D}\|) \cdot M_\varepsilon(s, t)$. Furthermore, by Lemma 4

$$
\lim_{\varepsilon \to 0+} D(K^*_{(\varepsilon)}) = D(K^*) = D^{-1}(K) \, . \tag{5.10}
$$

**6. Additional lemmas.** Utilizing (3.7), (3.9), (3.11), (5.8), and (5.9), we have the following

LEMMA 7. *If $K(x, y, s, t)$ satisfies the hypotheses of Theorem I, then*

$$
\begin{aligned}
&\left|\int_{I^2} [K_\varepsilon(x, y, s, t) - K(x, y, s, t)] f(s, t) ds dt\right| \\
&\qquad \leqq 4\varepsilon(1 + \varepsilon)\|f\| \cdot \|K\| \, , \\
&\int_{I^2}\Big[\int_{I^2} \frac{\partial^2}{\partial y \partial x} K_\varepsilon(x, y, s, t) f(s, t) ds dt\Big]^2 dx dy
\end{aligned}
\tag{6.1}
$$

$$\leqq \|f\|^2\left[\int_{I^2} M(s, t)dsdt + 2\beta + 4\|K\|\right]^2 ,$$

$$(6.2)\qquad \begin{aligned}&\lim_{\varepsilon\to 0+}\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)dsdt\right]^2 dxdy\\ &= \int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}\int_{I^2}K(x, y, s, t)f(s, t)dsdt\right]^2 dxdy ,\end{aligned}$$

$$(6.3)\qquad \begin{aligned}&\left|\int_{I^2}[K^*_{(\varepsilon)}(x, y, s, t) - K^*(x, y, s, t)\Big]f(s, t)dsdt\right|\\ &\leqq 4\varepsilon(1+\varepsilon)\|f\|\,\|K\|\,|D(K)|\cdot(|C| + \|\mathscr{D}\|) ,\end{aligned}$$

$$(6.4)\qquad \begin{aligned}&\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K^*_{(\varepsilon)}(x, y, s, t)f(s, t)dsdt\right]^2 dxdy\\ &\leqq (\|f\|D(K))^2(|C| + \|\mathscr{D}\|)\left[\int_{I^2}M(s, t)dsdt + 2\beta + 4\|K\|\right]^2 ,\end{aligned}$$

$$(6.5)\qquad \begin{aligned}&\lim_{\varepsilon\to 0+}\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K^*_{(\varepsilon)}(x, y, s, t)f(s, t)dsdt\right]^2 dxdy\\ &= \int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}K^*_{(\varepsilon)}(x, y, s, t)f(s, t)dsdt\right]^2 dxdy .\end{aligned}$$

The following two lemmas are the key results:

LEMMA 8. *Let $K(x, y, s, t)$ be as in Theorem I. Then*

$$\int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}\int_{I^2}K(x, y, s, t)f(s, t)dsdt\right]d^*f(x, y)$$

*defined as in* (1.2) *converges for a.a. $f$ in $C_Y$, and any two C.O.N. sets in the class lead to the same value for a.a. $f$ in $C_Y$. Furthermore, we have*

$$(6.6)\qquad \begin{aligned}&\underset{\varepsilon\to 0+}{\text{l.i.m.}}\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)dsdt\right]df(x, y)\\ &= \int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}\int_{I^2}K(x, y, s, t)f(s, t)dsdt\right]d^*f(x, y) \text{ on } C_Y ,\end{aligned}$$

*where the mean convergence is in $L^2$-sense.*

*Proof.* First we observe that

$$\int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}\int_{I^2}K(x, y, s, t)f(s, t)dsdt\right]d^*f(x, y)$$

$$
\begin{aligned}
&= \int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K(x, y, s, t)f(s, t)dsdt\right]d^*f(x, y) \\
(6.7)\qquad &+ \int_{I^2}\left[\int_0^1 B(y, x, t)f(x, t)dt\right]d^*f(x, y) \\
&+ \int_{I^2}\left[\int_0^1 A(x, s, y)f(s, y)ds\right]d^*f(x, y) \\
&+ \int_{I^2}(K_1 + K_2 - K_3 - K_4)(x, y, x, y)\cdot f(x, y)d^*f(x, y) ,
\end{aligned}
$$

and the existence and the consistency of each of last three expressions follow in the similar manner as in the generalized P.W.Z. integral $\int_{I^2} hfd^*f$. On account of (3.9) and Lemma 5 we may write

$$
\begin{aligned}
&\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)dsdt\right]df(x, y) \\
&= \int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K(x, y, s, t)f(s, t)dsdt\right]df(x, y) \\
(6.8)\qquad &+ \int_{I^2}f(s, t)\left[\int_{I^2}C_\varepsilon(x, s)B(y, s, t)df(x, y)\right]dsdt \\
&+ \int_{I^2}f(s, t)\left[\int_{I^2}C_\varepsilon(y, t)A(x, s, t)df(x, y)\right]dsdt \\
&+ \int_{I^2}(K_1+K_2-K_3-K_4)(s, t, s, t)\left[\int_{I^2}C_\varepsilon(x, s)C_\varepsilon(y, t)df(x, y)\right]dsdt .
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
&\int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K(x, y, s, t)f(s, t)dsdt\right]df(x, y) \\
&= \int_{I^2}\left[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K(x, y, s, t)f(s, t)dsdt\right]d^*f(x, y) \quad \text{for a.a.} f \in C_Y .
\end{aligned}
$$

Thus to establish (6.6) we need only show that

$$
\begin{aligned}
(6.9)\qquad &\underset{\varepsilon\to 0^+}{\text{l.i.m.}} \int_{I^2}f(s, t)\left[\int_{I^2}C_\varepsilon(x, s)B(y, s, t)\,df(x, y)\right]dsdt \\
&= \int_{I^2}\left[\int_0^1 B(y, x, t)f(x, t)dt\right]d^*f(x, y) \quad \text{on} \quad C_Y ,
\end{aligned}
$$

and so on. To see this we observe that the Yeh-Wiener integral

$$
\begin{aligned}
&\int_{C_Y}\left\{\int_{I^2}f(s, t)\left[\int_{I^2}C_\varepsilon(x, s)B(y, s, t)df(x, y)\right]dsdt\right\}^2 d_Y f \\
&= \frac{1}{4}\left\{\int_{I^4}\left[\int_t^1\int_s^1 C_\varepsilon(x, s')B(y, s', t')ds'dt'\right]^2 ds\,dt\,dx\,dy\right.
\end{aligned}
$$

$$+ \left[\int_{I^2}\left(\int_0^t\int_0^s C_\varepsilon(x, s)B(y, s, t)dxdy\right)dsdt\right]^2$$
$$+ \int_{I^4}\left(\int_0^t\int_0^s C_\varepsilon(x, s')B(y, s', t')dxdy\right)$$
$$\times \left(\int_0^{t'}\int_0^{s'} C_\varepsilon(x, s)B(y, s, t)dxdy\right)dsdtds'dt'\Bigg\} ,$$

$$\int_{C_Y}\left\{\int_{I_1}\left[\int_0^1 B(y, x, t)f(x, t)dt\right]d^*f(x, y)\right\}^2 d_Y f$$
$$= \frac{1}{4}\left\{\int_{I^3}\left[\int_x^1\left(\int_t^1 B(y, x', t')dt'\right)^2 dx'\right]dxdtdy\right.$$
$$\left. + \frac{1}{4}\left[\int_{I^2}\left(\int_y^1 B(y, x, t)dt\right)dxdy\right]^2\right\} ,$$

and

$$\int_{C_Y}\left\{\int_{I^2} f(s, t)\left[\int_{I^2} C_\varepsilon(x, s)B(y, s, t)df(x, y)\right]dsdt\right\}$$
$$\cdot\left\{\int_{I^2}\left[\int_0^1 B(y, x, t)f(x, t)dt\right]d^*f(x, y)\right\}d_Y f$$
$$= \frac{1}{4}\left\{\int_{I^3}\left[\int_{I^2} B(y, x, t')\min(s, x)C_\varepsilon(x, s)B(y, s, t)dxdy\right]\min(t, t')dsdtdt'\right.$$
$$+ \int_{I^3}\left(\int_0^t\int_0^s B(y', x', t_\prime)\left[\int_0^{t'}\int_0^{x'} C_\varepsilon(x, s)B(y, s, t)dxdy\right]dx'dy'\right)dsdtdt'$$
$$\left.+\frac{1}{2}\int_{I^4}\left(\int_0^{t'} B(y', x', t')\left[\int_0^t\int_0^s C_\varepsilon(s, x)B(y, s, t)dxdy\right]dy'\right)dx'dsdtdt'\right\} .$$

The techniques leading to the evaluation of above integrals can be found in [3] and [4]. Using these and then taking the limit as $\varepsilon\to 0^+$, we get

$$\lim_{\varepsilon\to 0^+}\int_{C_Y}\left\{\int_{I^2} f(s, t)\left[\int_{I^2} C_\varepsilon(x, s)B(y, s, t)df(x, y)\right]dsdt\right.$$
$$\left. - \int_{I^2}\left[\int_0^1 B(y, x, t)f(x, t)dt\right]d^*f(x, y)\right\}^2 d_Y f = 0 ,$$

which is exactly the same as (6.9).

LEMMA 8′. *Let $K(x, y, s, t)$ satisfy the hypotheses of Theorem I. Then for its resolvent kernel $K^*$ we have*

$$\int_{I^2}\left[\frac{\partial^2}{\partial y\partial x}\int_{I^2} K^*(x, y, s, t)f(x, t)dsdt\right]d^*f(x, y) \tag{6.10}$$

*converges for almost all $f$ in $C_Y$ and is essentially independent of the particular choice of the C.O.N. set, and*

$$
\text{(6.11)}\quad
\begin{aligned}
&\underset{\varepsilon\to 0^+}{\text{l.i.m.}}\int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K^*_{(\varepsilon)}(x, y, s, t)f(s, t)dsdt\Big]df(x, y)\\
&=\int_{I^2}\Big[\frac{\partial^2}{\partial y\partial x}\int_{I^2}K^*(x, y, s, t)f(s, t)dsdt\Big]d^*f(x, y) \quad on \quad C_Y .
\end{aligned}
$$

*Proof.* The claim on (6.10) can be treated as that in (6.7). As for (6.11) we may use (5.9) with $H_\varepsilon(x, y, s, t)\equiv\dfrac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)$ to see that

$$
\begin{aligned}
&\int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K^*_{(\varepsilon)}(x, y, s, t)f(s, t)dsdt\Big]df(x, y)\\
&= D(K)\Big\{C\int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)dsdt\Big]df(x, y)\\
&\quad-\int_{I^2}\Big[\int_{I^2}\Big(\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, u, v)\mathscr{D}(u, v, s, t)dudv\Big)f(s, t)dsdt\Big]df(x, y)\Big\} ,
\end{aligned}
$$

and the first term in the braces converges in the mean to

$$
C\int_{I^2}\Big[(\partial^2/\partial y\partial x)\int_{I^2}K(x, y, s, t)f(s, t)dsdt\Big]d^*f(x, y)
$$

on $C_Y$ by the preceding lemma. Therefore, in view of (5.3) and (5.7), it remains to show that

$$
\begin{aligned}
&\underset{\varepsilon\to 0^+}{\text{l.i.m.}}\int_{I^2}\Big[\int_{I^2}\Big(\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, u, v)\mathscr{D}(u, v, s, t)dudv\Big)f(s, t)dsdt\Big]df(x, y)\\
&=\int_{I^2}\Big[\frac{\partial^2}{\partial y\partial x}\int_{I^2}\Big(\int_{I^2}K(x, y, u, v)\mathscr{D}(u, v, s, t)dudv\Big)f(s, t)dsdt\Big]d^*f(x, y)
\end{aligned}
$$

on $C_Y$ whose proof is essentially on the same lines as that of Lemma 8.

LEMMA 9. *Let $F(f)$ be a nonnegative Yeh-Wiener measurable functional on $C_Y$, and let $K(x, y, s, t)$ satisfy the hypotheses of Theorem I. Then under the Fredholm transformation $T$ in* (1.1), *we have*

$$
\text{(6.12)}\quad \int_{C_Y}F(g)d_Y g \geqq |D(K)|\int_{C_Y}F(Tf)\cdot\exp\{-\Phi(f)\}d_Y f ,
$$

*where $\Phi(f)$ is given by* (2.8).

*Proof.* First we assume that $F(f)$ is a bounded nonnegative functional continuous on $C_Y$ with respect to the uniform topology, and is dependent only on the $n^2$ values of $f$ at $(x, y) = (i/n, j/n)$; $i, j = 1, 2, \cdots, n$. Since $\lim_{\varepsilon\to 0^+} D(K_\varepsilon) = D(K) \neq 0$ and $\lim_{n\to\infty} D(K_{\varepsilon(n)}) = D(K_\varepsilon)$

by Lemma 4, there exist $\varepsilon'$ and $N(\varepsilon)$ such that $0 < \varepsilon < \varepsilon'$ and $n \geqq N(\varepsilon)$ imply $D(K_\varepsilon) \neq 0$ and $D(K_{\varepsilon,(n)}) \neq 0$. For such $\varepsilon$ and $n$ we may apply Lemma 2 to obtain

$$\begin{aligned}\int_{C_Y} F(g_n)d_Y g = |\Delta(K_{\varepsilon,(n)})| \int_{C_Y} F\Big[f_{(n)} + \int_{I^2} K_{\varepsilon,(n)}(\cdot, \cdot, s, t) f_{(n)}(s, t)dsdt\Big] \\ \cdot \exp\Big\{-\int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y \partial x}K_{\varepsilon,(n)}(x, y, s, t)f_{(n)}(s, t)dsdt\Big]^2 dxdy \\ - 2\int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_{\varepsilon,(n)}(x,y,s,t)f_{(n)}(s, t)dsdt\Big]df_{(n)}x,y)\Big\}d_Y f\,.\end{aligned} \tag{6.13}$$

As $n \to \infty f_{(n)}(x, y) \rightrightarrows f(x, y)$ and $K_{\varepsilon,(n)}(x, y, s, t) \rightrightarrows K_\varepsilon(x, y, s, t)$. Hence $g_n(x, y) \equiv f_{(n)}(x, y) + \int_{I^2} K_{\varepsilon,(n)}(x, y, s, t)f_{(n)}(s, t)dsdt \rightrightarrows g(x, y) = f(x, y) + \int_{I^2} K_\varepsilon(x, y, s, t)f(s, t)dsdt$. Therefore $F(g_n) \to F(g)$ boundedly. Thus $\lim_n \int_{C_Y} F(g_n)d_Y g = \int_{C_Y} F(g)d_Y g$. Now, by Fatou's lemma it follows from (6.13) with the help of Lemma 3, (4.11), and (4.13) that

$$\begin{aligned}\int_{C_Y} F(g)d_Y g \geqq |D(K_\varepsilon)| \int_{C_Y} F\Big[f + \int_{I^2} K_\varepsilon(\cdot, \cdot, s, t)f(s, t)dsdt\Big] \\ \cdot \exp\Big\{-\int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)dsdt\Big]^2 dxdy \\ -2\int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_\varepsilon(x, y, s, t)f(s, t)dsdt\Big]df(x, y)\Big\}d_Y f\,.\end{aligned} \tag{6.14}$$

By (6.1) $\int_{I^2} K_\varepsilon(x, y, s, t)f(s, t)dsdt \rightrightarrows \int_{I^2} K(x, y, s, t)f(s, t)dsdt$ as $\varepsilon \to 0^+$, and hence

$$F\Big[f + \int_{I^2} K_\varepsilon(\cdot, \cdot, s, t)f(s, t)dsdt\Big] \to F\Big[f + \int_{I^2} K(\cdot, \cdot, s, t)f(s, t)dsdt\Big]$$

boundedly. Since "mean convergence" implies the existence of an almost everywhere convergent subsequence, it follows from (6.6) that there exists a monotone sequence $\varepsilon_n \downarrow 0$ such that

$$\begin{aligned}\lim_{n\to\infty} \int_{I^2}\Big[\int_{I^2}\frac{\partial^2}{\partial y\partial x}K_{\varepsilon_n}(x, y, s, t)f(s, t)dsdt\Big]df(x, y) \\ = \int_{I^2}\Big[\frac{\partial^2}{\partial y\partial x}\int_{I^2}K(x, y, s, t)f(s, t)dsdt\Big]d^*f(x, y) \text{ for a.a.} f \in C_Y\,.\end{aligned} \tag{6.15}$$

Now we replace $\varepsilon_n$ for $\varepsilon$ in (6.14), and then use Fatou's Lemma together with (6.2) and (6.15) to arrive at (6.12). This completes the proof for the case when $F(f)$ in nonnegative, bounded, and continuous in uniform topology. To obtain the result for arbitrary nonnegative measurable functionals, we go through, as usual, the following steps: after proving it for the preceding case, we go to characteristic functionals of intervals, then 0-sets, then $0_\delta$-sets, and then mull-sets. (In the last case we get equality rather than inequality, both sides being zero.) Then to characteristic functionals of measurable sets, nonnegative simple functionals, nonnegative bounded Yeh-Wiener measurable functionals, and then finally nonnegative Yeh-Wiener measurable functionals. For the details of these steps see [2: pp. 391–392].

On account of (5.9), (5.10), (6.3), (6.5), and (6.11) we can establish the following on the same lines as above:

LEMMA 9′. *Let $F(g)$ be a nonnegative Yeh-Wiener measurable functional on $C_Y$, and let $K(x, y, s, t)$ be as in Theorem I and $K^*(x, y, s, t)$ its resolvent kernel. Then under the transformation*

$$T^{-1}: g(x, y) \to f(x, y) = g(x, y) + \int_{I^2} K^*(x, y, s, t) g(s, t) ds dt ,$$

*we have*

$$\int_{C_Y} F(f) d_Y f \geqq |D^{-1}(K)| \int_{C_Y} F(T^{-1} g) \cdot \exp\{-\Phi^*(g)\} d_Y g ,$$

*where*

$$\Phi^*(g) = \int_{I^2} \left[\frac{\partial^2}{\partial x \partial y} \int_{I^2} K^*(x, y, s, t) g(s, t) ds dt\right]^2 dx dy + 2\int_{I^2} \left[\frac{\partial^2}{\partial y \partial x} \int_{I^2} K^*(x, y, s, t) g(s, t) ds dt\right] d^* g(x, y) . \tag{6.16}$$

7. **Proof of theorems.** We prove Theorem I for nonnegative Yeh-Wiener measurable functionals first. In this case by Lemma 9 we have under the transformation $T$ in (1.1),

$$\int_{C_Y} F(g) d_Y g \geqq |D(K)| \int_{C_Y} F[Tf] \exp\{-\Phi(f)\} d_Y f , \tag{7.1}$$

and upon applying Lemma 9′ on tne right-hand side of (7.1), we obtain under the transformation

$$T^{-1}: g(x, y) \to f(x, y) = g(x, y) + \int_{I^2} K^*(x, y, s, t) g(s, t) ds dt$$

$$\int_{C_Y} F[Tf] \exp\{-\Phi(f)\} d_Y f \geqq |D^{-1}(K)| \int_{C_Y} F(T \circ T^{-1} g) \exp\{-\Phi(T^{-1} g)\} \cdot \exp\{-\Phi^*(g)\} d_Y g . \tag{7.2}$$

Hence if we show that

$$\Phi(T^{-1}g) + \Phi^*(g) = 0 \quad \text{for a.a. } g \text{ in } C_Y \,, \tag{7.3}$$

then it will follow from (7.1) and (7.2) that

$$\begin{aligned}\int_{C_Y} F(g)d_Y g &\geqq |D(K)| \int_{C_Y} F[Tf] \exp\{-\Phi(f)\} d_Y f \\ &\geqq \int_{C_Y} F(g) d_Y g \,,\end{aligned}$$

and hence the theorem will follow for the case. Observe now that if $T(f) = g$ and so $T^{-1}(g) = f$, then

$$\begin{aligned}&\int_{I^2} K(x, y, s, t) f(s, t) dsdt = g(x, y) - f(x, y) \,, \\ &\int_{I^2} K^*(x, y, s, t) g(s, t) dsdt = f(x, y) - g(x, y) \,,\end{aligned} \tag{7.4}$$

and from (3.10) and (3.11), we see that the left-hand side of the first equation in (7.4) is absolutely continuous, and

$$\frac{\partial^2}{\partial y \partial x} \int_{I^2} K(x, y, s, t) f(s, t) dsdt$$

is continuous. Therefore $(\partial^2/\partial y\partial x)[g(x, y) - f(x, y)]$ is also continuous. Therefore by the corollary to Theorem 4 in [3], we see that

$$\begin{aligned}&\int_{I^2} \frac{\partial^2}{\partial y \partial x}[g(x, y) - f(x, y)] d^*[g(x, y) - f(x, y)] \\ &\qquad \equiv \int_{I^2} \left\{\frac{\partial^2}{\partial y \partial x}[g(x, y) - f(x, y)]\right\}^2 dxdy \,.\end{aligned} \tag{7.5}$$

Hence from (2.8), (7.4), and (7.5) it follows that

$$\begin{aligned}\Phi(T^{-1}g) &= \Phi(f) \\ &= \int_{I^2} \left\{\frac{\partial^2}{\partial y \partial x}[g(x, y) - f(x, y)]\right\}^2 dxdy \\ &\quad + 2\int_{I^2} \frac{\partial^2}{\partial y \partial x}[g(x, y) - f(x, y)] d^* f(x, y) \\ &= \int_{I^2} \left\{\frac{\partial^2}{\partial y \partial x}[g(x, y) - f(x, y)]\right\}^2 dxdy \\ &\quad + 2\int_{I^2} \frac{\partial^2}{\partial y \partial x}[g(x, y) - f(x, y)] d^* f(x, y) \\ &\quad - 2\int_{I^2} \left\{\frac{\partial^2}{\partial y \partial x}[g(x, y) - f(x, y)]\right\}^2 dxdy \,,\end{aligned} \tag{7.6}$$

and from (6.16) and (7.4)

$$\Phi^*(g) = \int_{I^2}\left\{\frac{\partial^2}{\partial y \partial x}[f(x, y) - g(x, y)]\right\}^2 dxdy \tag{7.7}$$
$$+ 2\int_{I^2}\frac{\partial^2}{\partial y \partial x}[f(x, y) - g(x, y)]d^*g(x, y) .$$

Therefore (7.3) follows from (7.6) and (7.7). Hence the theorem holds for any nonnegative Yeh-Wiener measurable functionals. For arbitrary real Yeh-Wiener measurable functionals the theorem will also hold by considering the positive part and the negative part separately. For complex functionals we get the same result once we consider real part and imaginary part separately. This completes the proof of Theorem I.

To prove Theorem II we consider the following two transformations

$$I_1\colon f(x, y) \to g(x, y) = f(x, y) + f_0(x, y) ,$$
$$I_2\colon f(x, y) \to h(x, y) = f(x, y) + \int_{I^2} K(x, y, s, t) f(s, t) dsdt .$$

The theorem now follows by the use of Cameron-Martin translation theorem (see [11] or more precisely Theorem 1.4 in [4] on $L_1$ and then our Theorem I for $L_2$.

REMARK. As mentioned in the introduction our theorems in one-dimensional version have slightly different forms from the ones given in [1], the difference being in the expressions of $\Phi(f)$ and $\Psi(f)$. The $\Phi(f)$ in [1] is given by

$$\Phi(f) = \int_0^1\left[\frac{d}{dx}\int_0^1 K(x, s)f(s)ds\right]^2 dx \tag{7.8}$$
$$+ 2\int_0^1\left[\int_0^1\frac{\partial}{\partial x}K(x, s)f(s)ds\right]df(x) + \int_0^1 J(x)d[f^2(x)] ,$$

and ours is in the form

$$\Phi(f) = \int_0^1\left[\frac{d}{dx}\int_0^1 K(x, s)f(s)ds\right]^2 dx + 2\int_0^1\left[\frac{d}{dx}\int_0^1 K(x, s)f(s)ds\right]d^*f(x) . \tag{7.9}$$

However by the assumptions given on $K(x, s)$ in [1], we have that

$$\frac{d}{dx}\int_0^1 K(x, s)f(s)ds = \frac{d}{dx}\int_0^x K_2(x, s)f(s)ds + \int_x^1 K_1(x, s)f(s)ds\Big]$$
$$= [K_2(x, x) - K_1(x, x)]f(x) + \int_0^1\frac{\partial}{\partial x}K(x, s)f(s)ds$$

$$= J(x)f(x) + \int_0^1 \frac{\partial}{\partial x} K(x, s)f(s)ds .$$

But assuming absolute continuity of $J(x)$ with $J'(x) \in L^2$, it follows from Theorem 5 in [3] that

$$\int_0^1 J(x)f(x)f(x)d^*f(x) = \frac{1}{2}\int_0^1 J(x)d[f^2(x)] \quad \text{for a.a.} \quad f \in C_W .$$

Also as mentioned earlier

$$\int_0^1 \left[\int_0^1 \frac{\partial}{\partial x} K(x, s)f(s)ds\right] d^*f(x) = \int_0^1 \left[\int_0^1 \frac{\partial}{\partial x} K(x, s)f(s)ds\right] df(x)$$

for almost all $f$ in $C_W$. Thus (7.8) and (7.9) represent essentially the same thing.

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