PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

LEFSCHETZ FIXED POINT THEOREMS FOR A NEW CLASS OF MULTI-VALUED MAPS

MICHAEL J. POWERS

The Lefschetz fixed point theorem states that whenever the Lefschetz number $\Lambda(f)$ of a map $f: X \to X$ is nonzero, then f must have a fixed point. The theorem is known to hold when X is an ANR and f is a compact continuous map. The theorem has been studied for compact, upper semi-continuous, acyclic multi-valued maps and is known to hold in this setting for topologically complete ANR's.

A more general class of multi-valued maps is considered in this paper: the class of compact upper semi-continuous maps which can be written as a composition of acyclic maps. Using this class of maps, a theorem is proved which generates spaces for which the Lefschetz theorem holds. In particular, the Lefschetz theorem holds for all (metric) ANR's.

Lefschetz fixed point theorems for multi-valued maps of compact spaces were first studied by Eilenberg and Montgomery [2]. Motivated by the LeRay-Schauder fixed point results, the concept of Lefschetz space has been generalized to that of Λ-space and $M\Lambda$-space for multi-valued maps ([3], [4], [6], [8]). A space X is a Λ-space if, for every compact map $f: X \to X$, the Lefschetz number is defined and the Lefschetz theorem holds. An $M\Lambda$-space is defined in a similar way for multi-valued maps. The multi-valued maps, however, are required to be both compact and acyclic (i.e., the image of each point is an acyclic subset). This additional condition is necessary in order to define the induced homomorphism on the homology groups.

A method of generating $M\Lambda$-spaces from known $M\Lambda$-spaces was presented in [8]. In applying the method, however, the acyclicity condition created a problem. It was possible to conclude, for example, that (metric) ANR's where Λ-spaces, but not that such spaces were $M\Lambda$-spaces. While enlarging the concept of $M\Lambda$-space to include maps which are compositions of acyclic maps, a more general generating theorem can be proved. The category of all metric ANR's is produced by this generating theorem.

2. Definitions. The image of a subset A of X under a multi-valued map $F: X \to Y$ is $F(A) = \bigcup_{x \in A} F(x)$. The map F is *compact* if $F(X)$ is contained in a compact subset of Y. Let $\Gamma(F)$ denote the graph of F. The projections $p: \Gamma(F) \to X$ and $q: \Gamma(F) \to Y$ are

called the *projections associated with F*. F is *upper semi-continuous* (u.s.c.) if:

(i) $F(x)$ is compact for each $x \in X$ and

(ii) for each $x \in X$ and each open set V containing $F(x)$, there is an open neighborhood U of x such that $F(U) \subset V$.

If $F: X \to Y$ and $G: Y \to Z$ are multi-valued maps, the *composition* of F and G is denoted $G \circ F: X \to Z$ and is defined by $G \circ F(x) = \bigcup_{y \in F(x)} G(y)$. The composition of u.s.c. maps is again u.s.c.

A point x is a *fixed point* for $F: X \to X$ if $x \in F(x)$.

Eilenberg and Montgomery [2] defined an induced homomorphism for certain multi-valued maps $F: X \to Y$ with X and Y compact. Their definition was extended to the non-compact case in [8]. Only the basic definitions are given here. Let $\mathscr{T}$ denote the category of Hausdorff spaces and continuous maps. Let $\mathscr{A}$ denote the category of graded vector spaces over the field $\mathscr{Q}$ of rational numbers and homomorphisms of degree zero. Let $\mathscr{H}: \mathscr{T} \to \mathscr{A}$ be a covariant functor which satisfies the homotopy axiom and agrees with the Čech homology functor $\check{H}$ on the subcategory of compact spaces. We also require that $\mathscr{H}$ satisfy a Vietoris mapping theorem of the following type:

If $f: X \to Y$ is a morphism in $\mathscr{T}$ such that

(i) f is inverse acyclic (i.e., for each $y \in Y$, $f^{-1}(y)$ is acyclic with respect to $\mathscr{H}$) and

(ii) f is proper (i.e., for each compact subset K of Y, $f^{-1}(K)$ is compact), then $\mathscr{H}(f)$ is an isomorphism in $\mathscr{A}$.

A functor $\vec{H}$ satisfying these conditions was exhibited in [8].

A multi-valued map $F: X \to Y$ is *acyclic* (with respect to $\mathscr{H}$) if for each $x \in X$, $F(x)$ is an acyclic subset of Y. When $F: X \to Y$ is u.s.c. and acyclic the projection $p: \Gamma(F) \to X$ satisfies the conditions for the Vietoris Theorem above.

DEFINITION 2.1. Let $F: X \to Y$ be an u.s.c. acyclic multi-valued map of spaces in $\mathscr{T}$. The *homomorphism induced by F* (with respect to $\mathscr{H}$) is defined $\mathscr{H}(F) = \mathscr{H}(q) \circ \mathscr{H}(p)^{-1}$, where p, q are the projections associated with F. We write $\mathscr{H}(F) = F_*$.

THEOREM 2.2. *Let $F: X \to Y$ and $G: Y \to Z$ be* u.s.c. *acyclic maps of spaces in $\mathscr{T}$. If $G \circ F$ is acyclic, then $(G \circ F)_* = G_* \circ F_*$.* (See [8], Theorem (3.8).)

Finally, we recall the definition of trace for endomorphisms of infinite dimensional vector spaces given by Leray [7]. Let ϕ be an endomorphism of a vector space V; let $N_\phi = \{v \mid \phi^p(v) = 0$ for some $p \geqq 1\}$. Suppose that V/N_ϕ has finite dimension. Then the trace of ϕ, $\mathrm{tr}(\phi)$ or $\mathrm{tr}_V(\phi)$, is defined by $\mathrm{tr}_V(\phi) = \mathrm{tr}_{V/N_\phi}(\phi')$, where ϕ' is induced by ϕ and $\mathrm{tr}(\phi')$ is the classical trace.

DEFINITION 2.3. Let $\phi = \{\phi_k\}$: $V \to V$ be an endomorphism (of degree zero) in $\mathscr{A}$. Then f is said to be of *finite type* if

(i) V_k/N_{ϕ_k} has finite dimension for each k and

(ii) $N_{\phi_k} = V_k$ in all but a finite number of dimensions k.

DEFINITION 2.4. Let ϕ: $V \to V$ be an endomorphism of finite type in $\mathscr{A}$. The *Lefschetz number* of ϕ is

$$\Lambda(\phi) = \sum_{k=0}^{\infty} (-1)^k \mathrm{tr}(\phi_k) .$$

LEMMA 2.5. *Let* ϕ: $V \to W$ *and* ψ: $W \to V$ *be homomorphisms* (of degree zero) *in* $\mathscr{A}$. *If* $\psi \circ \phi$: $V \to V$ *is of finite type, then* $\phi \circ \psi$: $W \to W$ *is also of finite type and in each dimension* k, $\mathrm{tr}(\psi_k \circ \phi_k) = \mathrm{tr}(\phi_k \circ \psi_k)$. (See [7], Prop. c.)

3. *$M\Lambda$*-spaces. We will consider u.s.c. maps F: $X \to X$ which can be factored into a sequence $G_n \circ \cdots \circ G_0$ of *acyclic* u.s.c. maps. We will then want to use $\Lambda(G_{n*} \circ \cdots \circ G_{0*})$ as a Lefschetz number for F. However, an easy example shows that some care must be exercised. It is possible to have

$$G_2 \circ G_1 = F = H_2 \circ H_1 \text{ and } \Lambda(G_{2*} \circ G_{1*}) \neq 0,\ \Lambda(H_{2*} \circ H_{1*}) = 0 .$$

Simply let F: $S^2 \to S^2$ by $F(x) = S^2$ for all $x \in S^2$. Define K: $S^2 \to S^2$ by $K(x) = \{y \in S^2 \mid \|x - y\| \leqq 7/4\}$. Then K is u.s.c. and acyclic. In fact, K is homotopic to the identity on S^2 and hence K_* is the identity. Let $G_1 = G_2 = K$. Let H_1 be defined by $H_1(x) = K(-x)$ and let $H_2 = K$. Then $G_2 \circ G_1 = F = H_2 \circ H_1$; but $\Lambda(G_{2*} \circ G_{1*}) = 2$, while $\Lambda(H_{2*} \circ H_{1*}) = 0$.

$\mathscr{T}_0$ will denote a subcategory of $\mathscr{T}$.

DEFINITION 3.1. An u.s.c. map F: $X \to X$ of a space in $\mathscr{T}$ is *admissible* (*relative to* $\mathscr{T}_0$) if there are maps

$$G_i\colon\ Y_i \longrightarrow Y_{i+1},\ i = 0, \cdots, n$$

(where $Y_0 = Y_{n+1} = X$) satisfying

(i) $F = G_n \circ \cdots \circ G_0$,

(ii) G_i is acyclic and u.s.c. for each $i = 0, \cdots, n$, and

(iii) Y_i is in $\mathscr{T}_0$, for $i = 1, \cdots, n$.
Each such sequence $G_0, \cdots, G_n$ is called an *admissible sequence* for F.

DEFINITION 3.2. An admissible map $F: X \to X$ is an *M-Lefschetz map* (*relative to* $\mathscr{T}_0$) if

(i) for each admissible sequence $G_0, \cdots, G_n$ for F, $G_{n*} \circ \cdots \circ G_{1*}$ has finite type and

(ii) whenever $G_0, \cdots, G_n$ is an admissible sequence with

$$\Lambda(G_{n*} \circ \cdots \circ G_{1*}) \neq 0 ,$$

then F must have a fixed point.

DEFINITION 3.3. A space X is an *M-Lefschetz space* [*MΛ-space*] (*relative to* $\mathscr{T}_0$) if each admissible [compact admissible] map $F: X \to X$ (relative to $\mathscr{T}_0$) is an M-Lefschetz map (relative to $\mathscr{T}_0$).

Note that a single-valued continuous map f is admissible iff f_* has finite type. Thus the theory here is no more general in the case of single-valued maps than that presented in [6].

The concept of M-Lefschetz space and $M\Lambda$-space is more general than that presented in [8]. We continue to use the same terminology since the definition presented here seems to be the one that most warrants further study. Moreover, it will be shown that spaces already known to be M-Lefschetz or $M\Lambda$-spaces remain M-Lefschetz and $M\Lambda$-spaces in this more general setting.

4. Generating theorems. The following theorems present a method for generating $M\Lambda$-spaces from known $M\Lambda$-spaces. Similar theorems hold for M-Lefschetz spaces. It was recently pointed out to the author that this method of generating $M\Lambda$-spaces was used by W. Hurewitz in his lectures for the proof of the Lefschetz theorem for compact ANR's.

DEFINITION 4.1. Let $F, G: X \to Y$ be multi-valued maps and $\alpha \in \operatorname{Cov} Y$, the set of open covers of Y. Then F *is* α-*near* G if for each $x \in X$ and each $y \in F(x)$, there is a U in α containing y and meeting $G(x)$.

THEOREM 4.2. *Let X be a T_3-space in $\mathscr{T}_0$. Suppose that for each $\alpha \in \operatorname{Cov} X$ there is an $M\Lambda$-space Y_α (relative to $\mathscr{T}_0$) and maps $H_\alpha: X \to Y_\alpha$, $K_\alpha: Y_\alpha \to X$ satisfying*

(i) *H_α, K_α are* u.s.c. *and acyclic,*

(ii) *$K_{\alpha*} \circ H_{\alpha*} = 1_{\mathscr{H}(X)}$, and*

(iii) *$K_\alpha \circ H_\alpha$ is α-near 1_X.*
Then X is an $M\Lambda$-space(relative to $\mathscr{T}_0$).

Proof. Let $F\colon X \to X$ be an u.s.c. compact map which is admissible (relative to $\mathscr{T}_0$). Let $G_0, \cdots, G_n$ be an admissible sequence for F. Take any $\alpha \in \operatorname{Cov} X$ and H_α, K_α, Y_α as in the hypothesis. Consider the diagram

$$\begin{tikzcd}
Y_\alpha \arrow[rr, "H_\alpha \circ G_n \circ \cdots \circ G_0 \circ K_\alpha"] \arrow[drr, dashed, "F \circ K_\alpha"] & & Y_\alpha \\
X \arrow[u, "H_\alpha"] \arrow[rr, "F = G_n \circ \cdots \circ G_0"'] & & X . \arrow[u, "H_\alpha"']
\end{tikzcd}$$

The map $H_\alpha \circ G_n \circ \cdots \circ G_0 \circ K_\alpha$ is compact and admissible (relative to $\mathscr{T}_0$). Since Y_α is an $M\Lambda$-space $H_{\alpha*} \circ (G_{n*} \circ \cdots \circ G_{0*} \circ K_{\alpha*})$ has finite type. Thus by (2.5) $(G_{n*} \circ \cdots \circ G_{0*} \circ K_{\alpha*}) \circ H_{\alpha*} = G_{n*} \circ \cdots \circ G_{0*}$ has finite type and $\Lambda(G_{n*} \circ \cdots \circ G_{0*}) = \Lambda(H_{\alpha*} \circ G_{n*} \circ \cdots \circ G_{0*} \circ K_{\alpha*})$.

Now suppose $\Lambda(G_{n*} \circ \cdots \circ G_{0*}) \neq 0$. Then for each $\alpha \in \operatorname{Cov} X$, $H_\alpha \circ G_n \circ \cdots \circ G_0 \circ K_\alpha$ is a compact admissible map and

$$\Lambda(H_{\alpha*} G_{n*} \circ \cdots \circ G_{0*} K_{\alpha*}) \neq 0 .$$

Thus the map has a fixed point $y_\alpha \in Y_\alpha$.

Since $y_\alpha \in H_\alpha \circ F \circ K_\alpha(y_\alpha)$, we can choose $\bar{x}_\alpha$ in $F(K_\alpha(y_\alpha)) \subseteqq \overline{F(X)}$ such that $y_\alpha \in H_\alpha(\bar{x}_\alpha)$. Choose x_α in $K_\alpha(y_\alpha)$ such that $\bar{x}_\alpha \in F(x_\alpha)$. Finally, since $K_\alpha \circ H_\alpha$ is α-near 1_X and since $x_\alpha \in K_\alpha(y_\alpha) \subseteqq K_\alpha \circ H_\alpha(\bar{x}_\alpha)$, then there is an element U_α of α which contains both x_α and $\bar{x}_\alpha$. Now $\{\bar{x}_\alpha\colon \alpha \in \operatorname{Cov} X\}$ is a net in the compact space $\overline{F(X)}$ and there is a subnet $T = \{T(m)\colon m \in E\}$ converging to a point $x_0 \in \overline{F(X)} \subset X$. (Then there is a map $N\colon E \to \operatorname{Cov} X$ of directed sets satisfying $T(m) = \bar{x}_{N(m)}$ and for each $\alpha \in \operatorname{Cov} X$, there is an $n \in E$ such that $m > n$ implies $N(m) > \alpha$.)

Consider the net $S = \{S(m)\colon m \in E\}$ defined by $S(m) = x_{N(m)}$. It suffices to prove that S also converges to x_0. For then $S \times T$ is a net in $\Gamma(F)$ converging to (x_0, x_0). But since F is u.s.c. its graph is closed. Hence $(x_0, x_0) \in \Gamma(F)$ and $x_0 \in F(x_0)$.

Let V be any neighborhood of x_0. To show that S is eventually in V, first recall that x_α and $\bar{x}_\alpha$ are both elements of U_α for each α in $\operatorname{Cov} X$. Since X is regular, there is an open neighborhood W of x_0 such that $\bar{W} \subset V$. Let $\alpha_0 = \{V, X - \bar{W}\}$ in $\operatorname{Cov} X$. Since the net T converges to x_0, there is an element n_W in E such that $m > n_W$ implies $T(m) \in W$. Moreover, there is an element n_0 of E such that

$N(m) > \alpha_0$ when $m > n_0$. Since E is directed, there exists an element n of E such that $n > n_0, n_W$. Now for any $m > n$ it is easily verified that $S(m) \in V$. For $T(m) = \bar{x}_{N(m)} \in W$ and $\bar{x}_{N(m)}, x_{N(m)}$ are both in $U_{N(m)}$. Then $U_{N(m)} \cap W \neq \emptyset$ and hence $U_{N(m)} \not\subset X - \bar{W}$. Since $m > n_0$, $N(m) > \alpha_0$ and $U_{N(m)} \subset V$. Thus $x_{N(m)} = S(m) \in V$.

Condition (iii) of the theorem is fairly restrictive. However, for the present applications H_α and K_α will be single-valued maps and the condition is perfectly reasonable.

COROLLARY 4.3. *A retract of an MA-space is again an MA-space.*

If X is a metric space, let $\{\varepsilon_k\}$ be a sequence of positive numbers converging to 0 and let α_k be some open cover of X by ε_k-balls, for each k. Then while $D = \{\alpha_k\}$ is not cofinal in Cov X in general, we still obtain the following theorem.

THEOREM 4.4. *Let X be a metric space in $\mathscr{T}_0$. Suppose that for each $\alpha_k \in D$ there is an MA-space Y_k(relative to $\mathscr{T}_0$) and maps H_k: $X \to Y_k$ and K_k: $Y_k \to X$ satisfying*

(i) *H_k, K_k are* u.s.c. *and acyclic,*

(ii) *$K_{k*} \circ H_{k*} = 1_{\mathscr{H}(X)}$, and*

(iii) *$K_k \circ H_k$ is ε_k-near 1_X(i.e., for each $x \in X$, $K_k \circ H_k(x) \subset B(x; \varepsilon_k)$).*

Then X is an MA-space (relative to $\mathscr{T}_0$).

Proof. Let F: $X \to X$ be an u.s.c. compact map which is admissible (relative to $\mathscr{T}_0$). Let $G_0, \cdots, G_n$ be an admissible sequence for F. Then for each integer k and H_k, K_k, Y_k as in the hypothesis it can be proved, just as in (4.2), that

$$\Lambda(G_{n*} \circ \cdots \circ G_{0*}) = \Lambda(H_{k*} \circ G_{n*} \circ \cdots \circ G_{0*} \circ K_{k*}) .$$

Suppose that $\Lambda(G_{n*} \circ \cdots \circ G_{0*}) \neq 0$. Then for each k, the compact admissible map $H_k \circ G_n \circ \cdots \circ G_0 \circ K_k$: $Y_k \to Y_k$ has nonzero Lefschetz number and hence has a fixed point $y_k \in Y_k$. Just as in the proof of (4.2), $y_k \in H_k \circ F \circ K_k(y_k)$ and there are points $\bar{x}_k$ in $F(K_k(y_k))$ and x_k in $K_k(y_k)$ such that $y_k \in H_k(\bar{x}_k)$ and $\bar{x}_k \in F(x_k)$. Now since $K_k \circ H_k$ is ε_k-near 1_X and since $x_k \in K_k(y_k) \subset K_k \circ H_k(\bar{x}_k)$, then $d(x_k, \bar{x}_k) < \varepsilon_k$. Now $\{\bar{x}_k\}$ is a sequence in the compact space $\overline{F(X)}$ and there is a subsequence, still denoted $\{\bar{x}_k\}$, which converges to a point $x_0 \in \overline{F(X)} \subset X$.

It suffices to prove that the corresponding subsequence of $\{x_k\}$ also converges to x_0. For then $\{(x_k, \bar{x}_k)\}$ is a sequence in $\Gamma(F)$ converging to (x_0, x_0) and $x_0 \in F(x_0)$. Given $\varepsilon > 0$, choose k_1 so that for $k > k_1$, $\varepsilon_k < \varepsilon/2$. Choose k_2 so that for $k > k_2$, $d(\bar{x}_k, x_0) < \varepsilon/2$. Then for $k > \max\{k_1, k_2\}$, $d(x_k, \bar{x}_k) < \varepsilon_k < \varepsilon/2$ and hence $d(x_k, x_0) < \varepsilon$. Then the subsequence $\{x_k\}$ converges to x_0.

5. **Applications.** In this section, $\mathscr{T}_0$ will be the category $\mathscr{T}_M$ of metric spaces and continuous maps.

LEMMA 5.1. *Let f: $X \to Y$ and g: $Y \to Z$ be maps in $\mathscr{T}$ which are proper and inverse acyclic (with respect to $\mathscr{H}$). Then $g \circ f$ is also proper and inverse acyclic (with respect to $\mathscr{H}$).*

Proof. $g \circ f$ is clearly proper. Take $z \in Z$ and let $A = g^{-1}(z)$. Then A is an acyclic subset of Y. Let f_1: $f^{-1}(A) \to A$ be defined by f. Then f_1 is a map in $\mathscr{T}$ satisfying conditions (i) and (ii) for the Vietoris Theorem. Hence $f_{1*} = \mathscr{H}(f^{-1}(A)) \to \mathscr{H}(A)$ is an isomorphism and $f^{-1}(A) = (g \circ f)^{-1}(z)$ is acyclic.

THEOREM 5.2. (Eilenberg and Montgomery). *Let X be a compact metric space and P a finite polyhedron. Let f, g: $X \to P$ be continuous maps such that f is inverse acyclic. Then $\Lambda(g_* \circ f_*^{-1})$ is defined and if $\Lambda(g_* \circ f_*^{-1}) \neq 0$, then f and g have a coincidence. (See Theorem 4, [2].)*

THEOREM 5.3. *Finite polyhedra are M-Lefschetz spaces (relative to $\mathscr{T}_M$).*

Proof. Let P be a finite polyhedron and F: $P \to P$ an u.s.c. admissible map. Let $G_0, \cdots, G_n$ be an admissible sequence for F, where G_i: $Y_i \to Y_{i+1}$ with $Y_0 = Y_{n+1} = P$ and all Y_i are metric spaces. Then $G_{n*} \circ \cdots \circ G_{0*}$ has finite type since P is finite.

Now suppose that $\Lambda(G_{n*} \circ \cdots \circ G_{0*}) \neq 0$. We prove that F has a fixed point for the case $n = 1$. The proof for arbitrary n is an exact generalization of this proof. We have G_0: $P \to Y$ and G_1: $Y \to P$, where Y is a metric space and $F = G_1 \circ G_0$. Let p_0, q_0 be the projections associated with G_0. Then p_0 is inverse acyclic. The composition $G_1 \circ q_0$: $\Gamma G_0 \to P$ is an u.s.c. acyclic map.

Let p_1, q_1 be the projections associated with $G_1 \circ q_0$; then p_1 is inverse acyclic. Then by (5.1) $p_0 \circ p_1$ is inverse acyclic and we have q_1, $p_0 \circ p_1$: $\Gamma G_1 \circ q_0 \to P$ where $\Gamma G_1 \circ q_0$ is a compact metric space (since P is finite). It is easy to check that $G_{1*} \circ G_{0*} = q_{1*} \circ (p_0 \circ p_1)_*^{-1}$ and hence $\Lambda(q_{1*} \circ (p_0 \circ p_1)_*^{-1}) \neq 0$. Then by (5.2) q_1 and $p_0 \circ p_1$ have a coincidence, say $(x, y, x') \in \Gamma G_1 \circ q_0$. Since (x, y, x') is a coincidence point for these maps, $x = x'$. And $(x, y, x') \in \Gamma G_1 \circ q_0$ implies that $x' = x \in G_1 \circ q_0(x, y) = G_1(y)$ and $y \in G_0(x)$. Thus $x \in G_1 \circ G_0(x) = F(x)$.

$$\begin{tikzcd}
 & \Gamma G_1\circ q_0 \arrow[dl, "p_1"'] \arrow[ddr, "q_1"] & \\
\Gamma G_0 \arrow[r, "q_0"] \arrow[d, "p_0"'] & Y \arrow[dr, "G_1"] & \\
P \arrow[ur, "G_0"] \arrow[rr, "F"'] & & P
\end{tikzcd}$$

THEOREM 5.4. *Every polyhedron with the Whitehead topology* ([5], p. 99) *is an MΛ-space (relative to* $\mathscr{T}_M$*).*

Proof. Let P be a polyhedron with the Whitehead topology and $F\colon P\to P$ an u.s.c. compact admissible map. Let $G_0, \cdots, G_n$ be an admissible sequence for F. As in the proof of (5.3) it suffices to consider $n=1$. Then we have u.s.c. and acyclic maps $G_0\colon P\to Y$ and $G_1\colon Y\to P$ where Y is a metric space and $F = G_1\circ G_0$.

Since P has the Whitehead topology and $\overline{F(P)}$ is compact, there is a finite subpolyhedron P' of P with $F(P)\subset P'$. Let $Y' = G_0(P')$ and define $F'\colon P'\to P'$, $G_0'\colon P'\to Y'$, $G_1'\colon Y'\to P'$ by F, G_0, G_1. Consider the commmutative diagram where G_0'', G_1'' are defined by G_0, G_1 and the single-valued maps are inclusions.

$$\begin{tikzcd}
P \arrow[r, "G_0"] \arrow[dr, "G_0''"] & Y \arrow[r, "G_1"] & P \\
 & G_0(P) \arrow[u, "k"] \arrow[dr, "G_1''"] & \\
P' \arrow[uu, "i"] \arrow[r, "G_0'"'] & Y' \arrow[u, "j"] \arrow[r, "G_1'"'] & P' \arrow[uu, "i"']
\end{tikzcd}\ .$$

Then $G'_{1*}\circ G'_{0*} = (G''_{1*}G''_{0*})\circ i_*$ and $G_{1*}\circ G_{0*} = i_*\circ(G''_{1*}\circ G''_{0*})$. Since $\Lambda(G'_{1*}G'_{0*})$ exists, (2.5) implies that $\Lambda(G_{1*}\circ G_{0*})$ exists and they are equal. Finally, if $\Lambda(G_{1*}\circ G_{0*})\neq 0$, then by (5.3) F' has a fixed point. This is also fixed point for F.

THEOREM 5.5. (a) *Every (metric) ANR is an MΛ-space (relative to* $\mathscr{T}_M$*).* (b) *Every compact (metric) ANR is an M-Lefschetz space (relative to* $\mathscr{T}_M$*).*

Proof. (a) Let X be a (metric) ANR and $\alpha \in \operatorname{Cov} X$. Then there is a polyhedron P_α (with the Whitehead topology) and continuous maps h_α: $X \to P_\alpha$, k_α: $P_\alpha \to X$ such that $k_\alpha \circ h_\alpha$ and 1_X are α-homotopic. (See 1, Theorem 14.3.) In particular, $k_{\alpha *} \circ h_{\alpha *} = 1_{\mathscr{H}(X)}$ and $k_\alpha \circ h_\alpha$ and 1_X are α-near. Then by (4.2) and (5.4) X is an $M\Lambda$-space.

(b) The proof follows as in (a) using finite polyhedra and (5.3).

As a final application, the results (6.1) and (6.2) of [8] can be strengthened.

THEOREM 5.6. *Let X be any convex subset of a Banach space and F: $X \to X$ a compact* u.s.c. *map which is admissible* (*relative to* $\mathscr{T}_M$). *Then F has a fixed point.*

Proof. X is an AR and hence is an $M\Lambda$-space. Let $G_0, \cdots, G_n$ be an admissible sequence for F, where G_i: $Y_i \to Y_{i+1}$ with $Y_0 = Y_{n+1} = X$ and all Y_i are metric spaces. It suffices to show

$$\Lambda(G_{n*} \circ \cdots \circ G_{0*}) \neq 0\ .$$

But since X is convex, the Lefschetz number of $G_{n*} \circ \cdots \circ G_0^*$ is simply its trace in dimension 0.

Take a point $y_0 \in Y_0$ and for $i = 0, \cdots, n$ let y_{i+1} be an element of $G_i(y_i)$. Consider the diagram.

$$\begin{array}{ccccccc}
\mathscr{H}_0(Y_0) & \xrightarrow{G_{0*}} & \mathscr{H}_0(Y_1) & \xrightarrow{G_{1*}} & \mathscr{H}_0(Y_2) \longrightarrow \cdots & \xrightarrow{G_{n*}} & \mathscr{H}_0(Y_{n+1}) \\
{\scriptstyle j_{0*}}\uparrow & & {\scriptstyle j_{1*}}\uparrow & & {\scriptstyle j_{2*}}\uparrow & & {\scriptstyle j_{n+1*}}\uparrow \\
\mathscr{H}_0(\{y_0\}) & \xrightarrow[c_{0*}]{} & \mathscr{H}_0(\{y_1\}) & \xrightarrow[c_{1*}]{} & \mathscr{H}_0(\{y_2\}) \longrightarrow \cdots & \xrightarrow[c_{n*}]{} & \mathscr{H}_0(\{y_{n+1}\})\ .
\end{array}$$

For each i, $j_{i+1} \circ c_i(y_i) = y_{i+1} \in G_i(y_i) = G_i \circ j_i(y_i)$. Thus $j_{i+1} \circ c_i$ is a cross-section of $G_i j_i$. But in general, if F and F' are acyclic maps and F is a cross-section of F', then $F_* = F'_*$ (see [8] (3.7)). Hence the diagram commutes. Since j_{0*} and j_{n+1*} are isomorphisms, we conclude that $G_{n*} \cdots G_{0*}$ is an isomorphism and hence its Lefschetz number is nonzero.

COROLLARY 5.7. *Let X be a metric AR and F: $X \to X$ a compact* u.s.c. *map which is admissible* (*relative to* $\mathscr{T}_M$). *Then F has a fixed point.*

Proof. X can be imbedded as a closed subset of a convex set C in a Banach space and there is a retraction r: $C \to X$. Then $F \circ r$ is a compact u.s.c. map and is admissible (relative to $\mathscr{T}_M$). By (5.8) $F \circ r$ has a fixed point. This must also be a fixed point for F.

References

1. J. Dugundji, *Absolute neighborhood retracts and local connectedness in arbitrary metric spaces*, Comp. Math., **13** (1958), 229-246.
2. S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. J. Math., **58** (1946), 214-222.
3. A. Granas, *Generalizing the Hopf-Lefschetz fixed point theorem for non-compact ANR's*, Proc. Symp. on Infinite Dimensional Topology, Baton Rouge, 1967.
4. A. Granas and L. Gorniewicz, *Fixed point theorems for multi-valued mappings of the absolute neighborhood retracts*, J. de Math. Pure et Appl. **49** (1970), 381-395.
5. S.-T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, 1965.
6. J. Jaworowski, and M. Powers, *Λ-spaces and fixed point theorems*, Fund. Math., **64** (1969), 157-162.
7. J. Leray, *Theorie des points fixes: Indice total et nombre de Lefschetz*, Bull. Soc. Math. France, **87** (1959), 221-233.
8. M. Powers, *Multi-valued mappings and Lefschetz fixed point theorems*, Proc. Camb. Phil. Soc., **68** (1970), 619-630.

Received January 5, 1971 and in revised form July 12, 1971.

NORTHERN ILLINOIS UNIVERSITY