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WILD ARCS IN THREE-SPACE

3: AN INVARIANT OF ORIENTED LOCAL TYPE FOR EXCEPTIONAL ARCS

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This paper continues the investigations of previous papers in this series, and attention is confined to exceptional arcs. Given a special constructing sequence for an exceptional arc, the associated sequence of local linking matrices is defined, and the cofinality class of this sequence is shown to be an invariant of the oriented local arc type of the exceptional arc. This paper also gives a set of sufficient conditions for an arc to have a constructing sequence.

The paper closes with examples which show that there exist uncountably many locally nonamphicheiral exceptional arcs of any penetration index. No two of the locally nonamphicheiral exceptional arcs exhibited here can be distinguished by the invariant of nonoriented local arc type developed previously.

An exceptional arc of penetration index three at its wild point is a Fox-Artin arc, and an invariant of oriented local type for such arcs has already been developed in [4]. This paper uses the techniques of [5] and [6] to find invariants of oriented local type for exceptional arcs whose penetration index is at least five. The paper concludes with examples of the application of this invariant and the invariant of [5].

The results in this paper are generalisations of the results in [2], pp. 45–70 and pp. 82–95. The notation and terminology used here has been explained in [3], [4], [5], and [6].

1. Preliminaries. Let k be an oriented arc in R^3 which is locally tame except at the endpoint p, where $P_0(k, p) \geqq 3$. Let

$$E_0 \supset E_1 \supset E_2 \supset \cdots$$

be a sequence of tame closed 3-cell neighborhoods of p such that

(i) $E_{i+1} \subset \operatorname{Int} E_i$ for each i, and $\bigcap E_j = p$,

(ii) the sets $A(E_i, E_{i+1})$ (of those subarcs of k in $E_i - E_{i+1}$ whose endpoints both lie on $\operatorname{Bd} E_i$) and $B(E_{i+1}, E_i)$ (of those subarcs of k in $E_i - \operatorname{Int} E_{i+1}$ whose endpoints both lie on $\operatorname{Bd} E_{i+1}$) are not empty for any i,

(iii) for each $\alpha \in A(E_i, E_{i+1})$ there exists a $\beta \in B(E_{i+1}, E_i)$ such that the pair (α, β) is not splittable (cf. [3], p. 230), and

(iv) for each $\beta \in B(E_{i+1}, E_i)$ there exists an $\alpha \in A(E_i, E_{i+1})$ such that the pair (α, β) is not splittable.

Then we shall say that the sequence $E_0 \supset E_1 \supset E_2 \supset \cdots$ has the (Fox-Artin) property $\mathscr{F}$.

The arc k is *exceptional* if it has a special constructing sequence, that is a sequence

$$\mathscr{C}\colon\ E_0 \supset V_0 \supset E_1 \supset V_1 \supset E_2 \supset V_2 \supset \cdots$$

of k-tori and tame closed 3-cells such that

(i) $E_0 \supset V_0 \succ V_1 \succ V_2 \succ \cdots$ is a constructing sequence for k in E_0 ([5]),

(ii) for each $i \geqq 1$, $\mathrm{Bd}\, E_i \subset \mathrm{Int}\,(V_{i-1} - V_i)$, and

(iii) the sequence $E_0 \supset E_1 \supset E_2 \supset \cdots$ has property $\mathscr{F}$.

Note that $P_1(k, p) = 1$ if k is exceptional. In this paper we will be assuming that $P_0(k, p) \geqq 5$, so we may use the results of [5] in our study of exceptional arcs; Theorem 1 of [5] is particularly important.

Let $L = l_1 \cup l_2$ be an oriented link of two components in an oriented 3-cell E, that is, an embedding of the disjoint union of two oriented 1-spheres in E. Each l_i bounds an orientable surface $S_i \subset E$. The *linking number* $\lambda(l_1, l_2)$ of l_1 with l_2 is the number of intersections of l_1 with S_2, counted algebraically; $\lambda(l_1, l_2) = \nu(l_1 \cap S_2)$. The value of $\lambda(l_1, l_2)$ is independent of the choice of the surface S_2, and

$$\lambda(l_1, l_2) = \nu(l_1 \cap S_2) = \nu(S_1 \cap l_2) = \lambda(l_2, l_1)$$

([8], p. 278). Since linking number is an invariant of the F-isotopy class of a link, it makes sense to define $\lambda(\alpha, \beta)$ for each

$$\alpha \in A(E_i, E_{i+1}) \qquad \text{and} \qquad \beta \in B(E_{i+1}, E_i)$$

whenever we have a sequence $E_0 \supset E_1 \supset E_2 \supset \cdots$ with property $\mathscr{F}$ (cf. [3], p. 230).

Suppose k is exceptional, and $\mathscr{C}$ is a special constructing sequence for k. Since k is oriented, the arcs of $A(E_i, E_{i+1})$ and $B(E_{i+1}, E_i)$ have a natural ordering for each i, namely the order in which they occur in the arc k. Henceforth, we shall assume that these sets have this natural ordering.

Then for each special constructing sequence $\mathscr{C}$, the *sequence of local linking matrices* is the sequence whose ith term is the *local linking matrix* $\Lambda(E_{i+1}, E_i) = [\lambda_{rs}]$, *where* $\lambda_{rs} = \lambda(\alpha_r, \beta_s)$ for $\alpha_r \in A(E_i, E_{i+1})$ and $\beta_s \in B(E_{i+1}, E_i)$. The rows and columns are ordered with the natural ordering of $A(E_i, E_{i+1})$ and $B(E_{i+1}, E_i)$. We will show in § 3 that the cofinality class of the sequence of local linking matrices is an invariant of the oriented local type of an exceptional arc.

2. The existence of constructing sequences. Our aim in this section is the proof of Theorem 1 below, which yields a partial answer to Problem 1 of [5]. We will need to use some cutting and pasting arguments in the proof of this and later theorems so, to save labor, we prove Lemma 1 first.

LEMMA 1. *Let $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$ be a sequence of k-tori and 3-cells, such that the sequence $E_0 \supset E_1 \supset \cdots$ has property $\mathscr{F}$, and let $V \subset \operatorname{Int} E_0$ be a k-torus. Let $A \subset \operatorname{Bd} V \cap \operatorname{Int}(V_{i-1} - V_i)$ be a disc or an annulus whose boundary lies on $\operatorname{Bd} E_i$ and whose interior is disjoint from $\operatorname{Bd} E_i$; if A is a disc, let A′ be one of the discs on $\operatorname{Bd} E_i$ bounded by the curve $\operatorname{Bd} A$, and if A is an annulus, let A′ be the annulus on $\operatorname{Bd} E_i$ which is bounded by the two boundary curves of A.*

Then if $A \cup \operatorname{Cl}(\operatorname{Bd} E_i - A')$ bounds a 3-cell E_i' which contains p (and therefore V_i) in its interior,

(i) *the sequence $E_0 \supset E_1 \supset \cdots \supset E_{i-1} \supset E_i' \supset E_{i+1} \supset \cdots$ has property $\mathscr{F}$,*

(ii) *$A(E_i', E_{i-1}) = A(E_i, E_{i-1})$ and $A(E_{i+1}, E_i') = A(E_{i+1}, E_i)$, and*

(iii) *$N(k \cap \operatorname{Bd} E_i') = N(k \cap \operatorname{Bd} E_i)$ (this is implied by* (ii)).

Proof. We have two cases to consider: (a) $A \subset \operatorname{Cl}(E_0 - E_i)$ and $E_i' \supset E_i$, and (b) $A \subset E_i$ and $E_i' \subset E_i$. We shall only prove the result in case (a); the proof for case (b) is similar. If we use K to denote the region bounded by $A \cup A'$ in $\operatorname{Int} E_0$, then $E_i' = E_i \cup K$.

Let $\alpha \in A(E_{i-1}, E_i)$, so that $\alpha \subset E_0 - E_i$. Then $\alpha \subset E_0 - E_i'$. For $\nu(\alpha \cap \operatorname{Bd} K) = \nu(\alpha \cap A) = 0$, and k meets A in at most one point, since A is part of the boundary of a k-torus; thus $N(\alpha \cap A) = 0$ and α meets neither A nor $\operatorname{Bd} E_i$, so $\alpha \subset E_0 - E_i'$. [Note that, in general, if V is a k-torus whose boundary lies in $\operatorname{Int}(V_j - V_{j+1})$, then $\alpha \cap \operatorname{Bd} V = \varnothing$ and $\beta \cap \operatorname{Bd} V = \varnothing$ for all $\alpha \in A(E_j, E_{j+1})$ and $\beta \in B(E_{j+1}, E_j)$, for both $\nu(\alpha \cap \operatorname{Bd} V)$ and $\nu(\beta \cap \operatorname{Bd} V)$ are zero.]

Therefore $A(E_{i-1}, E_i') = A(E_{i-1}, E_i)$. Now suppose there exists an arc $\beta \in B(E_i, E_{i-1})$ whose interior lies in $\operatorname{Int} K$ and whose endpoints both lie in $\operatorname{Int} A'$. Then we can join the endpoints of β by an arc β' lying in $\operatorname{Int} A'$, so that $\beta \cup \beta'$ is a tame knot, and note that the 3-cell E_i' splits $\beta \cup \beta'$ from $\alpha \cup \operatorname{Bd} E_{i-1}$ for each $\alpha \in A(E_{i-1}, E_i)$; this is impossible because the sequence $E_0 \supset E_1 \supset \cdots \supset E_{i-1} \supset E_i \supset \cdots$ has property $\mathscr{F}$. We conclude that no arc $\beta \in B(E_i, E_{i-1})$ can have its interior lying in $\operatorname{Int} K$. This implies that

$$N(k \cap A') \leqq N(k \cap A) \leqq N(k \cap \operatorname{Bd} V) = 1\ ;$$

hence $k \cap K$ is either empty, or a single arc joining the points $k \cap A$

and $k \cap A'$ (cf. the penultimate paragraph on p. 231 of [3]) — in particular, $N(k \cap A) = N(k \cap A')$, hence $N(k \cap \operatorname{Bd} E_i') = N(k \cap \operatorname{Bd} E_i')$. This proves part (iii) of the lemma.

If $k \cap (A \cup A') = \varnothing$, $B(E_i', E_{i-1}) = B(E_i, E_{i-1})$ and there is nothing to prove. So suppose $k \cap K$ consists of an arc γ which joins the points $k \cap A$ and $k \cap A'$, and that $k \cap A'$ is one endpoint of the arcs $\alpha_r \in A(E_i, E_{i+1})$ and $\beta_s \in B(E_i, E_{i-1})$.

Let $\alpha_t \in A(E_{i-1}, E_i)$ be chosen so that (α_t, β_s) is unsplittable; then we claim that (α_t, β_s') is unsplittable, where β_s' is obtained from β_s by removing $\operatorname{Int} \gamma$.

Suppose there exists a 2-sphere S which splits $\alpha_t \cup \alpha$ from $\beta_s' \cup \beta$, where α and β are suitably chosen arcs on $\operatorname{Bd} E_{i-1}$ and $\operatorname{Bd} E_i'$ respectively. S bounds a 3-cell C in E_0. If $\beta_s' \cup \beta \subset \operatorname{Int} C$ then we may use the usual cutting and pasting arguments, noting that (α_t, β_s) is unsplittable, to replace C by a 3-cell C' such that

$$C' \subset \operatorname{Int}(E_{i-1} - \alpha_t) ,$$

and then replace C' by a 3-cell C^* whose boundary lies in

$$\operatorname{Int}(E_{i-1} - E_i' - \alpha_t \cup \beta_s') .$$

If $\alpha \cup \alpha_t' \subset \operatorname{Int} C$, join $\operatorname{Bd} C$ to $\operatorname{Bd} E_0$ by an arc δ in $E_0 - (E_i' \cup \beta_s')$, and let C' be obtained from E_0 by removing an open regular neighborhood of $\delta \cup C \cup \operatorname{Bd} E_0$. Then C' is a 3-cell which contains $\beta \cup \beta_s'$ in its interior and, after cutting and pasting, we may replace C' by a 3-cell C^* whose boundary lies in $\operatorname{Int}(E_{i-1} - E_i' - \alpha_t \cup \beta_s')$.

Therefore the pair (α_t, β_s) is splittable if (α_t, β_s') is splittable, for there exists a 3-cell C^* such that

$$E_i \cup \beta_s \subset E_i' \cup \beta_s' \subset \operatorname{Int} C^* \subset C^* \subset \operatorname{Int}(E_{i-1} - \alpha_t) ;$$

this contradicts our choice of α_t, so (α_t, β_s') must be unsplittable whenever (α_t, β_s) in unsplittable.

Let $\beta_q \in B(E_{i+1}, E_i)$ be chosen so that (α_r, β_q) is unsplittable — we wish to prove that $(\alpha_r \cup \gamma, \beta_q)$ is also unsplittable. If $(\alpha_r \cup \gamma, \beta_q)$ is splittable, there exists a splitting 3-cell C in $\operatorname{Int}(E_i' - (\alpha_r \cup \gamma))$ and, after cutting and pasting, we may replace C by a 3-cell C^* whose boundary lies in $\operatorname{Int}(E_i - E_{i+1} - \alpha_r \cup \beta_q)$. Then

$$E_{i+1} \cup \beta_q \subset \operatorname{Int} C^* \subset C^* \subset \operatorname{Int}(E_i - \alpha_r) \subset \operatorname{Int}(E_i' - \alpha_r \cup \gamma) ;$$

which implies that (α_r, β_q) is splittable, contrary to our choice of β_q. Hence $(\alpha_r \cup \gamma, \beta_q)$ is unsplittable if (α_r, β_q) is unsplittable.

Now since

$$A(E_i', E_{i+1}) = (A(E_i, E_{i+1}) - \{\alpha_r\}) \cup \{\text{the arc } \gamma \cup \alpha_r\} ,$$

and

$$B(E_i', E_{i+1}) = (B(E_i, E_{i+1}) - \{\beta_s\}) \cup \{\text{the arc } \beta_s - \operatorname{Int} \gamma\} ,$$

these results show that the sequence

$$E_0 \supset E_1 \supset \cdots \supset E_{i-1} \supset E_i' \supset E_{i+1} \supset \cdots$$

has the property $\mathscr{F}$, which is what we needed to prove for part (i) of the lemma. To complete the proof of the lemma, then, we need to prove that $A(E_i', E_{i-1}) = A(E_i, E_{i-1})$ and $A(E_{i+1}, E_i') = A(E_{i+1}, E_i)$; note that $A(E_i', E_{i-1})$ will not differ from $A(E_i, E_{i-1})$ except perhaps in column r, and $A(E_{i+1}, E_i')$ will not differ from $A(E_{i+1}, E_i)$ except perhaps in row s.

For each $\beta \in B(E_{i+1}, E_i')$, let S_β be an oriented surface in the interior of E_i, bounded by β and an arc β' on $\operatorname{Bd} E_{i+1}$. Because $\gamma \subset \operatorname{Cl}(E_0 - E_i)$, γ does not meet S_β at all so $\nu(\gamma \cap S_\beta) = 0$; therefore

$$\lambda(\alpha_r \cup \gamma, \beta) = \nu(S_\beta \cap (\alpha_r \cup \gamma)) = \nu(S_\beta \cap \alpha_r) = \lambda(\alpha_r, \beta) .$$

Hence $A(E_{i+1}, E_i') = A(E_{i+1}, E_i)$.

Similarly, if we let S_α be an oriented surface in $E_{i-1} - E_i'$ bounded by $\alpha \in A(E_{i-1}, E_i')$ and an arc α' on $\operatorname{Bd} E_{i-1}$, then $\nu(\gamma \cap S_\alpha) = 0$ because $\gamma \subset E_i'$. Therefore

$$\lambda(\alpha, \beta_s - \operatorname{Int} \gamma) = \nu(S_\alpha \cap (\beta_s - \operatorname{Int} \gamma)) = \nu(S_\alpha \cap \beta_s) = \lambda(\alpha, \beta_s)$$

and $A(E_i', E_{i-1}) = A(E_i, E_{i-1})$. This completes the proof of the lemma.

THEOREM 1. *Let k be an arc which is wild at one endpoint p, at which $P_0(k, p) \geqq 5$. Let*

$$E_0 \supset V_0 \supset E_1 \supset V_1 \supset E_2 \supset V_2 \supset \cdots$$

be a sequence of tame closed 3-cell and tame closed solid torus neighborhoods of p, such that $N(k \cap \operatorname{Bd} V_i) = 1$ for all i. Then if the sequence $E_0 \supset E_1 \supset E_2 \supset \cdots$ of 3-cells has property $\mathscr{F}$, the sequence $E_0 \supset V_0 \succ V_1 \succ V_2 \succ \cdots$ is a constructing sequence for k in E_0 (and k is therefore exceptional).

Proof. We need to prove that for each index $i \geqq 1$, there is no k-torus $U(i)$ with $V_{i-1} \succ U(i) \succ V_i$, and that there is no k-torus $U(0)$ with $E_0 \supset U(0) \succ V_0$.

Suppose there was such a k-torus $U(0) \subset \operatorname{Int} E_0$. Since $V_0 \prec U(0)$, there exists a 3-cell C which contains V_0 and whose boundary lies in $\operatorname{Int}(U(0) - V_0)$. For each $\alpha \in A(E_0, E_1)$, $\alpha \cap \operatorname{Bd} U(0) = \varnothing$, and $\beta \cup E_1$ lies in $\operatorname{Int} V_0$ for each $\beta \in B(E_1, E_0)$; then the set

$$(\alpha \cup \operatorname{Bd} E_0) \cup (\beta \cup E_1)$$

is splittable by the 2-sphere $\operatorname{Bd} C$, contradicting the fact that

$$E_0 \supset E_1 \supset \cdots$$

has property $\mathscr{F}$. Therefore no such 3-cell C can exist; that is, if $U(0)$ is a k-torus whose boundary lies in $\operatorname{Int}(E_0 - V_0)$, then V_0 has nonzero order in $U(0)$.

Suppose $i \geqq 1$, and that there exists a k-torus $U(i)$ with

$$V_{i-1} \succ U(i) \succ V_i .$$

We put $\operatorname{Bd} U(i)$ into general position with respect to $\operatorname{Bd} E_i$, so that $\operatorname{Bd} U(i) \cap \operatorname{Bd} E_i$ consists of a finite number of simple closed curves, none of which meets k. Using the cutting and pasting arguments of the proof of Theorem 1 of [6], we may replace E_i by a 3-cell E_i^* whose boundary meets k in as many points as $\operatorname{Bd} E_i$, lies in

$$\operatorname{Int}(V_{i-1} - V_i) ,$$

and is disjoint from $\operatorname{Bd} U(i)$. It follows from Lemma 1, also, that the sequence $E_0 \supset E_1 \supset \cdots \supset E_{i-1} \supset E_i^* \supset E_{i+1} \supset \cdots$ still has property $\mathscr{F}$.

We then have two cases to consider. Either (i) $E_i^* \subset \operatorname{Int} U(i)$, or (ii) $U(i) \subset \operatorname{Int} E_i^*$.

(i) Since $U(i) \prec V_{i-1}$, there exists a 3-cell C which contains $U(i)$ in its interior, and whose boundary lies in the interior of $V_{i-1} - U(i)$. For each $\beta \in B(E_i^*, E_{i-1})$, $\beta \cup E_i^*$ lies in $\operatorname{Int} U(i)$ and therefore in $\operatorname{Int} C$, while $\alpha \cup \operatorname{Bd} E_{i-1}$ lies in $\operatorname{Int}(E_0 - V_{i-1})$ and therefore in $\operatorname{Int}(E_0 - C)$ for each $\alpha \in A(E_{i-1}, E_i^*)$. But this means that all the pairs (α, β) are splittable, contradicting the fact that the sequence $E_0 \supset \cdots \supset E_{i-1} \supset E_i^* \supset E_{i+1} \supset \cdots$ has property $\mathscr{F}$.

Therefore no such 3-cell C can exist, so if $E_i^* \subset \operatorname{Int} U(i)$, $U(i)$ must have nonzero order in V_{i-1}.

(ii) Similarly, if $U(i) \subset \operatorname{Int} E_i^*$, we contradict the fact that the sequence $E_0 \supset \cdots \supset E_{i-1} \supset E_i^* \supset E_{i+1} \supset \cdots$ has property $\mathscr{F}$ if we assume the existence of a 3-cell C such that

$$V_i \subset \operatorname{Int} C \subset C \subset \operatorname{Int} U(i) ,$$

so V_i has nonzero order in $U(i)$ if $U(i)$ lies in $\operatorname{Int} E_i^*$.

Cases (i) and (ii) show that there is no k-torus $U(i)$ such that $V_i \prec U(i) \prec V_{i-1}$. Therefore the sequence

$$E_0 \supset V_0 \succ V_1 \succ V_2 \succ \cdots$$

is a constructing sequence for k, and k is exceptional.

3. The invariance of the sequence of the local linking matrices. The result of this section is that the cofinality class of the sequence of local linking matrices is an invariant of the oriented local arc type of an exceptional arc. We need to start with a lemma.

LEMMA 2. *Let*

$$E_0 \supset T_0 \supset B_1 \supset T_1 \supset B_2 \supset \cdots \supset B_{n-1}$$
$$\supset T_{n-1} \supset B_n \supset V_n \supset E_{n+1} \supset V_{n+1} \supset E_{n+2} \supset \cdots$$

be a special constructing sequence for k, and let

$$E_0 \supset U_0 \succ U_1 \succ U_2 \succ \cdots \succ U_{n-1} \succ V_n$$

be a containing sequence for V_n. Then there exist 3-cells

$$E_1, E_2, \cdots, E_{n-1}, E_n$$

such that

(i) $E_0 \supset U_0 \supset E_1 \supset U_1 \supset \cdots \supset E_{n-1} \supset U_{n-1} \supset E_n \supset V_n \supset E_{n+1} \supset V_{n+1} \supset \cdots$ *is a special constructing sequence for k in E_0 and, for all $i = 0, 1, \cdots, n$,*

(ii) $\Lambda(E_{i+1}, E_i) = \Lambda(B_{i+1}, B_i)$, *and*

(iii) $N(k \cap \mathrm{Bd}\, E_i) = N(k \cap \mathrm{Bd}\, B_i)$.

Proof. (Note that (ii) implies (iii), so we only need to prove (i) and (ii).)

Let $\mathscr{V}$ be the class of all special constructing sequences

$$E_0 \supset T_0^* \supset B_1^* \supset T_1^* \supset \cdots \supset B_{n-1}^*$$
$$\supset T_{n-1}^* \supset B_n^* \supset V_n \supset E_{n+1} \supset V_{n+1} \supset E_{n+2} \supset \cdots$$

for which $\Lambda(B_{i+1}^*, B_i^*) = \Lambda(B_{i+1}, B_i)$, $i = 0, 1, 2, \cdots, n$, where we take $B_0^* = E_0$ and $B_{n+1}^* = B_{n+1} = E_{n+1}$. $\mathscr{V}$ is not empty, by hypothesis. Then there is a sequence in $\mathscr{V}$ whose boundary surfaces are in general position with respect to the surfaces $\mathrm{Bd}\, U_0, \cdots, \mathrm{Bd}\, U_{n-1}$, and meet those surfaces in a minimum number of intersection curves — we may denote this sequence by $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$. We are assuming, therefore, that

(a) $(\bigcup_{i=0}^{n-1} \mathrm{Bd}\, U_j) \cap (\bigcup_{i=0}^{n-1}(\mathrm{Bd}\, V_i \cup \mathrm{Bd}\, E_{i+1}))$ consists of a finite number of simple closed curves, none of which meets k, and

(b) there is no sequence in $\mathscr{V}$ whose boundary surfaces meet $\mathrm{Bd}\, U_0, \cdots, \mathrm{Bd}\, U_{n-1}$ in fewer curves than do the surfaces of the sequence $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$.

The major part of the proof (Parts 1 and 2) consists of showing that the boundary surfaces of the sequence $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$ are disjoint from the family $\bigcup_{j=0}^{n-1} \mathrm{Bd}\, U_j$. The proof of the lemma is then completed in Part 3 by showing that $U_i \subset E_i$ and $U_i \supset E_{i+1}$ for each $i = 0, 1, \cdots, n-1$ (we really show that $\mathrm{Bd}\, E_{i+1}$ lies in

$$\mathrm{Int}\,(U_i - U_{i+1}))\,.$$

Part 1. No intersection curve can be null-homologous on $\mathrm{Bd}\, U_j$, for any j.

Suppose there exists an intersection curve which bounds a disc on $\mathrm{Bd}\, U_h$, for some index h. We may choose an intersection curve $\sigma \subset \mathrm{Bd}\, U_h$ which bounds a disc $D \subset \mathrm{Bd}\, U_h$ containing no other intersection curves. We then have two cases to consider: either (i) $\sigma \subset \mathrm{Bd}\, V_s$, or (ii) $\sigma \subset \mathrm{Bd}\, E_s$, for some index s.

(i) If $\sigma \subset \mathrm{Bd}\, V_s$, then σ bounds a disc D' on $\mathrm{Bd}\, V_s$ (cf. part (a) of the proof of Lemma 5 of [5]); let S be the 3-cell bounded by $D \cup D'$, and let N be a (judiciously chosen) closed regular neighborhood of S. Then we write $V'_s = \mathrm{Cl}\,(V_s - N)$ if $D \subset V_s$, and $V'_s = V_s \cup N$ if $D \subset \mathrm{Cl}\,(E_0 - V_s)$; it follows from Lemma 3 of [5] that V'_s is a k-torus. Moreover,

$$\mathrm{Bd}\, U_j \cap \mathrm{Bd}\, V'_s \subset \mathrm{Bd}\, U_j \cap \mathrm{Bd}\, V_s$$

if $j \neq h$, and

$$\mathrm{Bd}\, U_h \cap \mathrm{Bd}\, V'_s \subset \mathrm{Bd}\, U_h \cap \mathrm{Bd}\, V_s - \{\sigma\}$$

(for we have eliminated all those intersection curves lying on D'). The sequence

$$E_0 \supset V_0 \supset E_1 \supset \cdots \supset V_{s-1} \supset E_s \supset V'_s \supset E_{s+1} \supset V_{s+1} \supset \cdots$$

is still a special constructing sequence for k in E_0, has the same sequence of local linking matrices as our original sequence in $\mathscr{V}$, yet meets the surfaces $\mathrm{Bd}\, U_0, \cdots, \mathrm{Bd}\, U_{n-1}$ in fewer intersection curves. This contradicts the minimality assumption (b) involved in the choice of our original sequence in $\mathscr{V}$, so we conclude that no curve σ of $\mathrm{Bd}\, V_s \cap \mathrm{Bd}\, U_h$ can be null-homologous on $\mathrm{Bd}\, U_h$.

(ii) $\sigma \subset \mathrm{Bd}\, \mathrm{E}_s$. We may choose one of the discs on $\mathrm{Bd}\, E_s$ which is bounded by σ, D' say, so that E_{s+1} does not lie in the 3-cell S bounded by $D \cup D'$. Let N be a closed regular neighborhood of S in E_0; write $E'_s = \mathrm{Cl}\,(E_s - N)$ if $D \subset E_s$, and $E'_s = E_s \cup N$ if $D \subset \mathrm{Cl}\,(E_0 - E_s)$.

It follows from Lemma 1 that $\Lambda(E_{s+1}, E'_s) = \Lambda(E_{s+1}, E_s)$ and $\Lambda(E'_s, E_{s-1}) = \Lambda(E_s, E_{s-1})$, for E'_s is the image of the 3-cell bounded by $D \cup (\mathrm{Bd}\, E_s - D')$ under a small isotopy (which fixes everything outside an open neighborhood of D). The sequence

$$E_0 \supset V_0 \supset \cdots \supset E_{s-1} \supset V_{s-1} \supset E'_s \supset V_s \supset E_{s+1} \supset V_{s+1} \supset \cdots$$

is therefore still a special constructing sequence for k in E_0, and is in $\mathscr{V}$; further, we have eliminated all those intersection curves which lie on D', without introducing any new intersection curves, so that the boundary surfaces of this new sequence in $\mathscr{V}$ meet the surfaces Bd $U_0, \cdots,$ Bd U_{n-1} in fewer curves than did the boundary surfaces of our original sequence. This contradicts the minimality assumption (b), and we conclude that no curve $\sigma \subset \mathrm{Bd}\, U_h \cap \mathrm{Bd}\, E_s$ can be null-homologous on Bd U_h.

Thus no intersection curve can be null-homologous on Bd U_j for any index j.

Part 2. Suppose there exists an index h such that the family of curves $\mathrm{Bd}\, U_h \cap (\bigcup_{i=0}^{n-1} (\mathrm{Bd}\, V_i \cup \mathrm{Bd}\, E_{i+1}))$ is not empty. Then these curves are parallel non-null-homologous curves on Bd U_h. There exists a largest index $M(h) = M < n$ such that either

(α) $\mathrm{Bd}\, U_h \cap \mathrm{Bd}\, V_M \neq \emptyset$ but $\mathrm{Bd}\, U_h \cap \mathrm{Bd}\, E_{M+1} = \emptyset$, or

(β) $\mathrm{Bd}\, U_h \cap \mathrm{Bd}\, E_{M+1} \neq \emptyset$.

(α) $\mathrm{Bd}\, U_h \cap \mathrm{Bd}\, V_M \neq \emptyset$ but $\mathrm{Bd}\, U_h \cap \mathrm{Bd}\, E_{M+1} = \emptyset$. Then there exists an annulus R in $V_M \cap \mathrm{Bd}\, U_h$ whose boundary lies in

$$\mathrm{Bd}\, U_h \cap \mathrm{Bd}\, V_M$$

and whose interior lies in Int V_M. Let σ and τ be the boundary curves of R. Then we must consider two cases, according as σ is not or is a meridian of Bd V_M (Part (a) of the proof of Lemma 5 of [5] shows that σ and τ cannot be null-homologous on Bd V_M).

(i) σ is not a meridian of Bd V_M. The annulus R splits V_M into two solid tori, T_1 and T_2, by Satz 1, p. 207 of [7]. T_1 has σ as a core (hence $O(T_1, V_M) \neq 0$) and $O(T_2, V_M) = 1$.

One of these tori, T_r say, contains E_{M+1} in its interior. We put Bd T_r into general position with respect to the surfaces

$$\mathrm{Bd}\, U_0, \cdots, \mathrm{Bd}\, U_{n-1}$$

by putting $V'_M = T_r - \{$an open regular neighborhood of $R\}$. Then V'_M is a k-torus, and the sequence

$$E_0 \supset V_0 \succ V_1 \succ \cdots \succ V_{M-1} \succ V'_M \succ V_{M+1} \succ \cdots \succ V_n$$

is a containing sequence for V_n (cf. [5], Lemma 4, and Part (b) of the proof of theorem 1). Then the sequence

$$E_0 \supset V_0 \supset E_1 \supset \cdots \supset V_{M-1} \supset E_M \supset V'_M \supset E_{M+1} \supset V_{M+1} \supset \cdots$$

is a special constructing sequence for k in $\mathscr{V}$, whose boundary surfaces meet the surfaces Bd $U_0, \cdots,$ Bd U_{n-1} in fewer curves than did our original sequence, for we have eliminated σ and τ and all the other intersection curves which were on the annulus

$$\operatorname{Bd}(V_M - T_r) - \operatorname{Int} R .$$

The existence of this sequence in $\mathscr{V}$ contradicts our minimality assumption (b), so σ must be a meridian of $\operatorname{Bd} V_M$ if $\operatorname{Bd} U_h \cap \operatorname{Bd} V_M \neq \emptyset$ but $\operatorname{Bd} U_h \cap \operatorname{Bd} E_{M+1} = \emptyset$.

(ii) σ is a meridian of $\operatorname{Bd} V_M$. R separates V_M into a solid torus $T(R)$ which has order one in V_M, and another space K which is a solid torus if and only if V_M and $T(R)$ are equally knotted. $T(R)$ and V_M share a meridian disc D. (Satz 2, p. 211 of [7]) Let $R' \subset \operatorname{Bd} V_M$ be chosen so that $R' \cup R = \operatorname{Bd} K$.

We will show that $p \notin K$ which, by Lemma 4 of [5], is sufficient to show that $T(R)$ is a k-torus.

If p lies in K, then $E_{M+1} \subset \operatorname{Int} K$; let $\beta \in B(E_{M+1}, E_M)$. Then β does not meet R', for $R' \subset \operatorname{Bd} V_M \subset \operatorname{Int}(E_M - E_{M+1})$. Since R meets k in at most one point, β cannot meet R at all because

$$\nu(\beta \cap (R \cup R')) = \nu(\beta \cap R) = 0 .$$

But then $\beta \cup E_{M+1} \subset \operatorname{Int} K$, so β does not meet the meridian disc D. Because β does not meet $\operatorname{Bd} V_M$, $\beta \cup E_{M+1}$ must lie in the interior of the 3-cell C obtained by removing an open regular neighborhood of D from V_M; then the pair (α, β) is splittable for each $\alpha \in A(E_M, E_{M+1})$, for

$$\beta \cup E_{M+1} \subset \operatorname{Int} C \subset C \subset V_M \subset \operatorname{Int}(E_M - \alpha) .$$

This contradicts the fact that the sequence

$$E_0 \supset \cdots \supset E_M \supset E_{M+1} \supset \cdots$$

has property $\mathscr{F}$.

This shows that $p \notin K$, that is that $p \in T(R)$ and $T(R)$ is a k-torus. We put $\operatorname{Bd} T(R)$ into general position with respect to the surfaces $\operatorname{Bd} U_0, \cdots, \operatorname{Bd} U_{n-1}$ by putting $V'_M = T(R) - \{$an open regular neighborhood of $R\}$. Then V'_M has order one in V_M, and the sequence

$$E_0 \supset V_0 \succ V_1 \succ \cdots \succ V_{M-1} \succ V'_M \succ V_{M+1} \succ \cdots \succ V_n$$

is therefore a containing sequence for V_n, by Theorem 1 of [5]. The sequence

$$E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots \supset V_{M-1} \supset E_M \supset V'_M \supset E_{M+1} \supset V_{M+1} \supset \cdots$$

is a special constructing sequence for k, in $\mathscr{V}$, whose boundary surfaces meet the surfaces $\operatorname{Bd} U_0, \cdots, \operatorname{Bd} U_{n-1}$ in fewer intersection curves than did our original sequence in $\mathscr{V}$ (for σ and τ and the other intersection curves lying in R' have been eliminated). This contradicts our minimality assumption (b) on the sequence

$$E_0 \supset V_0 \supset E_1 \supset V_1 \supset E_2 \supset V_2 \supset \cdots .$$

It follows that σ cannot be a meridian of Bd V_M and this, together with (i) above, shows that the situation (α) is impossible; that is, Bd U_h must meet Bd E_{M+1} if Bd $U_h \cap$ Bd $V_M \neq \emptyset$.

(β) Bd $U_h \cap$ Bd $E_{M+1} \neq \emptyset$. [Note: It would be nice to eliminate this case straight away by repeated use of Lemma 1, where we assume that A is a disc or annulus whose interior lies in Int E_{M+1}. However, the minimality assumption guarantees that A cannot be a disc, and that if A is an annulus, p lies outside the 3-cell bounded by $A \cup \mathrm{Cl}(E_{M+1} - A')$. We then have to "thicken" E_{M+1} by attaching annuli lying in Cl $(E_0 - E_{M+1})$; unfortunately, such annuli may meet Bd V_M.]

We will show first that Bd U_h cannot meet Bd E_M, and then that Bd U_h cannot meet Bd V_M. Finally, we will show that if Bd E_{M+1} is not disjoint from Bd U_h, then we can find a special constructing sequence

$$E_0 \supset V_0 \supset E_1 \supset \cdots \supset E_M \supset V_M \supset E'_{M+1} \supset V_{M+1} \supset E_{M+2} \supset \cdots$$

in $\mathscr{V}$ whose existence contradicts the choice of our original sequence in $\mathscr{V}$. This result, taken with the result of (α), will show that no such maximal index $M(h)$ can exist, and therefore that

$$(\mathrm{Bd}\ V_i \cup \mathrm{Bd}\ E_{i+1}) \cap \mathrm{Bd}\ U_j = \emptyset \quad \text{for all } i, j = 0, 1, \cdots, n-1 .$$

We start with a sublemma.

Sublemma. Bd $U_h \cap$ Bd $E_M = \emptyset$.

Proof. Suppose Bd U_h meets Bd E_M. Then Bd U_h must also meet Bd V_M; if a curve of Bd $U_h \cap$ Bd V_M is null-homologous on Bd V_M, it is also null-homologous on Bd U_h, so the result of Part 1 shows that we have an even number of curves of Bd $U_h \cap$ Bd V_M, and that these bound parallel annuli on Bd V_M. Also, we have an even number of curves of Bd $U_h \cap$ (Bd $V_M \cup$ Bd $E_M \cup$ Bd E_{M+1}), and these curves bound parallel annuli on Bd U_h.

Let λ and μ be generators of the homology group H_1(Bd V_M), representing the homology classes of a longitude and of a meridian of Bd V_M, respectively. The curves of Bd $U_h \cap$ Bd V_M all lie in the one homology class $\zeta = a\lambda + b\mu$, and $\zeta \neq 0$ in H_1(Bd V_M).

We may choose two curves $\alpha \subset$ Bd $U_h \cap$ Bd E_M and

$$\beta \subset \mathrm{Bd}\ U_h \cap \mathrm{Bd}\ V_M\ ,$$

so that α and β together bound and annulus $R_{\alpha\beta} \subset \mathrm{Bd}\ U_h$ which contains no other intersection curves. Let D_α be one of the discs on $\mathrm{Bd}\ E_M$ bounded by α, and note that D_α cannot meet $\mathrm{Bd}\ V_M$ because $V_M \subset \mathrm{Int}\ E_M$.

Then $D_\alpha \cup R_{\alpha\beta}$ is a disc in $\mathrm{Cl}\,(E_0 - V_M)$ which is bounded by the curve β, so $\zeta = [\beta]$ (the homology class of β on $\mathrm{Bd}\ V_M$) lies in the kernel of the map $H_1(\mathrm{Bd}\ V_M) \to H_1(\mathrm{Cl}\,(E_0 - V_M))$ induced by inclusion. Therefore $b = 0$ and $\zeta = a\lambda$.

We may choose two other curves $\alpha' \subset \mathrm{Bd}\ U_h \cap \mathrm{Bd}\ E_{M+1}$ and $\beta' \subset \mathrm{Bd}\ U_h \cap \mathrm{Bd}\ V_M$, which together bound an annulus $R'_{\alpha\beta} \subset \mathrm{Bd}\ U_h$, which contains no intersection curves in its interior. Let D'_α be one of the discs on $\mathrm{Bd}\ E_{M+1}$ bounded by α', and note that D'_α does not meet $\mathrm{Bd}\ V_M$.

Then $R'_{\alpha\beta} \cup D'_\alpha$ is a disc in V_M, which lies entirely in the interior of V_M except for its boundary curve β'. Since $[\beta'] = \zeta = [\beta]$, $\zeta = a\lambda$ must lie in the kernel of the map $H_1(\mathrm{Bd}\ V_M) \to H_1(V_M)$ induced by inclusion. Therefore $a = 0$, and $\zeta = a\lambda + b\mu = 0$, a contradiction.

Thus the boundary of U_h cannot meet the boundary of E_M if $\mathrm{Bd}\ U_h \cap \mathrm{Bd}\ E_{M+1} \neq \emptyset$.

Now we assume that $\mathrm{Bd}\ U_h$ meets $\mathrm{Bd}\ V_M$, even though it does not meet $\mathrm{Bd}\ E_M$. Then there exists a pair of curves σ and τ on $\mathrm{Bd}\ U_h \cap \mathrm{Bd}\ V_M$ which bound and annulus R lying in

$$\mathrm{Bd}\ U_h \cap \mathrm{Cl}\,(E_0 - V_M)$$

whose interior contains no intersection curves. We note that $R \subset \mathrm{Int}\ E_M$, by the sublemma. We have the usual two cases to consider: (i) σ is not a meridian of $\mathrm{Bd}\ V_M$, and (ii) σ is a meridian of $\mathrm{Bd}\ V_M$.

(i) σ is not a meridian of $\mathrm{Bd}\ V_M$. σ and τ separate $\mathrm{Bd}\ V_M$ into two disjoint annuli R_1 and R_2. One of these annuli, R_t say, together with R bounds a solid torus V which contains V_M([7], Satz 1, p. 213, Satz 2, p. 214). $\mathrm{Bd}\ V$ is put into general position with respect to the surfaces $\mathrm{Bd}\ U_0, \cdots, \mathrm{Bd}\ U_{n-1}$ by taking $V_M^* = V \cup$ (a closed regular neigborhood of R}. V_M^* is a k-torus, by Lemma 4 of [5].

V_M will have nonzero order in V_M^*, unless V_M is unknotted and σ is a longitude of $\mathrm{Bd}\ V_M$, when $O(V_M, V_M^*)$ may be zero ([7], loc. cit.). V_M^* lies in the interior of E_M, so $V_M^* \prec V_{M-1}$; so V_M will always have nonzero order in V_M^* because

$$E_0 \supset V_0 \succ V_1 \succ \cdots \succ V_{M-1} \succ V_M \succ \cdots \succ V_n$$

is a containing sequence for V_n.

Theorem 1 of [5] guarantees that

$$E_0 \supset V_0 \supset E_1 \supset \cdots \supset V_{M-1} \supset E_M \supset V_M^* \supset E_{M+1} \supset V_{M+1} \supset \cdots$$

is a special constructing sequence for k, in the class $\mathscr{V}$. But the boundary surfaces of this sequence meet the surfaces

$$\text{Bd } U_0, \cdots, \text{Bd } U_{n-1}$$

in fewer intersection curves than did our original sequence in $\mathscr{V}$, for we have eliminated σ and τ and all the intersection curves lying in $\text{Bd } V_M - R_t$, without introducing any new intersection curves. The existence of this sequence contradicts the minimality assumption (b) involved in the choice of our original sequence, and this contradiction shows that σ and τ must meridians of $\text{Bd } V_M$.

(ii) σ and τ are meridians of $\text{Bd } V_M$. σ and τ separate $\text{Bd } V_M$ into two disjoint annuli R_1 and R_2 and one of these, say R_t, together with R is the boundary of a solid torus V which contains V_M with order 1 ([7], Satz 3. p. 215). Lemma 4 of [5] shows that V is a k-torus. We put $\text{Bd } V$ into general position with respect to the surfaces $\text{Bd } U_0, \cdots, \text{Bd } U_{n-1}$ by taking $V_M^* = V \cup \{$a closed regular neighborhood of $R\}$. As above, we obtain a special constructing sequence

$$E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots \supset V_{M-1} \supset E_M \supset V_M^* \supset E_{M+1} \supset V_{M+1} \supset \cdots$$

for k in $\mathscr{V}$, whose existence contradicts the minimality assumption (b) involved in the choice of our original sequence. This forces us to conclude that $\text{Bd } U_h$ cannot meet $\text{Bd } V_M$ if $\text{Bd } U_h \cap \text{Bd } E_{M+1} \neq \emptyset$.

Then

$$\text{Bd } U_h \cap \bigcup_{i=0}^{n-1} (\text{Bd } V_i \cup \text{Bd } E_{i+1}) = \text{Bd } U_h \cap \text{Bd } E_{M+1} ;$$

and by using the cutting and pasting arguments of the proof of Theorem 1 of [6], Lemma 1, and the assumption (b) on the choice of our original sequence in $\mathscr{V}$, we may show that $\text{Bd } U_h$ cannot meet $\text{Bd } E_{M+1}$ at all. This result (β) together with the result (α) above shows that no such maximal index $M(h)$ can exist, so no surface $\text{Bd } U_h$ meets any of the surfaces $\text{Bd } V_0, \cdots, \text{Bd } V_{n-1}$ or

$$\text{Bd } E_1, \cdots, \text{Bd } E_n .$$

Part 3. For the sequence $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$ in $\mathscr{V}$, therefore, the family

$$\left(\bigcup_{j=0}^{n-1} \text{Bd } U_j\right) \cap \left(\bigcup_{i=1}^{n-1} \text{Bd } V_i \cup \text{Bd } E_{i+1}\right)$$

of intersection curves must be empty. From Theorem 1 of [5], it follows that either $V_i \subset \operatorname{Int} U_i$ and $O(V_i, U_i) \neq 0$, or $U_i \subset \operatorname{Int} V_i$ and $O(U_i, V_i) \neq 0$ for each $i = 0, 1, \cdots, n-1$. We wish to show that $\operatorname{Bd} U_i$ lies in $\operatorname{Int}(E_i - E_{i+1})$ for each i; it is sufficient to show that

$$\operatorname{Bd} \mathrm{E}_{i+1} \subset \operatorname{Int}(U_i - U_{i+1})$$

for all such i (where for U_n we take the k-torus V_n).

If $U_i \subset \operatorname{Int} V_i$, then $O(U_i, V_i) \neq 0$, so there is no 3-cell in $\operatorname{Int} V_i$ which contains U_i; so $E_{i+1} \subset \operatorname{Int} U_i$. Of course $E_{i+1} \subset \operatorname{Int} U_i$ if $V_i \subset \operatorname{Int} U_i$.

Either $U_{i+1} \subset \operatorname{Int} V_{i+1}$ or $V_{i+1} \subset \operatorname{Int} U_{i+1}$. In the former case, $\operatorname{Bd} E_{i+1}$ lies in $\operatorname{Int}(E_0 - V_{i+1})$ and therefore in $\operatorname{Int}(E_0 - U_{i+1})$. In the latter case, V_{i+1} has nonzero order in U_{i+1}, so $\operatorname{Bd} E_{i+1}$ cannot lie in $\operatorname{Int}(U_{i+1} - V_{i+1})$; consequently, $\operatorname{Bd} E_{i+1}$ must again lie in

$$\operatorname{Int}(E_0 - U_{i+1}) .$$

For each $i = 0, 1, \cdots, n-1$, therefore,

$$\operatorname{Bd} E_{i+1} \subset \operatorname{Int} U_i \cap \operatorname{Int}(E_0 - U_{i+1}) = \operatorname{Int}(U_i - U_{i+1}) .$$

This completes the proof of the lemma.

This brings us to the proof of this paper's main theorem.

THEOREM 2. *Let k_1 and k_2 be exceptional arcs with wild points p_1 and p_2 respectively, at which $P_0(k_i, p_i) \geqq 5$. If k_1 and k_2 have the same oriented local type at their wild points p_1 and p_2, then the sequences $\{\Lambda(E_{i+1}, E_i)\}$ and $\{\Lambda(B_{j+1}, B_j)\}$ of local linking matrices are cofinal, where $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$ is a special constructing sequence for k_1, and $B_0 \supset U_0 \supset B_1 \supset U_1 \supset \cdots$ is a special constructing sequence for k_2.*

Proof. Since the arcs have the same oriented local types at their respective wild endpoints, there exist oriented neighborhoods N_i of p_i and an orientation-preserving homeomorphism h which takes $(N_2, k_2 \cap N_2, p_2)$ to $(N_1, k_1 \cap N_1, p_1)$. We may assume that our special constructing sequence for k_2 lies entirely in N_2 and (by choosing a smaller B_0 if necessary) that $h(B_0) \subset \operatorname{Int} E_0$.

Given an index i, there exists an index $n(i)$ such that

$$V_{n(i)} \subset \operatorname{Int} h(U_i)$$

and, by Theorem 1 of [5], there exists a family of k_1-tori such that

$$E_0 \supset T_0 \succ T_1 \succ \cdots \succ T_l \succ h(U_0) \succ h(U_1) \succ \cdots \succ h(U_i)$$
$$\succ T_{l+i+2} \succ T_{l+i+3} \succ \cdots \succ T_{n(i)-1} \succ V_{n(i)}$$

is a containing sequence for $V_{n(i)}$. By Lemma 2, there exists a family of 3-cells such that

$$\text{(i)} \quad \begin{aligned} &E_0 \supset T_0 \supset C_1 \supset \cdots \supset C_l \supset T_l \supset C_{l+1} \supset h(U_0) \supset C_{l+2} \supset h(U_1) \supset \cdots \\ &\supset C_{l+i+1} \supset h(U_i) \supset C_{l+i+2} \supset T_{l+i+2} \supset \cdots \\ &\supset C_{n(i)-1} \supset T_{n(i)-1} \supset C_{n(i)} \supset V_{n(i)} \supset E_{n(i)+1} \supset V_{n(i)+1} \supset E_{n(i)+2} \supset \cdots \end{aligned}$$

is a special constructing sequence for k_1 in E_0, and

$$\text{(ii)} \qquad A(C_{j+1}, C_j) = A(E_{j+1}, E_j) \qquad j = 0, 1, \cdots, n(i) .$$

Our aim is to prove that $A(C_{j+1}, C_j) = A(B_{j-l}, B_{j-l-1})$ for all $j = l + 2, l + 3, \cdots, l + i$, for then the theorem follows by letting i take the values 3, 4, $\cdots$. Note that the matrices $A(B_{j-l}, B_{j-l-1})$ and $A(h(B_{j-l}), h(B_{j-l-1}))$ are identical.

A cutting and pasting argument of the type used in the proof of Theorem 2 of [6] shows that we may replace the 3-cells

$$C_{l+2}, C_{l+3}, \cdots, C_{l+i+1}$$

by 3-cells $C^*_{l+2}, \cdots, C^*_{l+i+1}$ such that $\operatorname{Bd} C^*_j \cap \operatorname{Bd} h(B_{j-l}) = \emptyset$ for $j = l + 2, \cdots, l + i + 1$. Further, for the reasons outlined below, it is also true that the matrices $A(C^*_{j+1}, C^*_i)$ and $A(C_{j+1}, C_j)$ are identical for $j = l + 1, \cdots, l + i + 1$, (where $C^*_{l+1} = C_{l+1}$ and $C^*_{l+i+2} = C_{l+i+2}$).

The proof of the preceding statement is as follows. We first apply the cut-and-paste to C_{l+2} to obtain C^*_{l+2}, then to C_{l+3} to obtain C^*_{l+3}, and so on inductively. Suppose $C^\sharp_{j+1}$ is obtained from C_{j+1} by attaching or removing a 3-cell S whose boundary is $D \cup D'$, where D is a disc on $\operatorname{Bd} h(B_{j-l+1})$ which contains no intersection curves in its interior, and D' is a disc (which has the same boundary as D) lying on $\operatorname{Bd} C_{j+1}$. Then $k_1 \cap S$ consists of at most $N(k_1 \cap \operatorname{Bd} C_{j+1})$ arcs running between $k_1 \cap D$ and $k_1 \cap D'$. Using this, it is easy to show that there is a one-to-one order-preserving correspondence $\alpha_r \leftrightarrow \alpha^\sharp_r$ between elements of $A(C_{j+1}, C_{j+2})$ and $A(C^\sharp_{j+1}, C_{j+2})$, and a one-to-one order-preserving correspondence $\beta_s \leftrightarrow \beta^\sharp_s$ between elements of $B(C_{j+1}, C^*_j)$ and $B(C^\sharp_{j+1}, C^*_j)$; under these correspondences, the pairs (α, β_s) and $(\alpha, \beta^\sharp_s)$ are F-isotopic for each $\alpha \in A(C^*_j, C_{j+1}) = A(C^*_j, C^\sharp_{j+1})$, and the pair (α_r, β) is F-isotopic to the pair $(\alpha^\sharp_r, \beta)$ for each $\beta \in B(C_{j+2}, C_{j+1})$ (two pairs are F-isotopic if their associated links — cf. p, 230 of [3] — are F-isotopic). Hence

$$A(C^\sharp_{j+1}, C^*_j) = A(C_{j+1}, C^*_j) = A(C_{j+1} C_j)$$

(by induction); which implies that

$$A(C^*_{j+1}, C^*_j) = A(C_{j+1}, C_j) \quad \text{for all } j = l + 1, \cdots, l + i + 1 .$$

To complete the proof of the theorem, therefore, we only need to prove that the matrices $\Lambda(C^*_{j+1}, C^*_j)$ and $\Lambda(h(B_{j-l}), h(B_{j-l-1}))$ are identical. For each $j = l+2, \cdots, l+i+1$, C^*_j either lies in $\operatorname{Int} h(B_{j-l})$, or contains $h(B_{j-l})$ in its interior; in both cases, there are $N(k_1 \cap \operatorname{Bd} h(B_{j-l}))$ arcs of k_1 which run between $\operatorname{Bd} C^*_j$ and $\operatorname{Bd} h(B_{i-l})$. There are four cases to be considered for each j, of which we shall consider only the first; the rest are similar.

(i) $C^*_j \subset \operatorname{Int} h(B_{j-l})$ and $C^*_{j+1} \subset \operatorname{Int} h(B_{j-l+1})$,
(ii) $C^*_j \subset \operatorname{Int} h(B_{j-l})$ and $h(B_{j-l+1}) \subset \operatorname{Int} C^*_{j+1}$,
(iii) $h(B_{j-l}) \subset \operatorname{Int} C^*_j$ and $C^*_{j+1} \subset \operatorname{Int} h(B_{j-l+1})$, and
(iv) $h(B_{j-l}) \subset \operatorname{Int} C^*_j$ and $h(B_{j-l+1}) \subset \operatorname{Int} C^*_{j+1}$.

In case (i), C^*_j lies in $\operatorname{Int} h(B_{j-l})$, and C^*_{j+1} lies in $\operatorname{Int} h(B_{j-l+1})$. For each $\alpha \in A(h(B_{j-l}), h(B_{j-l+1}))$, there exists a unique arc $\alpha^* \in A(C^*_j, C^*_{j+1})$, namely the arc $\alpha \cap C^*_j$; and for each $\beta \in B(h(B_{j-l+1}), h(B_{j-l}))$ there exists a unique arc $\beta^* \in B(C^*_{j+1}, C^*_j)$, which contains β as a subarc. There is an F-isotopy from the pair (α^*, β^*) to the pair (α, β) (composed of two simple F-isotopies from (α^*, β^*) to (α^*, β) and then from (α^*, β) to (α, β)), so $\lambda(\alpha^*, \beta^*) = \lambda(\alpha, \beta)$. In this case (i), therefore, $\Lambda(C^*_{j+1}, C^*_j) = \Lambda(h(B_{j-l}), h(B_{j-l+1}))$.

After consideration of the other cases, it follows that

$$\begin{aligned} \Lambda(E_{j+1}, E_j) &= \Lambda(C_{j+1}, C_j) \\ &= \Lambda(C^*_{j+1}, C^*_j) = \Lambda(h(B_{j-l}), h(B_{j-l+1})) \\ &= \Lambda(B_{j-l}, B_{j-l+1}) , \end{aligned}$$

and the sequences of local linking metrices are cofinal.

4. Some locally non-invertible exceptional arcs in R^3. Let E be a 3-cell in R^3 and α an arc whose endpoints lie on $\operatorname{Bd} E$ but whose interior is disjoint from $\operatorname{Bd} E$. If N is a suitable tame closed regular neighborhood of α, we shall say that α is unknotted with respect to E (or simply: α is unknotted) if either (i) $\operatorname{Int} \alpha \subset R^3 - E$ and $E \cup N$ is an unknotted solid torus, or (ii) $\operatorname{Int} \alpha \subset \operatorname{Int} E$ and $\operatorname{Cl}(E - N)$ is an unknotted solid torus.

Let x and y be two points in R^3. When we say "we join x to y by an oriented are α", it is understood that x is the starting point of the arc α, and y is the terminal point of α.

So let E_0 be a tame closed 3-cell in R^3, q a point in $R^3 - E_0$, p a point in $\operatorname{Int} E_0$, and V_0 an unknotted tame closed solid torus in $\operatorname{Int} E_0$ which contains p in its interior.

Let $n \geqq 1$ be fixed. Let D_{01} and D_{02} be discs on $\operatorname{Bd} E_0$, and choose $n+1$ points $x_{01}, x_{02}, \cdots, x_{0,n+1}$ in $\operatorname{Int} D_{01}$, and n points

$$x_{0,n+2}, \cdots, x_{0,2n+1}$$

in $\operatorname{Int} D_{02}$. For each $s \leqq n$, we join $x_{0,2n+2-s}$ to $x_{0,s}$ by an unknotted tame arc β_{0s} in $R^3 - q \cup E_0$, so that $\beta_{0s} \cap \beta_{0t} = \emptyset$ if $s \neq t$. We join q to $x_{0,n+1}$ by an arc γ_0 in $R^3 - E_0$.

Also, for each s such that $2 \leqq s \leqq n+1$, we join x_{0s} to $x_{0,2n+3-s}$ by an unknotted tame arc $\alpha_{0s} \subset \operatorname{Int} E_0 - V_0$ such that $\alpha_{0s} \cap \alpha_{0t} = \emptyset$ if $s \neq t$, and $\lambda(\sigma_0, \alpha_{0s}) = 1$ (where σ_0 is a longitude of V_0). The set so obtained when $n = 2$ is shown in Figure 1.

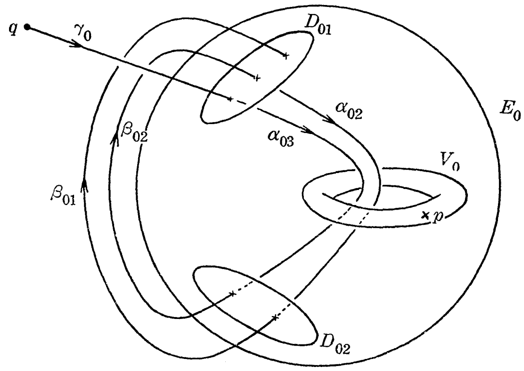

FIGURE 1

Let D_{11} and D_{12} be disjoint tame meridian discs of V_0, which do not contain p, and let E_1 be the closure of that component of

$$V_0 - D_{11} \cup D_{12}$$

which contains p. Let V_1 be an unknotted tame closed solid torus neighborhood of p in $\operatorname{Int} E_1$, and let σ_1 be a longitude of V_1. As before, we choose $n+1$ points $x_{11}, \cdots, x_{1,n+1}$ lying in $\operatorname{Int} D_{11}$, and n points $x_{1,n+2}, \cdots, x_{1,2n+1}$ in $\operatorname{Int} D_{12}$. For each $s \leqq n$, we join $x_{1,2n+2-s}$ to $x_{1,s}$ by an unknotted tame arc β_{1s} in $\operatorname{Int} V_0 - E_1$, so that $\beta_{1s} \cap \beta_{1t} = \emptyset$ if $s \neq t$; and for each s such that $2 \leqq s \leqq n+1$, we join x_{1s} to $x_{1,2n+3-s}$ by an unknotted tame arc α_{1s} in $\operatorname{Int} E_1 - V_1$ so that

$$\alpha_{1s} \cap \alpha_{1t} = \emptyset$$

if $s \neq t$, and $\lambda(\sigma_1, \alpha_{1s}) = 1$ for all s. We also join x_{01} to $x_{1,n+1}$ by a tame arc γ_1 such that $\operatorname{Int} \gamma_1 \subset \operatorname{Int} E_0 - E_1$ and $N(\gamma_1 \cap \operatorname{Bd} V_0) = 1$.

We note that for the 3-cell pair $E_0 \supset E_1$, $A(E_0, E_1) = \bigcup_{s=2}^{n+1} \alpha_{0s}$ and $B(E_1, E_0) = \bigcup_{t=1}^{n} \beta_{1t}$, and that $\lambda(\alpha_{0s}, \beta_{1t}) = 1$ for all s and t. Therefore none of the pairs (a_{0s}, β_{1t}) can be splittable.

We let D_{21}, D_{22} be meridian discs of V_1 which do not contain p, and we choose $n+1$ points $x_{21}, \cdots, x_{2,n+1}$ in $\operatorname{Int} D_{21}$, and n points $x_{2,n+2}, \cdots, x_{2,2n+1}$ in $\operatorname{Int} D_{22}$. We let E_2 be the closure of that component of $V_1 - D_{21} \cup D_{22}$ which contains p in its interior, and let V_2 be an unknotted tame closed solid torus neighborhood of p lying in Int E_2. σ_2 is a longitude of V_2. Then we may obtain the oriented arcs β_{2t}, $t = 1, 2, \cdots, n$ of $B(E_2, E_1)$ in a manner analogous to that described above, and note that none of the pairs $(\alpha_{1s}, \beta_{2t})$ is splittable because $\lambda(\alpha_{1s}, \beta_{2t}) = 1$ for all s and t. We join x_{11} to $x_{2,n+1}$ by a tame arc γ_2 whose interior lies in Int $E_1 - E_2$ and which meets Bd V_1 in precisely one point.

Proceeding in this way, then, we obtain an oriented arc

$$k_n = \bigcup_{i=0} \left\{\gamma_i \cup \left(\bigcup_{s=2}^{n+1} \alpha_{is}\right) \cup \left(\bigcup_{t=1}^{n} \beta_{it}\right)\right\}$$

(k_2 is shown in Figure 2 and in Figure 3 of [5]; k_1 is example 1.2 of [1]) and a sequence

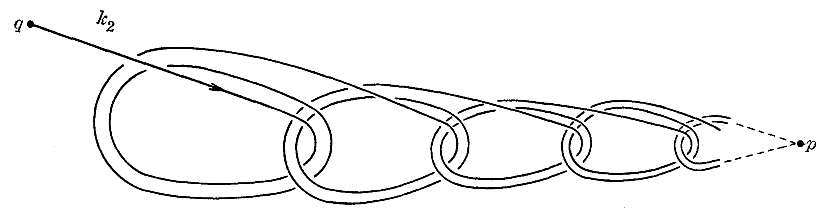

FIGURE 2

$$E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$$

of tame closed 3-cell and solid torus neighborhoods of p, with the following properties:

(i) $N(k_n \cap \text{Bd } V_i) = 1$ and $N(k_n \cap \text{Bd } E_i) = 2n + 1$ for all i,

(ii) $\bigcap V_i = p = \bigcap E_i$, and

(iii) the sequence $E_0 \supset E_1 \supset E_2 \supset \cdots$ has property $\mathscr{F}$.

By taking a tame closed regular neighborhood of each V_i, we obtain an unknotted tame closed solid torus which we shall also call V_i; judicious choice of these regular neighborhoods will ensure that $N(k_n \cap \text{Bd } V_i) = 1$ and that Bd E_{i+1} lies in Int $(V_i - V_{i+1})$.

Then (a) k_n is wild by [3] and, if $n \geqq 2$, a cut-and-paste argument will show that $P_0(k_n, p) \neq 3$; therefore $P_0(k_n, p) \geqq 5$.

(b) $E_0 \supset V_0 \succ V_1 \succ V_2 \succ \cdots$ is a constructing sequence for k_n in E_0, by Theorem 1.

(c) k_n is therefore exceptional, and $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$ is a special constructing sequence for k_n; Theorem 2 of [6] shows that $P_0(k_n, p) = 2n + 1$.

We have therefore proved the following theorem:

THEOREM 3. *For each integer $n \geqq 2$, there exists an exceptional arc k_n such that $P_0(k_n, p) = 2n + 1$ and $P_1(k_n, p) = 1$.*

Notice that for k_n, and the constructing sequence of (c) above, each of the matrices in the sequence $\{\Lambda(E_{i+1}, E_i)\}$ is an $n \times n$ matrix whose entry $\lambda_{st}(i) = \lambda(\alpha_{i,s-1}, \beta_{i+1,t}) = 1$ for all s and t.

We will now obtain some locally noninvertible arcs by varying the construction of k_n to obtain an arc k_n^*, as follows (we are only concerned with $n \geqq 2$; for $n = 1$ the existence of such arcs is guaranteed by Corollary 2 of [4]). k_n^* is obtained from k_n by replacing the arc α_{i2} with a tame unknotted arc α_{i2}^* so that $\lambda(\sigma_{i+1}, \alpha_{i2}^*) = 2$ for all $i = 0, 1, 2, \cdots$ (where σ_{i+1} is a longitude of V_{i+1}). Thus

$$k_n^* = \bigcup_{i=0}\left\{\gamma_i \cup \alpha_{i2}^* \cup \left(\bigcup_{s=3}^{n+1} \alpha_{is}\right) \cup \left(\bigcup_{t=1}^{n} \beta_{it}\right)\right\} ;$$

k_2^* is shown in Figure 3.

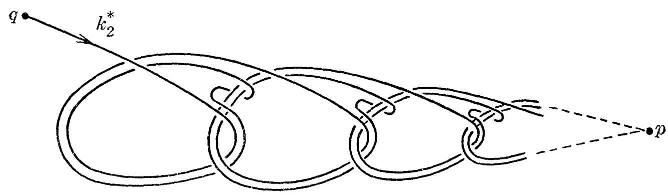

FIGURE 3

Let us denote the special constructing sequence for k_n^* obtained this way by

$$E_0^* \supset V_0^* \supset E_1^* \supset V_1^* \supset \cdots ,$$

even though $E_i^* = E_i$ and $V_i^* = V_i$ for all i.

If k_n and k_n^* had the same oriented local type, the sequences $\{\Lambda(E_{i+1}, E_i)\}$ and $\{\Lambda(E_{i+1}^*, E_i^*)\}$ would be cofinal, by Theorem 2. But there are no 2's occurring in the matrix $\Lambda(E_{i+1}, E_i)$ for any i, whereas each matrix $\Lambda(E_{i+1}^*, E_i^*)$ is an $n \times n$ matrix whose entries are all 1's except in the top row where they are all 2's. So k_n^* and k_n represent different oriented local arc types.

Let k be an exceptional arc and let $\Lambda(E_{i+1}, E_i)$ be one of the $n \times m$ local linking matrices associated with a special constructing sequence $E_0 \supset V_0 \supset E_1 \supset V_1 \supset \cdots$ for k. Because we have ordered the rows and columns of the matrix with the natural ordering that the sets $A(E_i, E_{i+1})$ and $B(E_{i+1}, E_i)$ inherit from k, the entry λ_{rs} becomes the entry $\lambda_{n-r+1,m-s+1}$ of the local linking matrix obtained when we reverse the orientation of k. Hence, if k is locally invertible, $\lambda_{rs} = \lambda_{n-r+1,m-s+1}$ for all r and s; in particular, if k_n^* is locally invertible, $\lambda_{11} = \lambda_{nn}$. k_n^* cannot be locally invertible, therefore, because $\lambda_{11} = 2$ and $\lambda_{nn} = 1$.

THEOREM 4. *For each integer* $n \geqq 2$, *there exist uncountably* *many locally noninvertible exceptional arcs, which have* 3*-cell penetration index* $2n+1$ *and total penetration index* 1.

Proof. Let $\pi_1, \pi_2, \cdots$ be an ordering of the prime positive integers, let $j(i)$ be a sequence of positive integers, and let $k_n(\{\pi_{j(i)}\})$ be the arc

$$\bigcup_{i=0}\left\{\gamma_i \cup \alpha_{i2}(\pi_{j(i)}) \cup \left(\bigcup_{s=3}^{n+1} \alpha_{is}\right) \cup \left(\bigcup_{t=1}^{n} \beta_{it}\right)\right\}$$

obtained by replacing the arc α_{i2} used above in the construction of k_n with an arc $\alpha_{i2}(\pi_{j(i)})$ such that $\lambda(\sigma_{i+1}, \alpha_{i2}(\pi_{j(i)})) = \pi_{j(i)}$ for all $i = 0, 1, 2, \cdots$. The entries of the ith local linking matrix of $k_n(\{\pi_{j(i)}\})$ are all 1's except in the top row, where the entries are all $\pi_{j(i)}$; thus $k_n(\{\pi_{j(i)}\})$ is locally noninvertible. (Note also that $P_0(k_n(\{\pi_{j(i)}\}), p) = 2n+1$ and $P_1(k_n(\pi_{j(i)}\}), p) = 1$.)

If the sequences $\{\pi_{j(i)}\}$ and $\{\pi_{l(i)}\}$ are not cofinal, then the arcs $k_n(\{\pi_{j(i)}\})$ and $k_n(\{\pi_{l(i)}\})$ represent different oriented local arc types, so the number of different local arc types is at least as large as the number of cofinality classes of sequences of primes. The number of cofinality classes of such sequences is easily shown to be uncountable.

5. An example of the use of k-sequences. The aim of this section is to show that (for $n \geqq 2$) the constructing sequence

$$E_0 \supset V_0 \succ V_1 \succ V_2 \succ \cdots$$

obtained in § 4 for k_n is actually a k_n-sequence in the sense of [5]; that is, that no k_n-torus $V \subset \operatorname{Int} E_0$ can be nontrivially knotted. This shows that the uncountably many arcs of Theorem 4 cannot be distinguished by the k-sequence invariant of Theorem 2 of [5].

Let V be a k_n-torus. There exists an index H such that $V_H \subset \operatorname{Int} V$, and there exist $H-1$ k_n-tori such that

$$E_0 \supset T_0 \succ \cdots \succ T_{h-1} \succ V \succ T_{h+1} \succ \cdots \succ T_{H-1} \succ V_H$$

is a containing sequence for V_H in E_0, by Theorem 1 of [5]. This same theorem guarantees the existence of a containing sequence

$$E_0 \supset T_0^* \succ \cdots \succ T_{h-1}^* \succ V^* \succ T_{h+1}^* \succ \cdots \succ T_{H-1}^* \succ V_H$$

such that $\operatorname{Bd} V^* \cap \operatorname{Bd} V_j = \varnothing$ (for all $j = 0, \cdots, H-1$), V^* is k_n-similar to V, and the knot type $\kappa(V)$ of V is a companion of $\kappa(V^*)$. There exists an index $s(=h-1$ or $h)$ such that $\operatorname{Bd} V^*$ lies in

$$\operatorname{Int}(V_s - V_{s+1}),$$

so that either $O(V^*, V_s) \neq 0$ or $O(V_{s+1}, V^*) \neq 0$. If V (and therefore

V^*) is nontrivially knotted, V_{s+1} must have zero order in V^* because V_{s+1} is unknotted and the trivial knot has no companions other than itself. Therefore $O(V^*, V_s) \neq 0$ (this implies, incidentally, that $s = h$).

Hence, to show that the sequence $E_0 \supset V_0 \succ V_1 \succ V_2 \cdots$ is a k_n-sequence, it is sufficient to show that for each index j it is impossible to find a knotted k_n-torus V which lies in the interior of V_j and has nonzero order in V_j.

Suppose such a knotted k_n-torus V does exist, and let $D_{j+1,1}$ and $D_{j+1,2}$ be the tame meridian discs of V_j used in the construction of k_n. Let D be a meridian disc of V.

Then $N(k_n \cap D) \geqq n = N(k_n \cap D_{j+1,2})$, so only one component D^* of $V \cap D_{j+1,2}$ can be a meridian disc of V. Hence $O(V, V_j) = 1$ and $k_n \cap D^* = k_n \cap D_{j+1,2}$.

Let $x_{j+1,n+2}$ and $x_{j+1,n+3}$ be points of k_n in $\operatorname{Int} D_{j+1,2}$ (and therefore in $\operatorname{Int} D^*$) and let β_{jn} and α_{jn} be the subarcs of k_n (in $\operatorname{Int} V$) which join $x_{j+1,n+2}$ to $x_{j+1,n}$ and run from there to $x_{j+1,n+3}$. We join $x_{j+1,n+3}$ to $x_{j+1,n+2}$ by an arc γ lying in $\operatorname{Int} D^*$.

Then $\gamma \cup \beta_{jn} \cup \alpha_{jn}$ is a tame knot κ $\operatorname{Int} V \subset \operatorname{Int} V_j$, and since κ meets $D_{j+1,1}$ in precisely one point, κ has order one in V_j. Then $O(\kappa, V) = 1$ because

$$1 = O(\kappa, V_j) = O(\kappa, V) \cdot O(V, V_j) = O(\kappa, V) .$$

$\kappa(V)$ is therefore a factor of κ; but κ is trivially knotted, so $\kappa(V)$ is trivial. This contradicts the assumption that V was knotted.

Therefore, for each index j, if V is a k_n-torus lying in $\operatorname{Int} V_j$ with nonzero order, V must be unknotted. It follows that no k_n-torus in $\operatorname{Int} E_0$ can be knotted, so the sequence

$$E_0 \supset V_0 \succ V_1 \succ V_2 \succ \cdots$$

is a k_n-sequence for the arc k_n.

It follows that k_2 has different local arc type at p to the arc shown in Figure 1 (a) of [5], which can be constructed using the method of § 4 above, except that we use kotted solid torus neighborhoods of p with $\kappa(V_i) =$ the trefoil knot, for all i (cf. Figure 1 (c) of [5]).

Added in proof. There are no locally invertible nearly polyhedral arcs. The proof is an easy application of the Invariance of Domain Theorem. The local noninvertibility results of [4], and Theorem 4 above, are therefore true but trite. Note however that the uncountably many arcs $k_n(\{\pi_{j(i)}\})$ are locally nonamphicheiral, because the local linking matrices of the mirror image of an arc are obtained from the local linking matrices for the arc itself by reversing the signs of all the entries. Note also that the arcs obtained by identifying the tame endpoints of $k_n(\{\pi_{j(i)}\})$ with that of the inverse of k_n (cf. Example 1.3 of [1]), for each n and sequence $\{\pi_{j(i)}\}$, are not invertible in R^3.

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