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KRULL-SCHMIDT AND CANCELLATION OVER LOCAL RINGS

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This paper proves a partial converse to the Krull-Schmidt theorem for Hensel local rings and a cancellation result for modules in terms of the endomorphism ring of the module. The second result is then used to prove the cancellation theorem for finitely generated modules over local rings.

All rings in this paper have units. A module M is indecomposable if $M \cong A \oplus B$ implies $A = 0$ or $B = 0$. A ring R satisfies the Krull-Schmidt Theorem if every finitely generated R module is uniquely a direct sum of indecomposable modules. A module M can be cancelled if $M \oplus A$ is isomorphic to $M \oplus B$ implies A is isomorphic to B. Usually one only considers cancellation when all the modules M, A, and B are finitely generated although this paper does not need these hypotheses. Local ring includes the noetherian property while quasi-local is used for a ring with only one maximal ideal.

This investigation began when the author noticed that Swan's proof of the Krull-Schmidt Theorem for complete local rings passes unchanged to Hensel local rings. Swan kindly supplied the author with an example of failure of Krull-Schmidt which generalizes to all not Hensel local rings. Finally we prove that the type of failure of Krull-Schmidt given by Swan's example is the only type possible by proving a cancellation theorem strong enough to prove that any finitely generated module over a local ring can be cancelled. I wish to thank Professors Artin, Kaplansky, and Swan whose lectures and discussions introduced me to much of this material.

The outline of this paper is as follows: Section 1 presents Swan's example of the failure of Krull-Schmidt, § 2 shows how for a given not Hensel local ring R we can find a local ring T which is a finite R algebra for which the Krull-Schmidt Theorem fails, and § 3 examines cancellation properties of modules in terms of their endomorphism rings.

1. Example of R.G. Swan. Let R be a local domain which is not a field. Assume $R/\mathfrak{m} = k$ has characteristic not equal to 2. Let $A = (R[x, y]/(x^3 - x^2 + y^2))_{(x,y)}$. Then the Krull-Schmidt theorem fails for A. Let $z = y/x$ in the quotient field of A and let $B = A[z]$. Then B has just 2 maximal ideals, $M_1 = (M, z - 1)$ and $M_2 = (M, z + 1)$ where M is the maximal ideal of A. Since $M_1 + M_2 = B$ we have an exact sequence of B modules

$$0 \to M_1 \cap M_2 \to M_1 \oplus M_2 \to B \to 0$$

which splits over B. Therefore, $M_1 \oplus M_2 \cong B \oplus (M_1 \cap M_2)$ over B and hence over A. All four modules here are torsion free rank 1 over A and so are indecomposable. If $B \cong M_1$ over A, then $B \cong M_1$ over B since B and M_1 are torsion free rank 1 modules and A and B have the same quotient field (any isomorphism would look like multiplication by an element of the quotient field). But this is impossible since B_{M_i} is local of dimension $\geqq 2$ so M_i needs at least 2 generators as a B module.

2. Swan's example generalized. To begin this section we first need some definitions. A quasi-local ring R with unique maximal ideal $\mathfrak{m}$ is called Hensel if for every monic polynomial $f(x)$ over R such that $f(x) \equiv g_0(x)h_0(x) \bmod \mathfrak{m}R[x]$ where $g_0(x)$ and $h_0(x)$ are monic and such that $g_0R[x] + h_0R[x] + \mathfrak{m}R[x] = R[x]$, then there exist monic polynomials $g(x)$ and $h(x)$ such that $f(x) = g(x)h(x)$, $g(x) \equiv g_0(x) \bmod \mathfrak{m}R[x]$ and $h(x) \equiv h_0(x) \bmod mR[x]$. This is equivalent to being able to lift idempotents from $A/\mathfrak{m}A$ to A for all finite R algebras A. The definition and basic properties are due to Azumaya [1]. Note that Azumaya allowed (and in fact required) A to be noncommutative. Several current treatments only consider commutative A's. This does not change the class of Hensel rings but does deaden the ability to see noncommutative applications such as the one in this section.

The integers localized at (5) is a local ring which is not Hensel for $x^2 + 1 \equiv (x + 2)(x + 3) \bmod 5$ is a factorization which cannot be lifted. But the integers localized at (5) satisfies the Krull-Schmidt theorem since it is a principal ideal domain. The following theorem shows that there is a strong connection between Hensel and Krull-Schmidt properties of local rings.

THEOREM 1. *Let R be a local ring. Then every R' which is a local ring and a finite R algebra satisfies the Krull-Schmidt theorem if and only if R is Hensel.*

Proof. If R is Hensel, then any commutative ring which is a finite R algebra is a direct sum of Hensel rings (see, for example [7, 43.1 and 43.16]). Hence for one half we only need to prove that Hensel rings satisfy Krull-Schmidt. Swan's proof of Krull-Schmidt for complete local rings R [10, Remark on page 566] only needs the ability to lift idempotents in finite algebras A over B from $A/\mathfrak{m}A$ to A. But this ability characterizes Hensel rings. For more details in Hensel case see [9].

To prove the other half we need to mimic Swan's example of failure of Krull-Schmidt. We first reduce to the case of domains.

LEMMA 1. *If R is a local ring which is not Hensel, then there exists a domain image of R which is not Hensel.*

Proof. R is Hensel if and only if $R_{\text{red}} = R/\text{nilpotent}$ is Hensel. [8, page 5]. So we can assume R is reduced. Then $0 = P_1 \cap \cdots \cap P_n$ with the P_i prime. Hence to prove the lemma it is enough to prove the following:

LEMMA 2. *If R is a local ring with ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} \cap \mathfrak{b} = 0$ such that $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are Hensel, then R is Hensel.*

Proof. Following Nagata's description of Henselization [7, pp. 179–188] we pick T an integrally closed domain with only one maximal ideal M such that T maps onto R with Kernel K. Let A and B be the complete inverse images of $\mathfrak{a}$ and $\mathfrak{b}$. Let T' be the integral closure of T in the algebraic closure L of the quotient field of T, H. Let M' be some maximal ideal of T. Let G be the elements of the Galois group of L over H which send M' to itself. Let T'' be the fixed ring of G. Then $\tilde{R}$, the Henselization of R, is $T''_{(M' \cap T'')}/KT''_{(M' \cap T'')}$. But since $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are Hensel we have $R/\mathfrak{a} = T''_{(M' \cap T'')}$ and $R/\mathfrak{b} = T''_{(M' \cap T'')}/BT''_{(M' \cap T'')}$. But since $T''_{(M' \cap T)}$ is faithfully flat over T we have

$$(AT''_{(M' \cap T'')}) \cap (BT''_{(M' \cap T'')}) = (A \cap B)T''_{(M' \cap T'')} = KT''_{(M' \cap T'')} .$$

Hence $\tilde{R}$, is a submodule of $R/\mathfrak{a} \oplus R/\mathfrak{b}$. Thus $\tilde{R}$ is finitely generated. But if $\mathfrak{m}$ is the maximal ideal of R, then $\tilde{R}/\mathfrak{m}\tilde{R} \cong R/\mathfrak{m}$ [7, 43.3]. Hence, by Nakayama's lemma $\tilde{R}$ is cyclic and hence equals R and R is Hensel as desired.

Hence to prove our result we can pass to R being a local domain which is not Hensel. Now we can apply Nagata's criteria that a quasi-local domain is Hensel if and only if every domain integral over it is quasi-local. [7, 43.12]. Hence we can find a domain R' integral over R which has at least two maximal ideals M_1 and M_2.

By picking $x_1 \in M_1 - M_2$ and $x_2 \in M_2 - M_1$ and passing to $R' = R[x_1, x_2]$ we can assume R' is finitely generated as an R module. Let A equal the intersection of the maximal ideals of R'. Then A is a finitely generated R module and so is $R[A]$. But $R[A]$ is local.

Except for minor pathology $R[A]$ and R' can be used as the A and B in Swan's example (respectively). That is, if M_1 and M_2 both needed 2 generators as R' modules we would be done. This would be true if the dimension of R'_{M_i} were at least 2.

In general we pass to $S = R[x]/(x^2)$ and $T = R'[x]/(x^2)$ and look at the maximal ideals $(M_1, x) = M'_1$ and $(M_2, x) = M'_2$. Then

$$0 \to M'_1 \cap M'_2 \to M'_1 \oplus M'_2 \to T \to 0$$

is exact and split. The modules are isomorphic over T if and only if they are isomorphic over S since it is enough to know what an isomorphism does on the part annihilated by x to know what it does on everything.

Finally we check that M'_i are not principal over T.

LEMMA 3. *If R is a domain and M is a proper ideal of R, then the ideal generated by M and x in $R[x]/(x^2)$ is not principal.*

Proof. Pick $m \in M - \{0\}$. Let $ax + b \in R[x]/(x^2)$ generate (M, a). Then there exist $r_i \in R$ such that

(1) $(r_1x + r_2)(ax + b) = m$ and

(2) $(r_3x + r_4)(ax + b) = x$

Expanding (1) we get

(i) $r_1b + r_2a = 0$ and

(ii) $r_2b = m$.

Expanding (2) we get

(iii) $r_2b = 0$

(iv) $r_3b + r_4a = 1$

From (iii) we get $r_4 = 0$ or $b = 0$. But if $b = 0$ then the ideal (ax) could not contain $m \neq 0$ *by* (ii). Hence $r_4 = 0$. Then from (iv) we get $r_3b = 1$. But then $b = (1 - r_3ax)(ax + b)$. Hence the ideal $(ax + b)$ contains b which is a unit whereas (M, x) is clearly proper.

Hence (M, x) is not principal.

REMARK. This is a simplification of a result suggested to me by Swan. His result which is proved by the same methods is that if an ideal (M, x) in $R[x]$ is principal then M is generated by an idempotent. On the other hand if M is generated by an idempotent e then $(M, x) = (e + (1 - e)x)$ as an ideal in $R[x]$.

This completes the proof of Theorem 1. One might ask for stronger results of this type. In particular you could ask if the dimension of R were big enough and R not Hensel then the Krull-Schmidt theorem fails for R itself. A difficulty involved is that R could be very close to being Hensel. Say, for example, if $R = k[[x_1, x_2, \cdots, x_n]][y]_{(x_1, \cdots x_n, y)}$.

3. The failure of Krull-Schmidt in the above section was always of the type $A \oplus B \cong C \oplus D$ all distinct and indecomposable. On the other hand we have

PROPOSITION 1. *Let R be a local ring and let A, B, and C be finitely generated R modules such that $A \oplus B$ is isomorphic to $A \oplus C$, then B is isomorphic to C.*

Proof. After completing and applying the Krull-Schmidt theorem over $\hat{R}$ we can conclude that $\hat{B}$ is isomorphic to $\hat{C}$. But then a theorem of Grothendeick [5, 2.5.8] asserts that B is isomorphic to C. The details are carried out in Vasconcelos's paper [12].

Of course, the above proof makes rather heavy use of finite generation of A, B, and C. For example $\hat{R}$ and R have isomorphic completions but are not isomorphic unless R is complete.

The next theorem strengthens the above cancellation result following ideas of Bass [2] and Dress [3].

If A is the endomorphism ring of a finitely generated module over a local ring, then A is itself finitely generated over R (since R is noetherian). Hence a theorem of Bass [2, Corollary 6.5] assures us that 1 is in the stable range for A. That is, if a and b are elements of A such that $Aa + Ab = A$, then there exists a $t \in A$ such that $a + tb$ is a unit of A.

The following theorem is a natural generalization of a remark of Kaplansky that if 1 is in the stable range of R and $R \oplus A$ is isomorphic to $R \oplus B$, then A and B are isomorphic. See also Swan [11, Proposition 11.7] for an earlier result of this type.

THEOREM 2. *Let R be any ring and A, B, and C any modules such that $A \oplus B$ is isomorphic to $A \oplus C$ and such that the endomorphism ring, T, of A has* 1 *in the stable range, then B is isomorphic to C.*

Proof. Let $f_{M,N}$ denote an R homomorphism from N to M.

We are presumed to have maps $f_{A\oplus C,A\oplus B}$ and $g_{A\oplus B,A\oplus C}$ whose composition in each direction are the identities. Thinking of f and g as 2×2 matrices we have

$$\begin{pmatrix} g_{A,A} & g_{A,C} \\ g_{B,A} & g_{B,C} \end{pmatrix} \begin{pmatrix} f_{A,A} & f_{A,B} \\ f_{C,A} & f_{C,B} \end{pmatrix} = \begin{pmatrix} I_A & 0 \\ 0 & I_B \end{pmatrix}$$

where I_M is the identity on M. Multiplying out the left hand side we get

$$\begin{pmatrix} g_{A,A} f_{A,A} + g_{A,C} f_{C,A} & g_{A,A} f_{A,B} + g_{A,C} f_{C,B} \\ g_{B,A} f_{A,A} + g_{B,C} f_{C,A} & g_{B,A} f_{A,B} + g_{B,C} f_{C,B} \end{pmatrix} = \begin{pmatrix} I_A & 0 \\ 0 & I_B \end{pmatrix}.$$

Hence $Tf_{A,A} + Tg_{A,C}f_{C,A} = T$. Hence there exist $t \in T$ with $f_{A,A} + tg_{A,C}f_{C,A} = u$ a unit.

Hence if we use the map

$$g' = \begin{pmatrix} I_A & tg_{A,C} \\ g_{B,A} & g_{B,C} \end{pmatrix}$$

in place of g we get

$$g'f = \begin{pmatrix} u & v_{A,B} \\ 0 & I_B \end{pmatrix}$$

$g'f$ is clearly an isomorphism since we can make u be the identity by multiplying by u^{-1} and then use elementary row operations to remove what is left over. But $g'f$ an isomorphism and f an isomorphism implies g' is. But then

$$g'' = \begin{pmatrix} g_{B,A} & I_B \\ -I_A & 0 \end{pmatrix}\begin{pmatrix} I_A & tg_{A,C} \\ g_{B,A} & g_{B,C} \end{pmatrix}\begin{pmatrix} I_A & -tg_{A,C} \\ 0 & I_C \end{pmatrix}$$

is an isomorphism.

$$g'' = \begin{pmatrix} I_A & 0 \\ 0 & g_{B,C} - g_{B,A}tg_{A,C} \end{pmatrix}$$

and hence $g_{B,C} - g_{B,A}tg_{A,C}$ is an isomorphism from C to B.

COROLLARY 1. *Let R be a local ring, A a finitely generated R module, and B and C any R modules such that $A \oplus B$ is isomorphic to $A \oplus C$. Then B is isomorphic to C.*

Proof. Immediate from Theorem 2 and the proceeding remarks.

REMARKS. We note that Theorem 2 applies even for non-noetherian rings. Estes and Ohm in [4] give examples of commutative rings R of any finite Krull dimension with 1 in the stable range. Heinzer in [6] gives such examples where the maximal spectrum is a noetherian space. Since any commutative ring equals its endomorphism ring, Theorem 2 shows that these rings are cancellable from any modules over them. I conjecture that these examples behave like the examples in Theorem 1 of discrete valuation rings which were not Hensel. More specifically, the conjecture is if R is ring with noetherian maximal spectrum such that every finite R algebra, T, has $d+1$ in the stable range, then the dimension of the maximal spectrum is less than or equal to d.

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