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ON A PROBLEM OF COMPLETION IN BORNOLOGY

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In this note an example is given to show that the bornological completion of a polar space need not be polar. Also, a theorem of Grothendieck's type is proved, from which necessary and sufficient conditions for the completion of a polar space to be again polar are derived.

1. Notation and terminology are as in [4]. In particular, b.c.s. means a locally convex, bornological linear space over the scalar field of real or complex numbers.

In [4, 5. p. 160] Hogbe-Nlend lists, among unsolved problems in bornology, the following one, which was first raised by Buchwalter in his thesis [1, Remarque, p. 26]:

Is the bornological completion of a polar b.c.s. again polar?

The purpose of this note is to exhibit an example that answers this question in the negative. We also prove a theorem of Grothendieck's type for regular b.c.s. with weakly concordant norms, which enables us to give necessary and sufficient conditions for the completion of a polar b.c.s. to be polar.

2. For each n let the double sequence $a^n = (a_{ij}^n)$ be defined by $a_{ij}^n = j$ for $i \leqq n$ and all j, $a_{ij}^n = 1$ for $i > n$ and all j, and denote by E_n the normed space of scalar-valued double sequences (x_{ij}) with only finitely many nonzero terms, under the norm

$$\| (x_{ij}) \|_n = \sup_{i,j} \frac{|x_{ij}|}{a_{ij}^n} . \tag{1}$$

Let E be the bornological inductive limit of the spaces E_n; thus $E = E_n$ algebraically, and a set $B \subset E$ is bounded for the inductive limit bornology if and only if there exist positive integers n, k such that $\| (x_{ij}) \|_n \leqq k$ for all $(x_{ij}) \in B$. It is easily seen that E is a polar b.c.s. whose dual $E^\times$ consists of all scalar-valued double sequences (u_{ij}) such that

$$\sum_{i,j=1}^{\infty} a_{ij}^n \, | u_{ij} | < \infty \qquad \text{for all } n .$$

By [1, Théorème (2.8.15)] the completion $\hat{E}$ of E is given by $\hat{E} = \varinjlim \hat{E}_n$ (bornological inductive limit), where $\hat{E}_n$ is the completion of the normed space E_n, i.e., the Banach space of scalar-valued double sequences (x_{ij}) such that $\lim_{i,j\to\infty} x_{ij}/a_{ij}^n = 0$ under the norm (1). It also follows from [1, Théorème (2.8.15)] that $\hat{E}^{\times} = E^{\times}$. Thus, it remains to show that the b.c.s. $\hat{E}$ is not polar with respect to the duality $\langle \hat{E}, E^{\times} \rangle$, i.e., that there is a bounded subset B of $\hat{E}$ whose bipolar B^{00} is unbounded. In fact, the set

$$B = \left\{ (x_{ij}) \in \hat{E} \colon \sup_{i,j} |x_{ij}| \leqq 1 \, , \quad \lim_{i,j\to\infty} x_{ij} = 0 \right\}$$

is bounded in the Banach space $\hat{E}_1$ and hence bounded in $\hat{E}$; however, since

$$B^{00} = \left\{ (x_{ij}) \in \hat{E} \colon \sup_{i,j} |x_{ij}| \leqq 1 \right\} ,$$

the sequence $\{(x_{ij}^n)\}$ with $x_{ij}^n = 0$ for $i \neq n$ and all j, $x_{ij}^n = 1$ for $i = n$ and all j, is contained in B^{00} and yet is unbounded, for

$$(x_{ij}^n) \in \hat{E}_n \sim \hat{E}_{n-1} \, .$$

Therefore, B^{00} is unbounded in $\hat{E}$.

3. Let E be a regular b.c.s. with dual $E^{\times}$. For a bounded, absolutely convex set $B \subset E$ we set:

E_B = the normed space spanned by B,
$\hat{B}$ = the completion of B in the Banach space $\hat{E}_B$,
E'_B = the dual of E_B,
B' = the unit ball of E'_B,
B^0 = the polar of B in $E^{\times}$,
B^{00} = the bipolar of B in $\hat{E}$,
p_B = the gauge of B^0 in $E^{\times}$,
$E_B^{\times}$ = the normed space $E^{\times}/p_B^{-1}(0)$.

Moreover, we denote by $E^{\times *}$ the algebraic dual of $E^{\times}$ and identify, as usual, $E_B^{\times}$ with a $\sigma(E'_B, E_B)$-dense subspace of E'_B.

THEOREM 1. *Let E be a regular b.c.s. with weakly concordant norms. The completion $\hat{E}$ of E consists, up to isomorphism, of all those linear functionals on $E^{\times}$ whose restrictions to B^0 are bounded and $\sigma(E^{\times}, E_B)$-continuous for some bounded, absolutely convex set $B \subset E$. Moreover, for every base $\mathscr{B}$ of the bornology of E, the family $\hat{\mathscr{B}} = \{\hat{B} \colon B \in \mathscr{B}\}$ is a base of the bornology of $\hat{E}$ and we have*

$$\hat{B} = \{x \in B^{00} \colon x \text{ is } \sigma(E^{\times}, E_B)\text{-continuous on } B^0\} \tag{2}$$

for every $\hat{B} \in \hat{\mathscr{B}}$.

Proof. If $x \in \hat{E}$, then by [3, Théorème 2, p. 221] there exists a bounded, absolutely convex subset B of E such that $x \in \hat{E}_B$; hence there is a sequence $\{x_n\} \subset E_B$ which converges to x in the Banach space $\hat{E}_B$. It is easily seen that $\{x_n\}$ converges to an element $y \in E^{\times *}$ for the topology $\sigma(E^{\times *}, E^{\times})$ and, therefore, $y = x$. Since $\{x_n\}$ is a bounded sequence in E_B, there is a positive number M such that $|\langle x_n, u\rangle| \leqq M$ for all n and all $u \in B^0$. It follows that $|\langle x, u\rangle \leqq M$ for all $u \in B^0$. It remains to show that the restriction of x to B^0 is $\sigma(E^{\times}, E_B)$-continuous. By Grothendieck's theorem x is $\sigma(E'_B, E_B)$-continuous on B'; hence x determines a unique bounded linear functional z on $E_B^{\times}$ whose restriction to the unit ball of $E_B^{\times}$ is $\sigma(E_B^{\times}, E_B)$-continuous. Let ϕ be the canonical map $E^{\times} \to E_B^{\times}$. Since $p_B^{-1}(0) = (E_B)^0$, ϕ is continuous from $(E^{\times}, \sigma(E^{\times}, E_B))$ to $(E_B^{\times}, \sigma(E_B^{\times}, E_B))$ and, therefore, the restriction of $x = z \circ \phi$ to B^0 is $\sigma(E^{\times}, E_B)$-continuous.

We have also proved that

$$(3) \qquad \hat{B} \subset \{x \in B^{00}\colon x \text{ is } \sigma(E^{\times}, E_B)\text{-continuous on } B^0\}\ .$$

Conversely, let $x \in E^{\times *}$ and suppose that, for some bounded, absolutely convex subset B of E, the restriction of x to B^0 is $\sigma(E^{\times}, E_B)$-continuous and satisfies

$$(4) \qquad |\langle x, u\rangle| \leqq M \qquad \text{for all } u \in B^0\ ,$$

with $M > 0$. By going through the mapping ϕ introduced above we see that x determines a unique bounded linear functional z on $E_B^{\times}$ ($z \circ \phi = x$) whose restriction to the unit ball $B^0/p_B^{-1}(0)$ of $E_B^{\times}$ is $\sigma(E_B^{\times}, E_B)$-continuous. Now $\sigma(E_B^{\times}, E_B)$ is the topology induced by $\sigma(E'_B, E_B)$ on $E_B^{\times}$, $B^0/p_B^{-1}(0)$ is a $\sigma(E'_B, E_B)$-dense subset of B' and B' is a complete uniform space for the uniformity induced by that of $(E'_B, \sigma(E'_B, E_B))$. It follows that z, being uniformly $\sigma(E'_B, E_B)$-continuous on $B^0/p_B^{-1}(0)$, has a unique extension $y \in (E'_B)^*$ which is uniformly $\sigma(E'_B, E_B)$-continuous on B'. By Grothendieck's theorem $y \in \hat{E}_B$ and, by (4),

$$|\langle y, u\rangle| \leqq M \qquad \text{for all } u \in B'\ .$$

This essentially proves the converse implication of (3). Thus (2) holds and the proof is complete, in virtue of the fact that if $\mathscr{B}$ is a base of the bornology of E, then $\widehat{\mathscr{B}} = \{\hat{B}\colon B \in \mathscr{B}\}$ is a base of the bornology of $\hat{E}$ by [3, Théorème 2, p. 221].

COROLLARY. *Let E be a regular b.c.s. with weakly concordant norms. Then E is complete if and only if every linear functional on $E^{\times}$ which is bounded and $\sigma(E^{\times}, E_B)$-continuous on B^0 for some bounded, absolutely convex subset B of E, is $\sigma(E^{\times}, E)$-continuous on $E^{\times}$.*

The referee has informed us of a Note [2] where Theorem 1 and its Corollary for polar b.c.s. are arrived at independently, and where counter examples to the same effect as that given in Section 2 are to be found. As every polar b.c.s. has weakly concordant norms (the converse being clearly false), the results in [2] are a particular case of the ones given here.

An immediate consequence of Theorem 1 is the following criterion for the completion of a polar b.c.s. to be again polar.

THEOREM 2. *Let E be a polar b.c.s. The completion $\hat{E}$ of E is polar if and only if every bounded subset B of E is contained in a bounded, absolutely convex set $C \subset E$ such that the restriction of every $x \in B^{00}$ to C^0 is $\sigma(E^\times, E_C)$-continuous.*

REFERENCES

1. H. Buchwalter, *Topologies, Bornologies et Compactologies*, (Thesis, Lyon, 1968).

2. J-F. Colombeau, M. Grange and B. Perrot, *Sur la complétion des espaces vectoriels bornologiques polaires*, C.R.A.S., **273** (1972), 1481–1483.

3. H. Hogbe-Nlend, *Complétion, tenseurs et nucléarité en bornologie*, J. Math. Pure et Appl., **49** (1970), 193–288.

4. ———, *Théorie des Bornologies et Applications*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

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