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MAPPING SPACES AND *CS*-NETWORKS

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In this paper the space of maps from an $\aleph_0$-space to a space Y is studied by means of convergent sequence-networks. The notion of a *cs*-σ-space, a simultaneous generalization of metric spaces and $\aleph_0$-spaces, is defined, and it is shown that if Y is a (paracompact) *cs*-σ-space then the mapping space from X to Y is a (paracompact) *cs*-σ-space when equipped with either the compact-open or the *cs*-open topology. It is proved that the compact sets are the same in the two topologies. The class of *cs*-σ-spaces and the class of $\aleph$-spaces introduced by O'Meara are shown to be identical in the presence of paracompactness.

In this paper all maps are continuous and all spaces Hausdorff.

1. *CS*-networks. We shall call a collection $\mathscr{P}$ of subsets of a space X a *k-network* for X if whenever $C \subset U$, with C compact and U open in X, there exist finitely many elements of $\mathscr{P}$ whose union covers C and lies in U. This is a slight modification of what E. Michael [2] called a *pseudobase*. We may define the $\aleph_0$-*spaces* of Michael as regular spaces with a countable k-network.

If X is a space with topology $\mathscr{T}$ we shall denote by $k(X)$ the k-space obtained by retopologizing X so that a set is closed if its intersection with every $\mathscr{T}$-compact set is $\mathscr{T}$-closed.

If $\{z_1, z_2, \cdots\}$ is a sequence of points which converges to a point z, then we call the set $Z = \{z, z_1, z_2, \cdots\}$ a *convergent sequence* and denote by Z_n the convergent sequence $\{z, z_n, z_{n+1}, \cdots\}$.

A collection $\mathscr{P}$ of subsets of a space X is a *convergent sequence-network* or, more conveniently, a *cs-network* for X if whenever $Z \subset U$, with Z a convergent sequence and U open in X, then $Z_n \subset P \subset U$ for some n and some $P \in \mathscr{P}$. We call a collection $\mathscr{P}$ of subsets of X a *network* for X if whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathscr{P}$.

The notion of *cs*-network was introduced in [1] where the following theorem was proved.

THEOREM 1. *For a topological space X the following are equivalent*:
(1) *X is an $\aleph_0$-space.*
(2) *X is a regular space with a countable cs-network.*

We shall call a regular space with a σ-locally finite *cs*-network a *cs-σ-space*. It is clear from Theorem 1 that every $\aleph_0$-space is a *cs*-

σ-space, and from the Nagata-Smirnov Metrization Theorem that all metric spaces are cs-σ-spaces.

2. Mapping spaces. We shall denote by $\mathscr{C}(X, Y)$ the space of all maps from X to Y with the compact-open topology, and by $\mathscr{C}_p(X, Y)$ the topology of pointwise convergence. The symbol $\mathscr{C}_{cs}(X, Y)$ will denote the space of maps from X to Y with the convergent sequence-open topology. This is the topology whose subbasic open sets are of the form $(Z, U) = \{f \mid f: X \to Y \text{ and } f(Z) \subset U\}$ where Z is a convergent sequence in X and U is open in Y.

The fact that many of the desirable properties of the compact-open topology are also enjoyed by the cs-open topology was asserted in [1]. Proofs may be found in [7] where O. Wyler shows that a category in which the cs-open topology appears naturally is convenient (in the technical sense of Steenrod [6]) for algebraic topology.

The class of $\aleph_0$-spaces appears to be especially suitable for the study of mapping spaces. For example, at the time he introduced $\aleph_0$-spaces Michael [2] showed that if X and Y are $\aleph_0$-spaces, so is $\mathscr{C}(X, Y)$. It is also true in this case [1] that $\mathscr{C}_{cs}(X, Y)$ is an $\aleph_0$-space. These two results and an unsolved problem form the basis of the present investigation. The problem, also stated by Michael [3], asks whether X compact metric and Y a CW-complex implies that $\mathscr{C}(X, Y)$ is paracompact. More generally one can ask what properties added to the paracompactness of Y will insure the paracompactness of $\mathscr{C}(X, Y)$.

LEMMA 1. *If $\mathscr{P}$ is a collection of subsets of a space X, which is closed under finite intersections, then $\mathscr{P}$ is a cs-network for X if whenever $Z \subset S$, with Z a convergent sequence and S a subbasic open set in X, then $Z_n \subset P \subset S$ for some n and some $P \in \mathscr{P}$.*

Proof. Suppose $Z \subset U$ with Z converging to z and U open in X. Then there exists a basic open set B such that $z \in B \subset U$. Now there exist finitely many subbasic open sets $S_1, \cdots, S_k$ such that $B = S_1 \cap \cdots \cap S_k$. Now $z \in S_i$ for each i, so there exist $n(i)$ and $P_i \in \mathscr{P}$ such that $Z_{n(i)} \subset P_i \subset S_i$ for $1 \leqslant i \leqslant k$. Now let $Z_n = Z_{n(1)} \cap \cdots \cap Z_{n(k)}$ and $P = P_1 \cap \cdots \cap P_k$. Then $Z_n \subset P \subset B \subset U$ and $\mathscr{P}$ is a cs-network for X.

THEOREM 2. *If X is an $\aleph_0$-space and Y is a cs-σ-space, then $\mathscr{C}(X, Y)$ is a cs-σ-space.*

Proof. By Theorem 11.4 (b) of [2] the $\aleph_0$-space X is the image of a separable metric space S under a compact-covering map. Thus by Lemma 1 of [5] $\mathscr{C}(X, Y)$ is homeomorphic to a subspace of $\mathscr{C}(S,$

Y). Since every subspace of a cs-σ-space is also a cs-σ-space, it will suffice to show that $\mathscr{C}(S, Y)$ is a cs-σ-space.

Let $\mathscr{P} = \{P_i\}$ be a countable open base for S which is closed under finite intersections, and let $\mathscr{R} = \bigcup_{j=1}^{\infty} \mathscr{R}_j$ be a σ-locally finite cs-network for Y. Let $[P_i, \mathscr{R}_j] = \{(P_i, R) \mid R \in \mathscr{R}_j\}$, where $(P_i, R) = \{f \in \mathscr{C}(S, Y) \mid f(P_i) \subset R\}$, and let $[\mathscr{P}, \mathscr{R}] = \bigcup_{i,j=1}^{\infty} [P_i, \mathscr{R}_j]$.

We first show that $[\mathscr{P}, \mathscr{R}]$ is σ-locally finite. Clearly $[\mathscr{P}, \mathscr{R}]$ is the union of countably many $[P_i, \mathscr{R}_j]$. To see that each $[P_i, \mathscr{R}_j]$ is locally finite, let $f \in \mathscr{C}(S, Y)$ and $x \in P_i$. Then $f(x) \in Y$, and there is a neighborhood V of $f(x)$ which intersects at most finitely many members of $\mathscr{R}_j$. Then (x, V) is a subbasic open neighborhood of f which meets only those elements (P_i, R) of $[P_i, \mathscr{R}_j]$ for which R intersects V. It is the set of all finite intersections of elements of $[\mathscr{P}, \mathscr{R}]$, which we will call $[\mathscr{P}, \mathscr{R}]'$, which is a σ-locally finite cs-network for $\mathscr{C}(S, Y)$.

By Lemma 1 we need consider only subbasic open sets in showing that $[\mathscr{P}, \mathscr{R}]'$ is a cs-network for $\mathscr{C}(S, Y)$. Let $F = \{f_0, f_1, f_2, \cdots\}$ be a sequence of maps converging to f_0 in $\mathscr{C}(S, Y)$. Let (C, U) be a subbasic open set containing F. Since F is compact, S is a k-space, and Y is regular, we may conclude by Lemma 9.2 of [2] that $F^{-1}(U) = \{x \in S \mid f_i(x) \in U \text{ for some } f_i \in F\}$ is open in S. Clearly $F^{-1}(U) \supset C$. Let $\mathscr{P}' = \{P \in \mathscr{P} \mid P \subset F^{-1}(U)\}$. For every $x \in C$, let $\mathscr{P}(x) = \{P \in \mathscr{P}' \mid x \in P \cap C\}$, and let $\mathscr{P}'(x) = \{P_i' \mid P_i' = \bigcup_{j=1}^{i} P_j, P_j \in \mathscr{P}(x)\}$. Also let $\mathscr{R}(x) = \{R \in \mathscr{R} \mid f_0(x) \in R \subset U\}$. Clearly $\mathscr{R}(x)$ is countable.

There must exist integers N, i, and j such that $F_N \subset (P_i', R_j) \subset (x, U)$. To see this, suppose not. Then since for every N, i, and j, $x \in P_i'$ and $R_j \subset U$, we have $(P_i', R_j) \subset (x, U)$. Therefore, it must be true for every N, i, and j that $F_N \not\subset (P_i', R_j)$. That is, there is some $n \geqslant N$ and some $x_{ij} \in P_i'$ such that $f_n(x_{ij}) \notin R_j$. We now extract a convergent subsequence of F using these results.

Choose $f_{n(1)}$ such that $f_{n(1)}(P_1') \not\subset R_1$. Then there is some $n(2) > n(1)$ such that $f_{n(2)}(P_2') \not\subset R_2$. Similarly choose $f_{n(3)}$ such that $n(3) > n(2)$ and $f_{n(3)}(P_3') \not\subset R_1$, and $f_{n(4)}$ so that $n(4) > n(3)$ and $f_{n(4)}(P_4') \not\subset R_2$. Note that the P_i' are being considered in order, but the R_j are being considered so that their subscripts form the sequence $1, 2, 1, 2, 3, 1, 2, 3, 4, 1, \cdots$. That is, at any place in the sequence of R_j, we proceed until we include the first R_j which had not been included before, and then start over with R_1.

Set $f_i' = f_{n(i)}$, and choose $x_i \in P_i'$ so that $f_i'(x_i)$ is not an element of the R_j which corresponds to $f_{n(i)}$ and P_i'. Now $\{f_i'\}$ is a subsequence of F, and hence it must converge to f_0. The collection $\mathscr{P}'(x)$ is a decreasing countable base for x in S. Thus $\{x_i\}$ converges to x.

Since convergence in the compact-open topology implies continuous convergence for sequences, $\{f'_i(x_i)\}$ converges to $f_0(x)$. Thus all but finitely many elements of $\{f'_i(x_i)\}$ lie in U. Therefore, there exist an integer N and an $R_k \in \mathscr{R}(x)$ so that $f'_i(x_i) \in R_k$ for all $i \geqslant N$. But by the construction of the sequences $\{f_i\}$ and $\{x_i\}$ there is some $m > N$ such that $f'_m(x_m) \notin R_k$. This contradiction means that there do exist some $N(x)$, $i(x)$, and $j(x)$ such that $F_{N(x)} \subset (P'_{i(x)}, R_{j(x)}) \subset (x, U)$. Now $\{P'_{i(x)} \mid x \in C\}$ covers C; therefore, some finite number of the $P'_{i(x)}$ cover C, say $P'_{i(x_0)}, P_{i(x_1)}, \cdots, P'_{i(x_r)}$. Take $M = \max_{0 \leq t \leq r}\{N(x_t)\}$. Then $F_M \subset \bigcap_{t=0}^{r} (P'_{i(x_t)}, R_{j(x_t)}) \subset (C, U)$, and $\mathscr{C}(S, Y)$ has a σ-locally finite *cs*-network. Since Y is regular, $\mathscr{C}(S, Y)$ is regular, and hence is a *cs*-σ-space. Thus $\mathscr{C}(X, Y)$ is also a *cs*-σ-space.

Now note that we could have obtained the collection of sets which forms the *cs*-network for $\mathscr{C}(X, Y)$ in another way. Let f be the compact-covering map such that $f(S) = X$. Then for every $P \in \mathscr{P}$ and $R \in \mathscr{R}$, $(P, R) \cap \mathscr{C}(X, Y) = (f(P), R)$. Thus if we are interested in actually exhibiting a σ-locally finite *cs*-network for $\mathscr{C}(X, Y)$ we may be assured one can be constructed from a countable k-network $\mathscr{P}$ for X and a σ-locally finite *cs*-network $\mathscr{R}$ for Y by forming $[\mathscr{P}, \mathscr{R}]'$ as above.

We now turn our attention to the *cs*-open topology. This topology is compared to the compact-open topology in the following.

LEMMA 2. *Let X be a space in which every compact set is sequentially compact. Then $\mathscr{C}(X, Y)$ and $\mathscr{C}_{cs}(X, Y)$ have the same convergent sequences.*

Proof. Clearly any sequence converging in the compact-open topology converges in the coarser topology. Conversely, let $\{f_n\}$ be a sequence converging to f_0 in $\mathscr{C}_{cs}(X, Y)$. We will show that every subbasic open set in $\mathscr{C}(X, Y)$ which contains f_0 contains all but finitely many f_n. Let $f_0 \in (C, U)$. Suppose there are infinitely many $f_{i(n)}$ for which $f_{i(n)} \notin (C, U)$. Then for every n there exists $x_n \in C$ such that $f_{i(n)}(x_n) \notin U$. But C is sequentially compact, so $\{x_n\}$ has a convergent subsequence $Z \subset C$. Now $f_0 \in (Z, U)$, but for infinitely many f_n, $f_n(Z) \not\subset U$. Thus $\{f_n\}$ converges in $\mathscr{C}(X, Y)$.

THEOREM 3. *If X is an $\aleph_0$-space and Y is a cs-σ-space, $\mathscr{C}_{cs}(X, Y)$ is a cs-σ-space.*

Proof. By Theorem 2 $\mathscr{C}(X, Y)$ has a σ-locally finite *cs*-network $\mathscr{P}$. This same collection of sets forms a *cs*-network for $\mathscr{C}_{cs}(X, Y)$ since $\mathscr{C}(X, Y)$ and $\mathscr{C}_{cs}(X, Y)$ have the same convergent sequences and $\mathscr{C}(X, Y)$ has at least as many open sets as $\mathscr{C}_{cs}(X, Y)$. The neighborhoods used in Theorem 2 to show that the *cs*-network for $\mathscr{C}(S, Y)$

was σ-locally finite were of the form (x, U). Thus the restrictions of these open sets to the subspace $\mathscr{C}(X, Y)$ will illustrate the σ-locally finiteness of $\mathscr{P}$. Sets of the form (x, U) are also open in $\mathscr{C}_{cs}(X, Y)$. Thus $\mathscr{C}_{cs}(X, Y)$ has a σ-locally finite *cs*-network, and since, by Proposition 1 of [1] $\mathscr{C}_{cs}(X, Y)$ is regular, $\mathscr{C}_{cs}(X, Y)$ is a *cs*-σ-space.

LEMMA 3. *If X is separable and Y has each point a G_δ, then $\mathscr{C}_p(X, Y)$ has each point a G_δ.*

Proof. Let $\{x_i\}$ be a countable dense subset of X and let $f \in \mathscr{C}_p(X, Y)$. For every i, let $\{U_{ij}\}$ be a countable collection of open sets whose intersection is $f(x_i)$. Define $V_{ij} = (x_i, U_{ij})$. Clearly $f \in \bigcap_{i,j=1}^{\infty} V_{ij}$. Conversely, suppose $g \neq f$. Then there is some x_k such that $f(x_k) \neq g(x_k)$ and some V_{kj} such that $g(x_k) \notin V_{kj}$. Thus $g \notin \bigcap_{i,j=1}^{\infty} V_{ij}$ and f is a G_δ.

THEOREM 4. *If X is a separable space in which every compact set is sequentially compact and Y has each point a G_δ, then $\mathscr{C}(X, Y)$ and $\mathscr{C}_{cs}(X, Y)$ have the same compact sets.*

Proof. $\mathscr{C}(X, Y)$ and $k(\mathscr{C}(X, Y))$ have the same compact subsets. Also $\mathscr{C}_{cs}(X, Y)$ has the same compact subsets as $k(\mathscr{C}_{cs}(X, Y))$. Now points are G_δ-sets in $\mathscr{C}(X, Y)$ and $\mathscr{C}_{cs}(X, Y)$ and hence points are G_δ's in the associated k-spaces. But a k-space in which every point is a G_δ is a sequential space [4]. Thus $k(\mathscr{C}(X, Y))$ and $k(\mathscr{C}_{cs}(X, Y))$ are each sequential spaces, obtained by expanding the topologies of spaces which had the same convergent sequences. Thus $k(\mathscr{C}(X, Y))$ and $k(\mathscr{C}_{cs}(X, Y))$ are homeomorphic under the identity map, and therefore have the same compact subsets. The conclusion of the theorem now follows.

COROLLARY. *If X is an $\aleph_0$-space and Y is a cs-σ-space, then $\mathscr{C}(X, Y)$ and $\mathscr{C}_{cs}(X, Y)$ have the same compact sets.*

Another simultaneous generalization of $\aleph_0$-spaces and metric spaces has been introduced by P. O'Meara [5]. He calls a regular space an $\aleph$-*space* if it has a σ-locally finite k-network. Because of Theorem 1 it may be expected that there be some relation between *cs*-σ-spaces and $\aleph$-spaces. That this is, in fact, the case is established in the following two theorems.

THEOREM 5. *Every cs-σ-space is an $\aleph$-space.*

Proof. A straightforward adaptation of the relevant part of the proof of Theorem 1 in [1] suffices.

THEOREM 6. *In a paracompact space X the following are equivalent*:

(1) *X is a cs-σ-space.*

(2) *X is an $\aleph$-space.*

Proof. In light of Theorem 5 we need to show only that (2) implies (1). Let $\mathscr{P} = \bigcup_{i=1}^{\infty} \mathscr{P}_i$ be a σ-locally finite k-network for X such that $\mathscr{P}_i \subset \mathscr{P}_{i+1}$ and each $P \in \mathscr{P}$ is closed. For every natural number i and every $x \in X$, let $V_{ix} = X \backslash \bigcup \{P \in \mathscr{P}_i \mid x \notin P\}$. Set $\mathscr{V}_i = \{V_{ix} \mid x \in X\}$. Then $\mathscr{V}_i$ is an open cover of X for every i, and hence it has a precise locally finite open refinement $\mathscr{G}_i = \{G_{ix} \mid x \in X\}$ with $G_{ix} \subset V_{ix}$ for every x. Now for every $P \in \mathscr{P}_i$ such that $x \in P$, define $P_{ix} = P \cap G_{ix}$. For a fixed i and x there are at most finitely many P_{ix}. Denote the finite unions of these P_{ix} by $R_{ix1}, \cdots, R_{ixk}$.

Now the collection $\mathscr{R}_i = \{R_{ixn} \mid x \in X, 1 \leqslant n < \infty\}$ is locally finite. For if $y \in X$ there exists an open neighborhood $N(y)$ which intersects at most finitely many $G_{ix} \in \mathscr{G}_i$. But each G_{ix} intersects only those finitely many R_{ixn} which it contains, and hence $N(y)$ intersects at most finitely many R_{ixn} for each i.

It remains to be shown that $\mathscr{R} = \bigcup_{i=1}^{\infty} \mathscr{R}_i$ is a cs-network for X. Suppose Z is a sequence converging to z and U is an open set such that $Z \subset U$. Then since Z is compact there exists a natural number j and finitely many $P \in \mathscr{P}_j$, say $P_{j1}, \cdots, P_{jm}$, such that $Z \subset \bigcup_{i=1}^{m} P_{ji} \subset U$. We may assume that $z \in P_{ji}$ for $1 \leqslant i \leqslant m$.

Since $\mathscr{G}_j$ is an open cover of X there is some $G_{jx} \in \mathscr{G}_j$ such that $z \in G_{jx}$. Each P_{ji} must contain x, for if $x \notin P_{ji}$ then $z \notin V_{jx} \supset G_{jx}$. Thus $\bigcup_{i=1}^{m} (P_{ji} \cap G_{jx}) \in \mathscr{R}_j$. But $G_{jx} \cap U$ is an open neighborhood of z and hence there exists an r such that $Z_r \subset G_{jx} \cap U$. Therefore, $Z_r \subset \bigcup_{i=1}^{m} (P_{ji} \cap G_{jx}) \subset U$, and $\mathscr{R}$ is a cs-network for X.

The following lemma and theorem were obtained by O'Meara [5].

LEMMA 4. *Let X be a regular space with a σ-locally finite network $\mathscr{T} = \bigcup_{n=1}^{\infty} \mathscr{T}_n$. Suppose for every n there is a locally finite family of neighborhoods $\{V_n(x) \mid x \in X\}$ such that $Cl\,(V_n(x))$ meets only finitely many $T \in \mathscr{T}_n$. Then X is paracompact.*

THEOREM 7. *If X is an $\aleph_0$-space and Y is a paracompact $\aleph$-space, then $\mathscr{C}(X, Y)$ is a paracompact $\aleph$-space.*

We have a similar result if the mapping space is equipped with the cs-open topology.

THEOREM 8. *Let X be an $\aleph_0$-space and let Y be a paracompact cs-σ-space. Then $\mathscr{C}_{cs}(X, Y)$ is a paracompact cs-σ-space.*

Proof. Let $\mathscr{P} = \{P_i\}$ be a countable k-network for X, and let $\mathscr{R} = \bigcup_{i=1}^{\infty} \mathscr{R}_i$ be a σ-locally finite cs-network for Y. Let $[P_i, \mathscr{R}_j] = \{(P_i, R) \mid R \in \mathscr{R}_j\}$, and let $[\mathscr{P}, \mathscr{R}] = \bigcup_{i,j=1}^{\infty} [P_i, \mathscr{R}_j]$. By Theorem 3 and the remarks at the end of the proof of Theorem 2, it may be seen that the set of all finite intersections of $[\mathscr{P}, \mathscr{R}]$ forms a σ-locally finite cs-network for $\mathscr{C}_{cs}(X, Y)$. We now show that Lemma 4 may be applied to this family.

For every $f \in \mathscr{C}_{cs}(X, Y)$, choose $x \in P_i$ and let $V_{ij}(f)$ be an open neighborhood of $f(x)$ which intersects at most finitely many $R \in \mathscr{R}_j$. Consider the open cover $\{V_{ij}(f) \mid f \in \mathscr{C}_{cs}(X, Y)\}$ of Y. By the paracompactness of Y there exists a locally finite open refinement $\mathscr{W}_{ij} = \{W_{ij}(f) \mid f \in \mathscr{C}_{cs}(X, Y)\}$ such that $W_{ij}(f) \subset \mathrm{Cl}\,(W_{ij}(f)) \subset V_{ij}(f)$ for every f. Then $\mathrm{Cl}\,(x, W_{ij}(f)) \subset (x, \mathrm{Cl}\,(W_{ij}(f)))$ which intersects at most finitely many $(P_i, R) \in [P_i, R_j]$. Thus $\mathrm{Cl}\,(x, W_{ij}(f))$ meets at most finitely many of the finite intersections of $[P_i, \mathscr{R}_j]$ and by Lemma 4, $\mathscr{C}_{cs}(X, Y)$ is paracompact.

It can be seen from Example 1 of [1] that despite Theorem 4 the spaces $\mathscr{C}(X, Y)$ and $\mathscr{C}_{cs}(X, Y)$ considered in Theorems 2, 3, 7, and 8 need not be homeomorphic even in the special case where both X and Y are separable metric spaces.

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