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DIMENSION THEORY IN ZERO-SET SPACES

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The main purpose of this paper is to show that the zero-set spaces of Gordon provide a natural and very general setting in which to develop dimension theory. Defining covering dimension for zero-set spaces in the natural way, it is shown that the subspace theorem, the product theorem, and sum theorem hold. As a consequence it is possible to give a subspace theorem for arbitrary topological spaces.

1. A subspace theorem for arbitrary topological spaces. From the general theory in zero-set spaces it is possible to deduce a subspace theorem in arbitrary topological spaces. To express the result it is convenient to have on hand a definition of the dimension of a ring as defined in [1]. This notion also allows the simplification of certain proofs in dimension theory.

Let R be a commutative ring with identity. By a *basis* of R we mean a finite set of elements which generate R. The *order* of a basis is the largest integer n for which there exist $n+1$ members of the basis with nonzero product. A basis $\{a_i\}$ of R is said to *refine* the basis $\{b_j\}$ of R if each a_i is a multiple of some b_j. The *dimension* of R, denoted by $d(R)$, is the least cardinal $\mathfrak{m}$ such that every basis of R has a refinement of order at most $\mathfrak{m}$.

Let $C(X)$ denote the ring of continuous real-valued functions on a topological space X. It is shown in [1] that $\dim X = d(C(X))$. For subspaces A of X, the statement $d(C(A)) \leqq d(C(X))$ is equivalent to $\dim A \leqq \dim X$, the assertion of the subspace theorem. These statements are not always true in arbitrary topological spaces [9, p. 264]. However we obtain a subspace theorem by replacing $C(A)$ by another ring of functions associated with A. Let $C_Z(A)$ denote the set of all real-valued functions f defined on A such that for each real number r, the sets $\{x \in A \mid f(x) \leqq r\}$ and $\{x \in A \mid f(x) \geqq r\}$ are the intersections with A of zero-sets of X. Here a zero-set of X is the set of zeros of a continuous real-valued function on X. For general information about zero-sets the reader is referred to [9]. It follows from Theorem 3.5 of [10] that $C_Z(A)$ is a uniformly closed ring and is also a lattice.

The proof of the following subspace theorem will be discussed after Theorem 10.

THEOREM 1. *If A is a subspace of an arbitrary topological space X, then $d(C_Z(A)) \leqq d(C(X))$.*

It is also possible to formulate analogous versions of the product theorem and the sum theorem in arbitrary topological spaces. However, it will become apparent that the most natural way to formulate these results is in the context of zero-set spaces.

The ring $C_Z(A)$ introduced above seems to have been little studied in the literature and we digress to give some further information about it.

It is clear that $C_Z(A)$ is contained in $C(A)$. A condition for equality may be given using the concept of "z-embedding" from [11]. A subspace A of a topological space X is said to be *z-embedded* in X if every zero-set of A is the intersection with A of a zero-set of X. In particular, if X is perfectly normal, every closed set of X is a zero-set of X, and it follows that every subspace of X is z-embedded in X.

THEOREM 2. *Let A be a subspace of a topological space X. Then $C_Z(A) = C(A)$ if and only if A is z-embedded in X.*

Proof. Suppose A is z-embedded in X and $f \in C(A)$. For each real number r, the sets $\{x \in A \mid f(x) \leqq r\}$ and $\{x \in A \mid f(x) \geqq r\}$ are zero-sets of A and by hypothesis are the intersections with A of zero-sets of X. Hence $f \in C_Z(A)$. It follows that $C_Z(A) = C(A)$.

Conversely, suppose $C_Z(A) = C(A)$ and let Z be a zero-set of A. Then $Z = \{x \in A \mid f(x) = 0\}$ for some $f \in C(A)$. Since $f \in C_Z(A)$, it follows that Z is the intersection with A of a zero-set of X. Hence A is z-embedded in X.

If $f \in C(X)$, the restriction of f to a subspace A clearly belongs to $C_Z(A)$. In the case when A is a zero-set of X, we have a Urysohn extension theorem for members of $C_Z(A)$.

THEOREM 3. *Let A be a zero-set of an arbitrary topological space X. Then each $f \in C_Z(A)$ has a continuous extension to X.*

Proof. Except for notation, the proof is the same as in Gillman and Jerison [9, pp. 18, 19].

2. Zero-set spaces. In the following, $\boldsymbol{R}$ will denote the real numbers and N the positive integers. For information on zero-set spaces, the reader is referred to Gordon [10]. In this section we review some of the main facts and give some further results.

A zero-set space is a pair $(X, \mathscr{Z})$ where $\mathscr{Z}$ is a family of subsets of X satisfying certain axioms. The sets in $\mathscr{Z}$ are called zero-sets and their complements with respect to X are called cozero-sets. The first of the axioms given by Gordon concerning the separation of distinct points of X by a set in $\mathscr{Z}$ is not needed in dimension theory and we will omit this requirement. Further, we find it convenient to give the axioms for the cozero-sets rather than the zero-sets. Thus by a zero-set space we mean a pair $(X, \mathscr{Z})$ where $\mathscr{Z}$ is a family of subsets of X such that the family $\mathscr{C}$ of all complements in X of members of $\mathscr{Z}$ satisfies the following conditions.

(1) $\mathscr{C}$ is closed under countable unions; in particular, $X \in \mathscr{C}$.

(2) $\mathscr{C}$ is closed under finite intersections; in particular, $\phi \in \mathscr{C}$.

(3) Whenever A and B are disjoint zero-sets of X, there exist disjoint cozero-sets C and D with $A \subset C$ and $B \subset D$.

(4) Each cozero-set of X is the countable union of zero-sets of X.

It is interesting to note that these conditions differ from the axioms for the open sets of a perfectly normal topological space only in the requirement that we have closure under countable unions rather than arbitrary unions. It is significant that a good theory of dimension is available for perfectly normal spaces [2], [3].

The family of all cozero-sets in an arbitrary topological space satisfies the conditions (1)—(4), [9, Chapter 1]. It is again significant that the most satisfactory theory of dimension in topological spaces involves the use of cozero-sets [9, p. 243], [8].

If X and Y are zero-set spaces, a mapping $\alpha: X \rightarrow Y$ is called a zero-set mapping if the inverse image of every cozero-set is again a cozero-set. When $Y = \boldsymbol{R}$, the cozero-sets are taken to be the open sets of $\boldsymbol{R}$. If X is a zero-set space, $S(X)$ will denote the set of real-valued zero-set mappings on X. It is shown in [9] that

(i) $S(X)$ is a uniformly closed ring and is also a lattice:

(ii) For each cozero-set C of X, there is a zero-set function $f \in S(X)$ such that $C = \{x \in X \mid f(x) \neq 0\}$.

Part (ii) is the justification for calling members of $\mathscr{C}$ cozero-sets, and may be derived from a Urysohn's lemma argument in almost the same way that an open set in a perfectly normal topological space is shown to be the cozero-set of a continuous real-valued function as in [6, p. 148].

With subspaces defined in the natural way as in [10], we also have the Urysohn extension theorem holding for zero-set spaces. Again the proof is the same as in [9].

THEOREM 4. *Let X be a zero-set space and let A be a zero-set of X. Then each $f \in S(A)$ has an extension to a zero-set function of $S(X)$.*

By a *basic cover* of a zero-set space, we mean a covering of X by cozero-sets. In defining locally finite families in zero-set spaces, we use a condition which is somewhat stronger than the condition for local finiteness in topological spaces.

DEFINITION. A family $\{U_\alpha | \alpha \in A\}$ of subsets of a zero-set space X is said to be *locally finite* if there exists a countable basic cover $\{V_i | i \in N\}$ of X such that each V_i meets at most finitely many of the U_α.

We now give a result which is crucial in the development of dimension theory in zero-set spaces. This says in effect that every zero-set space is countably paracompact. Countably paracompact spaces were discussed by Dowker [4] who showed that each perfectly normal topological space is countably paracompact.

THEOREM 5. *Let $\{U_i\}$ be a countable basic cover of a zero-set space. Then there exists a countable locally finite basic cover $\{V_i\}$ of X such that $V_i \subset U_i$ for each i.*

Proof. Choose zero-set functions f_j such that

$$\bigcup_{k \leqq j} U_k = \{x | f_j(x) > 0\} ,$$

and let

$$B_{ji} = \{x | f_j(x) < 1/i\} ,$$

$$V_i = U_i \cap (\bigcap_{j<i} B_{ji}) ,$$

and

$$W_{jk} = \{x | f_j(x) > 1/k\} .$$

It is clear that V_i is a cozero-set and that $V_i \subset U_i$. For each $x \in X$, there is some first i for which $x \in U_i$ but $x \notin U_k$ for $k < i$. Suppose that $x \notin V_i$. Then $x \notin B_{ji}$ for some $j < i$ and therefore $f_j(x) \geqq 1/i > 0$. Thus $x \in U_k$ for some $k \leqq j < i$. This contradiction shows that $x \in V_i$ and so $\{V_i\}$ is a basic cover of X.

Given $x \in X$ there is some j for which $f_j(x) > 0$. Hence $x \in W_{jk}$ for some k and therefore $\{W_{jk}\}$ is a countable basic cover of X. If $i > j$ and $i > k$, then $y \in W_{jk}$ implies that $f_j(y) > 1/k > 1/i$ so that $y \notin B_{ji}$. Thus $y \notin V_i$ and $W_{jk} \cap V_i = \phi$. Therefore, W_{jk} meets only finitely many V_i and the proof is complete.

REMARK. It is possible to show as in [7, p. 221] that each countable basic cover has a star-finite countable basic refinement.

3. Covering dimension for zero-set spaces. The *order* of a basic cover is the largest integer n for which there exist $n+1$ members of the cover with nonempty intersection.

DEFINITION. Let X be a zero-set space. The dimension of X, denoted by $\dim X$, is the least cardinal $\mathfrak{m}$ such that every finite basic cover of X has a finite basic refinement of order at most $\mathfrak{m}$.

If $(X, \mathscr{T})$ is a topological space, then the family $\mathscr{Z}$ of zero-sets of X form a zero-set structure on X. The definition of dimension for a topological space $(X, \mathscr{T})$ as given in [9, p. 243] is the above definition for the case of the zero-set space $(X, \mathscr{Z})$.

The subspace theorem for zero-sets of a zero-set space is an easy consequence of the definition. If X is a zero-set space with $\dim X \leqq n$, and if A is a zero-set of X, then $\dim A \leqq n$.

We remark that if $\{U_i\}$ is a basic cover and $\{V_j\}$ is a countable basic refinement of order $\leqq n$, then there exists a basic cover $\{W_i\}$ of order $\leqq n$ such that $W_i \subset U_i$ for each i. In fact we only have to set $W_i = \bigcup \{V_j \mid V_j \subset U_i,\ V_j \not\subset U_k \text{ for } k < i\}$. Some of the W_i may possibly be empty.

The following lemma is proved using an argument due to deVries and given in Nagata [13, p, 22].

LEMMA 6. *Let $\{U_i \mid i \in N\}$ be a countable basic cover of a zero-set space X, and let $\{Z_j \mid j \in N\}$ be a countable locally finite cover of X consisting of zero-sets Z_j such that $\dim Z_j \leqq n$ for each j and such that each Z_j meets at most finitely many U_i. Then there exists a basic cover $\{V_i \mid i \in N\}$ of order $\leqq n$ such that $V_i \subset U_i$ for each $i \in N$.*

Proof. We construct by induction a sequence of basic covers $\{U_{j,i} \mid i \in N\}$ such that $U_{0,i} = U_i$ and $U_{j,i} \subset U_{k,i}$ for $j > k$, and such that each point of Z_j is contained in at most $n+1$ members of $\{U_{j,i} \mid i \in N\}$. Suppose that $\{U_{k,i} \mid i \in N\}$ has been constructed. To get $\{U_{k+1,i} \mid i \in N\}$ we restrict $\{U_{k,i} \mid i \in N\}$ to Z_{k+1} and choose a basic refinement $\{W_{k,i} \mid i \in N\}$ of it of order $\leqq n$ such that $W_{k,i} \subset U_{k,i}$ for each i. If we put $U_{k+1,i} = (U_{k,i} \setminus Z_{k+1}) \cup W_{k,i}$, the induction step is completed.

Let $V_i = \bigcap \{U_{k,i} \mid k \in N\}$. To see that V_i is a cozero-set, we let $\{A_s\}$ be a countable basic cover such that each A_s meets at most finitely many Z_k. Since $U_{k+1,i}$ was obtained from $U_{k,i}$ by removing part of Z_{k+1}, it follows that for some integer K we have $U_{k,i} \cap A_s = U_{K,i} \cap A_s$ for all $k > K$. Since V_i is a countable union of sets of the form $U_{K,i} \cap A_s$, it is a cozero-set. Finally it is clear that each point of X is contained in at most $n+1$ of the V_i.

LEMMA 7. *Let $\{V_k\}$ be a countable basic cover of a zero-set space*

X. There exists a basic cover $\{W_k\}$ *and zero-sets* Z_k *such that* $W_k \subset Z_k \subset V_k$ *for each* k.

Proof. The sets W_k and Z_k may be found inductively as in [9, p. 243].

For normal topological spaces, Dowker [5] has shown that one may use locally finite covers instead of finite covers in the definition of dimension. For zero-set spaces one may use countable basic covers instead of finite basic covers as shown in the next result.

THEOREM 8. *Let X be a zero-set space. Then* $\dim X \leqq n$ *if and only if every countable basic cover of X has a countable basic refinement of order* $\leqq n$.

Proof. The sufficiency of the condition follows from the remarks preceding Lemma 6.

Suppose now that $\dim X \leqq n$. In view of Theorem 5, it is sufficient to prove the result for locally finite covers. Let $\{U_i\}$ be a countable locally finite basic cover of X and let $\{A_k\}$ be a countable basic cover of X such that each A_k meets at most finitely many U_i. Let $\{V_k\}$ be a locally finite basic cover of X such that $V_k \subset A_k$ for each k. Choose zero-sets Z_k as in Lemma 7. Since $\{V_k\}$ is locally finite, so is $\{Z_k\}$ and since $Z_k \subset A_k$, each Z_k meets at most finitely many U_i. We have $\dim Z_k \leqq n$ for each k. An application of Lemma 6 now yields a basic refinement of $\{U_i\}$ of order $\leqq n$. This completes the proof.

Further characterizations of dimension in terms of mappings and separation of zero-sets may be obtained for zero-set spaces. These results, and their proofs, are similar to the results in topological spaces as in [12] and [8].

4. The sum, subspace, and product theorems.

THEOREM 9. (The sum theorem.) *Let X be a zero-set space and let* $X = \bigcup_{i=1}^{\infty} Z_i$ *where each Z_i is a zero-set. If* $\dim Z_i \leqq n$ *for each i, then* $\dim X \leqq n$.

Proof. This result may be proved as in Hemmingsen [12, Theorem 4.2]. For the case when the family $\{Z_i\}$ is locally finite, the result is immediate from Lemma 6. It is only the latter case which is needed in the proof of the subspace theorem.

THEOREM 10. (The subspace theorem.) *If X is a zero-set space and A is a subspace of X, then* $\dim A \leqq \dim X$.

Proof. We have already observed that the result holds when A is a zero-set of X.

Suppose next that A is a cozero-set of X. Choose a zero-set function f such that $0 \leqq f \leqq 1$ and $A = \{x \mid f(x) > 0\}$. Let $Z_i = \{x \mid 1/(i+1) \leqq f(x) \leqq 1/i\}$, $(i = 1, 2, \cdots)$, and let $U_i = \{x \mid f(x) > 1/i\}$. Then $\{U_i\}$ is a basic cover of A and each U_i meets finitely many Z_i. Thus $\{Z_i\}$ is a locally finite collection in A. Also $A = \bigcup_{i=1}^{\infty} Z_i$ and since $\dim Z_i \leqq n$ for each i, we have $\dim A \leqq n$ by Theorem 9.

Finally we consider an arbitrary subspace A of X. Let $\{U_i\}$ be a finite basic cover of A. We can write $U_i = V_i \cap A$ for cozero-sets V_i of X. The cozero-set $V = U_i V_i$ has dimension $\leqq n$ (by the preceding paragraph) and so the basic cover $\{V_i\}$ of V has a basic refinement $\{W_j\}$ (in V) of order $\leqq n$. The trace of $\{W_j\}$ on A is then a basic refinement of $\{U_i\}$ of order $\leqq n$. Thus $\dim A \leqq n$ and the proof is complete.

If X is a zero-set space, we may show, as in [1], that $\dim X = d(S(X))$. Moreover, if $S^*(X)$ denotes the bounded functions in $S(X)$, the same argument may be used to show that $\dim X = d(S^*(X))$.

Theorem 1 now follows from Theorem 10. If $\mathscr{T}$ is a topology on X, we let $\mathscr{Z}$ denote the family of zero-sets in X of elements of $C(X)$. If A is a zero-set subspace of $(X, \mathscr{Z})$, then $\dim A \leqq \dim X$. Hence $d(S(A)) \leqq d(S(X))$. The result follows if we observe that $C_Z(A) = S(A)$ and $C(X) = S(X)$.

The product of zero-set spaces is discussed in [10, p. 152]. The product theorem for zero-set spaces will be derived from the subspace theorem and the product theorem for compact topological spaces.

Let X be a zero-set space for which $S(X)$ separates the points of X. Gordon [10] has shown that such a space has a compactification βX with the property that $S^*(X)$ is isomorphic to $S(\beta X)$. Moreover, if βX is given the weak topology induced by $S(\beta X)$, then $C(\beta X)$, the ring of continuous functions on βX for this topology, coincides with $S(\beta X)$. Thus $d(C(\beta X)) = d(S(\beta X))$ and it follows that the dimension of βX is the same whether considered as a zero-set space or as a topological space.

THEOREM 11. *If X is a zero-set space, then* $\dim X = \dim \beta X$.

Proof. Since $S^*(X)$ and $S(\beta X)$ are isomorphic, $\dim X = n \Leftrightarrow d(S^*(X)) = n \Leftrightarrow d(S(\beta X)) = n \Leftrightarrow \dim \beta X = n$.

THEOREM 12. (The product theorem.) *If X and Y are zero-set spaces, then*

$$\dim (X \times Y) \leqq \dim X + \dim Y .$$

Proof. Suppose first that $S(X)$ and $S(Y)$ separate the points of X and Y respectively. Since $X \times Y \subset \beta X \times \beta Y$, the subspace theorem yields $\dim (X \times Y) \leqq \dim (\beta X \times \beta Y)$. By the product theorem for compact topological spaces [12], we have $\dim (\beta X \times \beta Y) \leqq \dim \beta X + \dim \beta Y$. Since $\dim X = \dim \beta X$ and $\dim Y = \dim \beta Y$, it follows that $\dim (X \times Y) \leqq \dim X + \dim Y$.

Now suppose that X and Y are arbitrary zero-set spaces. We may define, as in the construction on p. 41 of [9], zero-set spaces X' and Y' such that $S(X)$ is isomorphic to $S(X')$ and $S(Y)$ is isomorphic to $S(Y')$, and $S(X')$ and $S(Y')$ separate the points of X' and Y' respectively. It is clear from the construction that $S(X \times Y)$ is isomorphic to $S(X' \times Y')$. It follows that $\dim (X \times Y) = \dim (X' \times Y')$. The result follows on applying the first part of the theorem to X' and Y'.

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