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THE SCHOLZ-BRAUER PROBLEM ON ADDITION CHAINS

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An addition chain for a positive integer n is a set $1 = a_0 < a_1 < \cdots < a_r = n$ of integers such that every element a_i is the sum $a_j + a_k$ of two preceding members (not necessarily distinct) of the set. The smallest length r for which an addition chain for n exists is denoted by $l(n)$. Let $\lambda(n) = [\log_2 n]$, and let $\nu(n)$ denote the number of ones in the binary representation of n. The purpose of this paper is to show how to establish the result that if $\nu(n) \geqq 9$ then $l(n) \geqq \lambda(n) + 4$. This is the $m = 3$ case of the conjecture that if $\nu(n) \geqq 2^m + 1$ then $l(n) \geqq \lambda(n) + m + 1$ for which cases $m = 0, 1, 2$ have previously been estabished. The fact that the conjecture is true for $m = 3$ leads to the theorem that $n = 2^m(23) + 7$ for $m \geqq 5$ is an infinite class of integers for which $l(2n) = l(n)$. The paper concludes with this result.

An addition chain for a positive integer n is a set $1 = a_0 < a_1 < a_2 < \cdots < a_r = n$ of integers such that every element a_i is the sum $a_j + a_k$ of two preceding members (not necessarily distinct) of the set. The smallest length r for which an addition chain for n exists is denoted by $l(n)$. Let $\lambda(n) = [\log_2 n]$, and let $\nu(n)$ denote the number of ones in the binary representation of n. Step i of an addition chain is $a_i = a_j + a_k$ for some $k \leqq j < i$. Since $a_i \leqq 2a_j \leqq 2a_{i-1}$, either $\lambda(a_i) = \lambda(a_{i-1})$ or $\lambda(a_i) = \lambda(a_{i-1}) + 1$. Step i is called a small step in the former case and a big step in the latter case. Since $a_i \leqq 2a_{i-1}$, a member of the chain must occur in each of the half-open intervals $[2^k, 2^{k+1})$ for $0 \leqq k \leqq \lambda(n)$. Every time a step takes the chain from one interval to the next it is a big step; otherwise, it is a small step. There are $\lambda(n)$ big steps in the chain, and the remaining steps are small steps. If $N(a_i)$ represents the number of small steps in the chain to a_i, then the length r of the chain may be expressed as $r = \lambda(n) + N(n)$.

A conjecture which is equivalent to one made by K. B. Stolarsky [10] states that if $\nu(n) \geqq 2^m + 1$, then $l(n) \geqq \lambda(n) + m + 1$. That is to say if $\nu(n) \geqq 2^m + 1$, then there are at least $m + 1$ small steps in any chain for n. The conjecture is true for $m = 0, 1, 2$. These results may be found in [8] with the case $m = 2$ being part of D. E. Knuth's Theorem C. The primary purpose of this paper is to show how to establish the conjecture for $m = 3$ and to show this case leads to the result that there is an infinite class of integers for which $l(2n) = l(n)$.

If a_j and a_k are two integers written in binary notation and placed one on top of the other in order to add or subtract, the resultant figure is called a configuration and is designated by a_j/a_k. The configuration is divided up into slots numbered from left to right. If $a_j = 101100111$ and $a_k = 10101110$, then a_j/a_k is as follows:

$$\begin{array}{rccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ a_j = & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ a_k = & & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0\,. \end{array}$$

The slot numbers are written above. Slot 4 is called a 1/1 slot, slot 9 is a 1/0 slot etc. Two lemmas which involve integers written in their binary notation are the following:

LEMMA 1. *If* $a_i = a_j + a_k$ *and if* c *represents the number of carries in* $a_j + a_k$, *then* $\nu(a_i) = \nu(a_j) + \nu(a_k) - c$.

LEMMA 2. *If* $a_t = a_j - a_k$ *and there are* s 1/1 *slots in* a_j/a_k *and a one appears in* a_t p *times under either a* 1/1 *slot or a* 0/0 *slot, then* $\nu(a_t) = \nu(a_j) - s + p$.

Two further lemmas will now be given which involve numbers in an addition chain.

LEMMA 3. *If* a_j *and* a_k *are two members of an addition chain and if* $\lambda(a_j) = \lambda(a_k) + m (m \geqq 0)$ *and* $2^m a_k < a_j$, *then* $N(a_j) \geqq N(a_k) + 1$.

Proof. Since $\lambda(a_j) = \lambda(a_k) + m$, there are precisely m big steps from a_k to a_j in the chain, but $2^m a_k < a_j$ implies that there are at least $m + 1$ steps in the chain from a_k to a_j; hence, at least one of them is a small step.

LEMMA 4. *If* a_j *and* a_k *are two members of an addition chain and if* $\lambda(a_j) = \lambda(a_k) + m (m \geqq 2)$ *and* $a_j > 2^{m-1}a_k + 2^{m-2}a_k$, *then* $N(a_j) \geqq N(a_k) + 1$ *unless* $a_j = 2^{m-1}a_{k+1}$.

Proof. Suppose that there are no small steps from a_k to a_j. Assume that there is at least one t such that $2 \leqq t \leqq m$ and $a_{k+t} \neq 2a_{k+t-1}$. Then $a_{k+t} \leqq a_{k+t-1} + a_{k+t-2} \leqq 2^{t-1}a_k + 2^{t-2}a_k$ which implies that $a_{k+m} = a_{k+t+(m-t)} \leqq 2^{m-t}a_{k+t} \leqq 2^{m-t}(2^{t-1}a_k + 2^{t-2}a_k) = 2^{m-1}a_k + 2^{m-2}a_k < a_j$. Thus, $a_{k+m} < a_j$ which implies that there is at least one small step from a_k to a_j which is a contradiction. Therefore, if there are no small steps from a_k to a_j, then $a_{k+t} = 2a_{k+t-1}$ for $2 \leqq t \leqq m$ which implies that $a_j = 2^{m-1}a_{k+1}$. It follows that if $a_j \neq 2^{m-1}a_{k+1}$, then $N(a_j) \geqq N(a_k) + 1$.

Knuth's Theorem C [8] along with the four previous lemmas will be much used in the work that follows. The statement of Theorem

C follows with the integers being expressed in binary form.

THEOREM C. *If* $\nu(n) \geqq 4$, *then* $l(n) \geqq \lambda(n) + 3$ *except when* $\nu(n) = 4$ *and* n *has one of the four following forms*: (A) $n = 1 \cdots d \cdots 1 \cdots 1 \cdots d \cdots 1 \cdots$ *where* d *indicates the number of zeros between the first and second one and between the third and fourth one.* (B) $n = 1 \cdots d \cdots 1 \cdots 1 \cdots e \cdots 1 \cdots$ *where* d *and* e *again indicate zeros and* $e = d - 1$. (C) $n = 1001 \cdots 11 \cdots$. (D) $n = 10000111 \cdots$. *In these four cases* $l(n) = \lambda(n) + 2$.

The $m = 3$ case of the conjecture will now be stated as a theorem, and the method of proof will be described.

THEOREM 1. *If* $\nu(n) \geqq 9$, *then* $l(n) \geqq \lambda(n) + 4$.

Proof. Let $1 = a_0 < a_1 < \cdots < a_r = n$ be an addition chain for an integer n for which $\nu(n) \geqq 9$. Let a_i denote the first member of the chain for which $\nu(a_i) \geqq 9$. Then $a_i = a_j + a_k$ where $k < j$ since if $k = j$, then $a_i = 2a_j$ which would mean that $\nu(a_i) = \nu(a_j)$. Thus, a_j and a_k are distinct members of the chain, and since $\nu(a_j) \leqq 8$ and $\nu(a_k) \leqq 8$, it follows from Lemma 1 that $9 \leqq \nu(a_i) \leqq 16$. Each of the eight cases for $\nu(a_i)$ must be considered, and for each of these cases the possibilities for $\nu(a_j)$ and $\nu(a_k)$ must be considered. For convenience the various cases will be listed as ordered triples $(\nu(a_i), \nu(a_j), \nu(a_k))$. There are 120 cases altogether. The case (9, 5, 4) will be considered first.

By Lemma 1 $c = 0$ for (9, 5, 4), and the only possibility for a_j/a_k is:

$$\begin{aligned} a_j &= 1 \cdots\cdots \\ +a_k &= \cdots 1 \cdots \\ a_i &= 1 \cdots\cdots\ . \end{aligned}$$

As can be seen $\lambda(a_i) = \lambda(a_j)$ and, thus, there is at least one small step from a_j to a_i. Case $m = 2$ of the conjecture implies that $N(a_j) \geqq 3$ since $\nu(a_j) = 5$. Thus, $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq 4$.

Case (9, 4, 5) is virtually the same as (9, 5, 4) except that it is $N(a_k)$ which is greater than or equal to 3. Since $N(a_j) \geqq N(a_k)$, it follows as before that $N(n) \geqq 4$.

The 34 additional cases for which $c = 0$ are handled in the same manner as these cases.

For $c = 1$ there are 28 cases for $(\nu(a_i), \nu(a_j), \nu(a_k))$. Since $a_i \leqq 2a_j$, either $\lambda(a_i) = \lambda(a_j)$ or $\lambda(a_i) = \lambda(a_j) + 1$. If $\lambda(a_i) = \lambda(a_j)$, then as in the cases where $c = 0$ it may be concluded that $N(n) \geqq 4$. If $\lambda(a_i) = \lambda(a_j) + 1$, then with $c = 1$ the only possibility for a_j/a_k is:

$$\begin{aligned} a_j &= 1\cdots \\ +a_k &= \ \ 1\cdots \\ a_i &= 10\cdots . \end{aligned}$$

As previously noted a_j and a_k are distinct members of the chain, and since $\lambda(a_j) = \lambda(a_k)$ it follows that $N(a_j) \geqq N(a_k) + 1$. For those cases where $\nu(a_k) \geqq 5$, $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq 4$. When $\nu(a_k) \leqq 4$, some further work is necessary.

The cases where $3 \leqq \nu(a_k) \leqq 4$ shall first be considered. By Lemma 1 $\nu(a_j) \geqq 6$ since $c = 1$. $a_j \neq 2a_k$ since $\nu(a_j) \neq \nu(a_k)$, and it follows that either $a_j = a_m + a_s$ where $s \leqq m$ and $a_m \neq a_k$ or $a_j = a_k + a_t$ where $t < k$. Suppose $a_j = a_m + a_s$ where $a_m \neq a_k$. Since $a_j \leqq 2a_m$, the possibilities on the number line are:

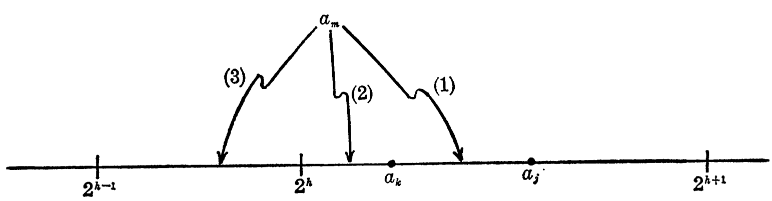

FIGURE 1

In case (1) $N(a_j) \geqq N(a_k) + 2 \geqq 4$ since $\nu(a_k) \geqq 3$. In case (2) $N(a_m) \geqq 2$ for if $N(a_m) \leqq 1$, then $1 = a_0 < a_1 < \cdots < a_m < a_j$ is an addition chain for a_j with less than three small steps contradicting the fact that $\nu(a_j) \geqq 5$ implies $N(a_j) \geqq 3$. Thus, $N(a_j) \geqq N(a_m) + 2 \geqq 4$. In case (3) similar reasoning shows that $N(a_m) \geqq 3$, and, consequently, $N(a_j) \geqq N(a_m) + 1 \geqq 4$. In all three cases $N(n) \geqq N(a_j) \geqq 4$.

Suppose $a_j = a_k + a_t$ where $t < k < j$. Then $a_t = a_j - a_k$. Since $c = 1$ there is only one 1/1 slot in a_j/a_k. When a_j/a_k is considered from a subtraction point of view, it follows from Lemma 2 that $\nu(a_t) \geqq 5$ which means that $N(a_t) \geqq 3$. Thus, $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq N(a_t) + 1 \geqq 4$.

All cases for $c = 1$ have been dispensed with except (9, 8, 2). In this case $\nu(a_k) = 2$ implies $N(a_k) \geqq 1$. If $N(a_k) = 1$, then it may be concluded that all members of the chain preceding a_k have two or less ones in their binary representation. Thus, $\nu(a_{k+1}) \leqq 4$ and $\nu(a_{k+2)}) \leqq 6$. Since $\lambda(a_j) = \lambda(a_k)$, this means that $N(n) \geqq N(a_j) \geqq N(a_k) + 3 \geqq 4$. If $N(a_k) \geqq 2$, then $N(n) \geqq 4$ in the same manner as when $3 \leqq \nu(a_k) \leqq 4$.

For $c = 2$ the cases where $\nu(a_j) \geqq 5$, $\nu(a_k) \geqq 5$, and $\nu(a_j) \neq \nu(a_k)$ are handled rather easily. As with the $c = 1$ cases it may be supposed that $\lambda(a_i) = \lambda(a_j) + 1$. If $\lambda(a_j) = \lambda(a_k)$, then $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq 4$. Thus, it may be supposed that $\lambda(a_j) > \lambda(a_k)$, and the only possibility for a_j/a_k with $c = 2$ is:

$$\begin{aligned} a_j &= \ 11\cdots \\ +\, a_k &= \ \ \ 1\cdots \\ a_i &= 100\cdots . \end{aligned}$$

If $a_j = a_m + a_s$ where $s \leqq m < j$ and $a_m \neq a_k$, then there are three possibilities on the number line:

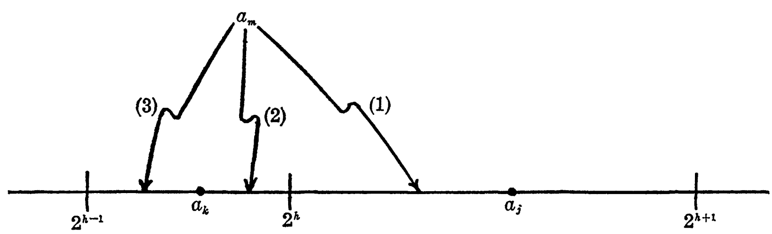

FIGURE 2

In cases (1) and (2) $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq 4$ since $\nu(a_k) \geqq 5$. In case (3) $N(a_m) \geqq 3$ or else $1 = a_0 < a_1 < \cdots < a_m < a_j$ is a chain for a_j with less than three small steps which contradicts $\nu(a_j) \geqq 5$. Thus, $N(n) \geqq N(a_j) \geqq N(a_m) + 1 \geqq 4$. If $a_j = a_k + a_t$, then $a_t = a_j - a_k$. Since $c = 2$, there can be no more 1/1 slots in a_j/a_k, and since $\nu(a_j) \neq \nu(a_k)$, $a_j \neq 2a_k$ which means that a_k and a_t are distinct members of the chain. a_j/a_k then looks as follows:

$$\begin{aligned} a_j &= 11\cdots \\ -a_k &= \ 1\cdots \\ a_t &= \ 1\cdots . \end{aligned}$$

By Lemma 2 $\nu(a_t) \geqq 5$ since $\nu(a_j) \geqq 5$. Since $\lambda(a_k) = \lambda(a_t)$, $N(n) \geqq N(a_k) \geqq N(a_t) + 1 \geqq 4$.

There are 12 cases for which $c = 2$, $\nu(a_j) \geqq 5$, $\nu(a_k) \geqq 5$, and $\nu(a_j) \neq \nu(a_k)$. Thus, 76 of the 120 cases for $(\nu(a_i), \nu(a_j), \nu(a_k))$ have been dispensed with so far. In (10, 6, 6), (12, 7, 7), and (14, 8, 8) $\nu(a_j) = \nu(a_k)$, and it is possible that $a_j = 2a_k$. This means that $a_k = a_t$; hence, a_k and a_t are not distinct members of the chain. Thus, the statement that $N(a_k) \geqq N(a_t) + 1$ cannot be made as with the other cases where $c = 2$ and $\nu(a_j) \geqq 5$ and $\nu(a_k) \geqq 5$. Some additional concepts need to be discussed at this point which make it possible to dispense with cases such as these.

Let $l_8(n)$ denote the minimal length of an addition chain for an integer n all of whose members have eight or less ones in their binary representation. A list of propositions concerning $l_8(n)$ will now be given. The proof of one of these propositions will then be given. The proofs of the others are similar.

PROPOSITION 1. *If* $\nu(n) = 7$ *and* $n = 111\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 2. *If* $\nu(n) = 8$ *and* $n = 111\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$ *unless* $n = 1111\cdots1111\cdots$.

PROPOSITION 3. *If* $\nu(n) = 7$ *and* $n = 110\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$ *unless* $n = 11001\cdots1111\cdots$.

PROPOSITION 4. *If* $\nu(n) = 8$ *and* $n = 110\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$ *unless*

$$n = 11\cdots d\cdots 11\cdots 11\cdots e\cdots 11\cdots \quad \textit{where } e = d \textit{ or } e = d - 1\,.$$

(Note: The d and e again stand for d and e zeros respectively between the ones.)

PROPOSITION 5. *If* $\nu(n) = 6$ *and* $n = 111\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$ *unless* $n = 111\cdots111\cdots$, $111001011\cdots$, $1111\cdots1001\cdots$, $1111\cdots101\cdots$, *or* $1111\cdots11\cdots$.

PROPOSITION 6. *If* $\nu(n) = 7$ *and* $n = 10111\cdots01\cdot\cdot01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 7. *If* $\nu(n) = 8$ *and* $n = 1011111\cdots01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 8. *If* $\nu(n) = 8$ *and*

$$\begin{aligned} n = 10111\cdots01\cdots&01\cdots0011\cdots\,,\\ &01\cdots0011\cdots01\cdots\,,\\ &0011\cdots01\cdots01\cdots\,,\end{aligned}$$

then $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 9. *If* $\nu(n) = 8$ *and*

$$\begin{aligned} n = 1011\cdots01\cdots&01\cdots00111\cdots\,,\\ &01\cdots00111\cdots01\cdots\,,\\ &00111\cdots01\cdots01\cdots\,,\end{aligned}$$

then $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 10. *If* $\nu(n) = 8$ *and* $n = 1010111\cdots01\cdots01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 11. *If* $\nu(n) = 8$ *and* $n = 1011011\cdots01\cdots01\cdots01$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 12. *If* $\nu(n) = 6$ *and* $n = 11\cdots01\cdots01\cdots01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 13. *If* $\nu(n) = 7$ *and* $n = 1011\cdots01\cdots01\cdots01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 14. *If* $\nu(n) = 8$ *and* $n = 101111\cdots01\cdots01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 15. *If* $\nu(n) = 8$ *and* $n = 101011\cdots01\cdots01\cdots01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 16. *If* $\nu(n) = 8$ *and*

$$\begin{aligned} n = 1011\cdots01\cdots01\cdots01\cdots0011\cdots&\ , \\ 01\cdots01\cdots0011\cdots01\cdots&\ , \\ 01\cdots0011\cdots01\cdots01\cdots&\ , \\ 0011\cdots01\cdots01\cdots01\cdots&\ , \end{aligned}$$

then $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 17. *If* $\nu(n) = 8$ *and* $n = 10111\cdots01\cdots01\cdots01\cdots 01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 18. *If* $\nu(n) = 8$ *and* $n = 1011\cdots01\cdots01\cdots01\cdots 01\cdots01\cdots$, *then* $l_8(n) \geqq \lambda(n) + 4$.

PROPOSITION 19. *If* $\nu(n) = 7$ *and* $n = 1011100\cdots111$, *then* $l_8(n) \geqq \lambda(n) + 4$.

Proof. (Prop. 1) Let $1 = a_0 < a_1 < \cdots < a_r = n$ be an addition chain for n where $\nu(n) = 7$ and $n = 111\cdots$. It shall be assumed that all members of the chain have eight or less ones in their binary representation. Let a_i denote the first member of the chain for which $\nu(a_i) = 7$ and $a_i = 111\cdots$. $a_i = a_j + a_k$ for some $k \leqq j < i$. In fact $k < j$ for if $a_j = a_k$ then $a_i = 2a_j$ which would mean that $\nu(a_j) = 7$ and $a_j = 111\cdots$ contradicting the fact that a_i was chosen as the first member of the chain having these properties. Thus, a_j and a_k are distinct members of the chain and $1 \leqq \nu(a_j), \nu(a_k) \leqq 8$. The 49 cases for $(\nu(a_j), \nu(a_k))$ must be considered.

$a_i \leqq 2a_j$ implies that $\lambda(a_i) = \lambda(a_j)$ or $\lambda(a_i) = \lambda(a_j) + 1$. If $\nu(a_k) \geqq 5$, it may be assumed that $\lambda(a_i) = \lambda(a_j) + 1$; otherwise, $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq N(a_k) + 1 \geqq 4$. However, if $\lambda(a_i) = \lambda(a_j) + 1$, the only way to obtain $a_i = 111\cdots$ is if a_j/a_k is as follows:

$$\begin{aligned} a_j &= \ 111\cdots \\ +a_k &= \ 11\cdots\cdot \\ a_i &= 111\cdots\cdot\ . \end{aligned}$$

The arrows indicate that at least three carries are needed with this configuration. As can be seen $\lambda(a_j) = \lambda(a_k)$, and it follows that $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq 4$ for all cases where $\nu(a_k) \geqq 5$. If $\nu(a_k) \leqq 4$ and $\nu(a_j) \geqq 5$, then the configuration still holds, and all cases where $\nu(a_j) = 7$ may be dispensed with since $a_j = 111\cdots$ again contradicts the "firstness" of a_i. The cases (8, 1), (6, 3), (6, 2), (6, 1), (5, 4), (5, 3), and (5, 2) all have less than three carries in $a_j + a_k$ by Lemma 1 while at least three carries are needed in the configuration. In case (8, 2) only two carries are possible while three are needed. In (8, 3) it may be assumed as with case (10, 8, 3) of Theorem 1 that $a_j = a_k + a_t$ (see Figure 1). $a_t = a_j - a_k$, and by Lemma 2 $\nu(a_t) \geqq 5$ which implies that $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq N(a_t) + 1 \geqq 4$. In (8, 4) it may be assumed that a_k is one of the four special types in Theorem C; otherwise, $N(a_k) \geqq 3$ which implies $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq 4$. Since $a_k = 11\cdots$, this means that $a_k = 11\cdots 11\cdots$. As in (8, 3) it may be assumed that $a_j = a_k + a_t$, and as with (8, 3) $\nu(a_t) \geqq 5$ unless there are four 1/1 slots in a_j/a_k. By Lemma 1 $c = 5$ in $a_j + a_k$, and the only way to meet all of these requirements is if a_j/a_k is as follows:

$$\begin{aligned} a_j &= \ 11111\cdots 1\cdots 1\cdots 1\cdots \\ +\ a_k &= \ 11011\cdots 0\cdots 0\cdots 0\cdots \\ a_i &= 111010\cdots 1\cdots 1\cdots 1\cdots \end{aligned} \quad \text{implies} \quad \begin{aligned} a_j &= 11111\cdots 1\cdots 1\cdots 1\cdots \\ -\ a_k &= 11011\cdots 0\cdots 0\cdots 0\cdots \\ a_t &= \quad 100\cdots 1\cdots 1\cdots 1\cdots\ . \end{aligned}$$

$\lambda(a_k) = \lambda(a_t) + 2$ while $2^2 a_t < a_k$, and so by Lemma 3 $N(a_k) \geqq N(a_t) + 1 \geqq 3$. Thus, $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq 4$. In (6, 4) $c = 3$ by Lemma 1. Therefore, a_j/a_k must be:

$$\begin{aligned} a_j &= \ 111\cdots 1\cdots 0\cdots \\ +a_k &= \ 111\cdots 0\cdots 1\cdots \\ a_i &= 1110\cdots 1\cdots 1\cdots\ . \end{aligned}$$

By Theorem C $N(a_k) \geqq 3$; hence, $N(n) \geqq N(a_j) \geqq N(a_k) + 1 \geqq 4$.

The only remaining cases to be considered are (4, 4), (4, 3), and (3,4). $\lambda(a_i) = \lambda(a_j) + 1$ is not possible since at least three carries are needed while these cases by Lemma 1 have less than two. When either $\nu(a_j) = 4$ or $\nu(a_k) = 4$, it may be assumed that a_j and a_k are what shall be called "special fours" meaning that they are one of the types in Theorem C. Otherwise, $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq 4$ since it may be assumed that $\lambda(a_i) = \lambda(a_j)$. In (4, 4) the possible configurations a_j/a_k for obtaining $a_i = 111\cdots$ with $c = 1$ are:

$$\begin{array}{llll}
(1) & \begin{aligned} a_j &= 1001\cdots \\ +a_k &= 101\cdots \\ a_i &= 1110\cdots \end{aligned} & (2) & \begin{aligned} a_j &= 101\cdots01\cdots \\ +a_k &= 10\cdots01\cdots \\ a_i &= 111\cdots10\cdots \end{aligned} \\
(3) & \begin{aligned} a_j &= 100\cdots01\cdots \\ +a_k &= 11\cdots01\cdots \\ a_i &= 111\cdots10\cdots \end{aligned} & (4) & \begin{aligned} a_j &= 11\cdots \\ +a_k &= 00\cdots \\ a_i &= 111\cdots . \end{aligned}
\end{array}$$

In (1), (2), and (3) either $a_j = a_m + a_s$ where $a_m \neq a_k$ or $a_j = a_k + a_t$. If $a_j = a_m + a_s$, then $N(a_j) \geqq 3$ by reasoning similar to that used in (9, 6, 5) of Theorem 1 (see Figure 2). Thus, $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq 4$. It shall be assumed then that $a_j = a_k + a_t$. In (1) there are two possibilities for $a_t = a_j - a_k$:

$$\begin{array}{llll}
(a) & \begin{aligned} a_j &= 1001\cdots \\ -a_k &= 101\cdots \\ a_t &= 100\cdots \end{aligned} & (b) & \begin{aligned} a_j &= 1001\cdots \\ -a_k &= 101\cdots \\ a_t &= 11\cdots . \end{aligned}
\end{array}$$

Since $c = 1$, there can be no further 1/1 slots in a_j/a_k. Thus, in (a) $\nu(a_t) \geqq 3$ by Lemma 2, and since $\lambda(a_k) = \lambda(a_t)$ and $a_t \neq a_k$, this means $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq N(a_k) + 1 \geqq N(a_t) + 2 \geqq 4$. In (b) $\nu(a_t) \geqq 5$ by Lemma 2, and, so, $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq N(a_t) + 1 \geqq 4$. (2) may be dispensed with in the same manner as (1) part (a) while in (3) since a_k is a "special four" a_j/a_k becomes:

$$\begin{aligned} a_j &= 100\cdots010\cdots \\ -a_k &= 11\cdots011\cdots \\ a_t &= \cdots\cdots111\cdots . \end{aligned}$$

By Lemma 2 $\nu(a_t) \geqq 5$; hence, $N(n) \geqq 4$ as in (1) part (b).

In (4) it may be assumed that the first two digits in a_k are ones; otherwise, $\lambda(a_j) = \lambda(a_k) + m$ for some positive integer m while $2^m a_k < a_j$. By Lemma 3 this would mean $N(a_j) \geqq N(a_k) + 1 \geqq 3$, and, hence, $N(n) \geqq 4$. Since a_j and a_k both start with two ones and are "special fours", they must both have the form $11\cdots11\cdots$, but in this event it is not possible to have $c = 1$ in $a_j + a_k$.

In (4, 3) and (3, 4) $c = 0$ which means that there are no 1/1 slots in a_j/a_k. The possibilities for a_j/a_k are the following:

$$\begin{array}{llllllll}
(1) & \begin{aligned} a_j &= 101\cdots \\ +a_k &= 10\cdots \\ a_i &= 111\cdots \end{aligned} & (2) & \begin{aligned} a_j &= 100\cdots \\ +a_k &= 11\cdots \\ a_i &= 111\cdots \end{aligned} & (3) & \begin{aligned} a_j &= 110\cdots \\ +a_k &= 1\cdots \\ a_i &= 111\cdots \end{aligned} & (4) & \begin{aligned} a_j &= 111\cdots \\ +a_k &= 000\cdots \\ a_i &= 111\cdots . \end{aligned}
\end{array}$$

In (1) $N(n) \geqq 4$ for both (4, 3) and (3, 4) by the same reasoning used in (4, 4) with configuration (1) part (a). The remaining configurations will now be discussed for (4,3).

In (2) $a_t = a_j - a_k$ and by Lemma 2 $\nu(a_t) \geqq 4$. Thus, $N(a_t) \geqq 3$ which implies $N(n) \geqq 4$ unless a_t is a "special four". Since $a_k = 11\cdots$, it may be assumed that a_t also starts with two ones by the same reasoning that was used for a_k in (4, 4) configuration (4). Thus, $a_t = 11\cdots11\cdots$. Since there can be no ones under a 0/0 slot in a_j/a_k (otherwise $\nu(a_t) \geqq 5$), there are only two possibilities for a_j/a_k:

$$\begin{array}{rrl} \text{(a)} & a_j = & 1001\cdots101\cdots \\ & -a_k = & \;110\cdots010\cdots \\ & a_t = & \;\;11\cdots011\cdots \end{array} \qquad \begin{array}{rrl} \text{(b)} & a_j = & 1000111\cdots \\ & -a_k = & \;111000\cdots \\ & a_t = & \;\;\;\;1111\cdots . \end{array}$$

In (a) $N(a_k) \geqq 3$ by arguments used before unless $a_k = a_t + a_u$ for some $u \leqq t < k$. If a_k/a_t is examined, it may be seen that $\nu(a_u) \geqq 4$, $\lambda(a_t) = \lambda(a_u)$ and $a_u \neq a_t$. Thus, $N(a_t) \geqq N(a_u) + 1 \geqq 3$ which implies $N(n) \geqq 4$. In (b) a_j is not a "special four" and, so, $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq 4$.

In (3) $a_j = 11\cdots11\cdots$ since a_j is a "special four". As in configuration (4) of (4, 4) it may be assumed that a_k starts with two ones. a_j/a_k is then:

$$\begin{aligned} a_j &= 1100\cdots11\cdots \\ +a_k &= \quad 11\cdots00\cdots \\ a_i &= 1111\cdots11\cdots . \end{aligned}$$

As can be seen $a_j > 2a_k + a_k$, and, so, by Lemma 4 $N(a_j) \geqq N(a_k) + 1 \geqq 3$ unless $a_j = 2a_{k+1}$. Since $\nu(a_{k+1}) = 4$ and $\lambda(a_{k+1}) = \lambda(a_k) + 1$, it follows as before that $N(a_{k+1}) \geqq 3$ unless $a_{k+1} = a_k + a_t$ for some $t \leqq k$. From a_j/a_k and the fact that $a_j = 2a_{k+1}$ it may be determined that a_{k+1}/a_k is as follows:

$$\begin{aligned} a_{k+1} &= 1100\cdots11\cdots \\ -a_k &= \;11\cdots00\cdots\cdot \\ a_t &= \;1\cdots\cdot\cdot1\cdots\cdot . \end{aligned}$$

By Lemma 2 $\nu(a_t) \geqq 3$. Thus, $N(n) \geqq N(a_i) \geqq N(a_j) + 1 \geqq N(a_k) + 1 \geqq N(a_t) + 2 \geqq 4$.

In (4) $a_j = 1111\cdots$ since a_j is a "special four", and since $\nu(a_k) = 3$, it follows that $\lambda(a_j) = \lambda(a_k) + m$ for some positive integer m while $2^m a_k < a_j$. By Lemma 3 $N(a_j) \geqq N(a_k) + 1 \geqq 3$ which implies $N(n) \geqq 4$. Configurations (2), (3), and (4) will now be discussed for (3, 4).

In (2) it may again be assumed that $a_j = a_k + a_t$, and $a_k = 11\cdots11\cdots$ since a_k is a "special four". By Lemma 2 $\nu(a_t) \geqq 3$, and a one can occur in a_t at most once under a 0/0 slot in a_j/a_k or else

$\nu(a_t) \geqq 5$. The possibilities for a_j/a_k are:

$$\begin{array}{llll} \text{(a)} & a_j = 10000\cdots & \text{(b)} & a_j = 100\cdots100\cdots \\ & -a_k = \ \ 1111\cdots & & -a_k = \ \ 11\cdots011\cdots \\ & a_t = \qquad 1\cdots & & a_t = \ \ \ 1\cdots001\cdots \\ \text{(c)} & a_j = 100000\cdots & \text{(d)} & a_j = 100\cdots1000\cdots \\ & -a_k = \ \ 11011\cdots & & -a_k = \ \ 11\cdots0011\cdots \\ & a_t = \quad 101\cdots & & a_t = \ \ \ 1\cdots0101\cdots . \end{array}$$

In (a) and (b) $\nu(a_t) = 3$, and no matter where the remaining ones in a_t are placed the conditions of Lemma 3 will apply. In (d) $\nu(a_t) = 4$, and, so it may be assumed that a_t is a "special four" in which case a_t must start as $a_t = 10\cdots$. Thus, the conditions of Lemma 3 also apply to (c) and (d), and in all four cases $N(a_k) \geqq N(a_t) + 1 \geqq 3$ which implies that $N(n) \geqq 4$.

In (3) it may again be assumed as in configuration (4) of (4, 4) that the first two digits of a_k are ones, and since a_k is a "special four", this means that $a_k = 11\cdots11\cdots$. As in (4, 3) configuration (3) it may also be assumed that $a_j = 2a_{k+1}$ and that $a_{k+1} = a_k + a_t$ for some $t \leqq k$. These facts together with a_j/a_k determine a_{k+1}/a_k:

$$\begin{array}{lll} a_j = 1100\cdots00\cdots & \text{implies} & a_{k+1} = 1100\cdots00\cdots \\ +a_k = \quad 11\cdots11\cdots & & -a_k = \ \ 11\cdots110\cdots \\ a_i = 1111\cdots11\cdots & & a_t = \ \ 1\cdots\cdots10\cdots . \end{array}$$

No matter where the other one in a_{k+1} is placed, it can be seen that $\nu(a_t) \geqq 3$, $\lambda(a_k) = \lambda(a_t)$ and $a_t \neq a_k$. Thus, $N(a_k) \geqq N(a_t) + 1 \geqq 3$ which implies $N(n) \geqq 4$.

In (4) a_k is a "special four", and the conditions of Lemma 3 will apply unless $a_k = 111\cdots$. $\lambda(a_j) = \lambda(a_k) + m$ for some $m \geqq 2$ while $a_j > 2^{m-1}a_k + 2^{m-2}a_k$, and, so, by Lemma 4 $N(a_j) \geqq N(a_k) + 1 \geqq 3$ unless $a_j = 2^{m-1}a_{k+1}$. As before it may be assumed that $a_{k+1} = a_k + a_t$ for some $t \leqq k$, and these facts together with a_j/a_k determine a_{k+1}/a_k:

$$\begin{array}{lll} a_j = 111\cdots0000\cdots & \text{implies} & a_{k+1} = 11100\cdots \\ +a_k = \qquad\quad 1111\cdots & & -a_k = \ \ 1111\cdots \\ a_i = 111\cdots1111\cdots & & a_t = \ \ 1101\cdots \end{array}$$

$N(a_k) \geqq N(a_t) + 1 \geqq 3$; hence, $N(n) \geqq 4$.

In all 49 cases it has been shown that $N(n) \geqq 4$, and, so, it may be concluded that if $\nu(a_i) = 7$ and $a_i = 111\cdots$, then $l_8(n) \geqq \lambda(n) + 4$.

In Proposition 2 a_i denotes the first member of the chain for which $\nu(a_i) = 8$, $a_i = 111\cdots$ but $a_i \neq 1111\cdots1111\cdots$. The proof is then carried out in the same manner as the proof of Proposition 1. The proofs of the remaining propositions are similar, and as each one is proved it may be used in the proof of the next one. Propositions 1 to 5 are extremely helpful in the proofs of the remaining propositions and in that part of the proof of Theorem 1 that remains. We shall now return to the proof of Theorem 1 to demonstrate how the propositions are used. As an example of the remaining cases (9, 7, 7) will be examined.

To recall a_i is the first member of an addition chain for n for which $\nu(a_i) \geqq 9$. $a_i = a_j + a_k$ where $\nu(a_j) \leqq 8$ and $\nu(a_k) \leqq 8$. The propositions concerning $l_8(n)$ are applicable to a_j and a_k and all other members of the chain preceding a_i. As in (9, 6, 5) it may be assumed in (9, 7, 7) that $\lambda(a_i) = \lambda(a_j) + 1$ and $\lambda(a_j) > \lambda(a_k)$. Also if $\lambda(a_j) = \lambda(a_k) + m$, it may be assumed that $a_j \geqq 2^m a_k$ or else by Lemma 3 $N(a_j) \geqq N(a_k) + 1 \geqq 4$ which implies $N(n) \geqq 4$. In (9, 7, 7) $c = 5$, and the possibilities for a_j/a_k are now listed. These possibilities are the ways to proceed from left to right to the first 1/1 slot in a_j/a_k without exceeding five carries and with the previously mentioned restrictions kept in mind.

$$\begin{array}{llll}
(1) & a_j = 11\cdots\cdot & (2)\ a_j = 101\cdots & (3)\ a_j = 111\cdots\cdots \\
 & +a_k = \ \ 11\cdots & \ \ +a_k = \ \ 11\cdots & \ \ +a_k = \ \ \ 111\cdots \\
 & a_i = 10\cdots\cdots & \ \ a_i = 100\cdots\cdot & \ \ a_i = 100\cdots\cdots\cdot \\
(4) & a_j = 1011\cdots\cdot & (5)\ a_j = 1001\cdots & (6)\ a_j = 1101\cdots \\
 & +a_k = \ \ 1011\cdots & \ \ +a_k = \ \ 111\cdots & \ \ +a_k = \ \ \ 11\cdots \\
 & a_i = 1000\cdots\cdots & \ \ a_i = 1000\cdots\cdot & \ \ a_i = 1000\cdots\cdot \\
(7) & a_j = 1111\cdots\cdots\cdot & (8)\ a_j = 10101\cdots & (9)\ a_j = 10011\cdots \\
 & +a_k = \ \ \ \ 1111\cdots & \ \ +a_k = \ \ 1011\cdots & \ \ +a_k = \ \ 1101\cdots \\
 & a_i = 1000\cdots\cdots\cdots & \ \ a_i = 100000\cdots & \ \ a_i = 100000\cdots \\
(10) & a_j = 10001\cdots & (11)\ a_j = 11001\cdots & (12)\ a_j = 111010\cdots \\
 & +a_k = \ \ 1111\cdots & \ \ +a_k = \ \ \ 111\cdots & \ \ +a_k = \ \ \ \ 111\cdots \\
 & a_i = 100000\cdots & \ \ a_i = 100000\cdots & \ \ a_i = 1000001\cdots \\
(13) & a_j = 111110000\cdots & & \\
 & +a_k = \ \ \ \ 11111\cdots & & \\
 & a_i = 1000001111\cdots\ . & &
\end{array}$$

In configurations (3), (5), (6), (7), (9), (10), (11), (12), and (13) Propositions 1 and 3 imply that either $N(a_j) \geqq 4$ or $N(a_k) \geqq 4$. In either event this means that $N(n) \geqq N(a_j) \geqq 4$. In (1) $N(n) \geqq 4$ in the same manner unless a_j and a_k both have the binary form $11001\cdots1111\cdots$, but in this event it is impossible to arrange a_j/a_k so that $c = 5$. In (2) it may be assumed that $a_k = 11001\cdots1111\cdots$ and that $a_j = a_k + a_t$ for some $t \leqq k$ (see Figure 2). Since $c = 5$ there can be at most two more 1/1 slots in a_j/a_k. There are two possibilities for a_j/a_k:

$$\text{(a)}\quad \begin{aligned} a_j &= 101\cdots\cdots\cdots 00\cdots \\ -a_k &= \ \ 11001\cdots 1111\cdots \\ a_t &= \ \ 10\cdots\cdots\cdots 01\cdots \end{aligned} \qquad \text{(b)}\quad \begin{aligned} a_j &= 101\cdots\cdots\cdots 00\cdots \\ -a_k &= \ \ 11001\cdots 1111\cdots \\ a_t &= \ \ \ \ 1\cdots\cdots\cdots 01\cdots . \end{aligned}$$

In (a) it is impossible to have two further 1/1 slots in a_j/a_k with zeros under them. Thus, $\nu(a_t) \geqq 5$ by Lemma 2, and since $\lambda(a_k) = \lambda(a_t)$ and $a_t \neq a_k$, $N(n) \geqq N(a_k) \geqq N(a_t) + 1 \geqq 4$. Configuration (b) can be filled out a little further by realizing that the 1 can occur under the 1/1 slot only if a_j/a_k is as follows:

$$\begin{aligned} a_j &= 10100\cdots\cdots 00\cdots \\ -a_k &= \ \ 11001\cdots 1111\cdots \\ a_t &= \ \ \ \ 111\cdots\cdots 01\cdots . \end{aligned}$$

It is impossible to have zeros in a_t under any further 1/1 slots in a_j/a_k, and, so, by Lemma 2 $\nu(a_t) \geqq 9$ which contradicts the fact that a_i is the first member of the chain for which $\nu(a_i) \geqq 9$. In (4) it may again be assumed that $a_j = a_k + a_t$ for some $t \leqq k$. $c = 5$ implies that there is one more 1/1 slot in a_j/a_k; hence, $\nu(a_t) \geqq 5$ by Lemma 2. It is evident that $\lambda(a_k) = \lambda(a_t)$, and if $a_t \neq a_k$, $N(n) \geqq N(a_k) \geqq N(a_t) + 1 \geqq 4$. It is possible in this case, however, that $a_t = a_k$ which means $a_j = 2a_k$. With $c = 5$ the configuration would be:

$$\begin{aligned} a_j &= \ \ 101110\cdots 10\cdots 10\cdots 10\cdots \\ +a_k &= \ \ \ \ 101110\cdots 1\cdots 01\cdots 01\cdots \\ a_i &= 1000101\cdots 11\cdots 11\cdots 11\cdots . \end{aligned}$$

By Proposition 6 $N(n) \geqq N(a_j) \geqq 4$. In (8) it is not possible that $a_j = 2a_k$, and, so, $N(n) \geqq 4$ as in (4) when $a_t \neq a_k$. This concludes the proof of (9, 7, 7).

The proof of the remaining cases is similar. Once Theorem 1 is established it follows that the propositions concerning $l_8(n)$ are true in general. That is $l(n)$ may be used in the statements of all of the propositions instead of $l_8(n)$. The reason for this is that if an integer with more than eight ones in its binary representation does occur in one of the chains then by Theorem 1 there are at least four small steps in the chain up to that integer which means that $N(n) \geqq 4$. In particular Proposition 19 may be restated to say that if $\nu(n) = 7$ and $n = 1011100\cdots 111$, then $l(n) \geqq \lambda(n) + 4$. This leads to the result that there exists an infinite class of integers for which $l(2n) = l(n)$. This is the essence of the following theorem.

THEOREM 2. *If* $n = 2^m(23) + 7$ *where* $m \geqq 5$, *then* $l(2n) = l(n) = m + 8$.

Proof. n has the binary form $n = 1011100\cdots111$, and by the restatement of Proposition 19 $l(n) \geqq \lambda(n) + 4$. On the other hand,

$$1, 2, 3, 4, 7, 14, 21, 23, 2(23), \cdots, 2^m(23), 2^m(23) + 7 = n$$

is a chain for n with only four small steps. Thus, $l(n) = \lambda(n) + 4$.

$2n = 2^{m+1}(23) + 14 = 1011100\cdots1110$. $\nu(2n) = 7$ implies that $l(2n) \geqq \lambda(2n) + 3$ while

$$1, 2, 4, 5, 9, 14, 23, 2(23), \cdots, 2^{m+1}(23), 2^{m+1}(23) + 14 = 2n$$

is a chain for $2n$ with only three small steps. Thus, $l(2n) = \lambda(2n) + 3$.

Since $\lambda(2n) = \lambda(n) + 1 = m + 5$, it follows that $l(2n) = \lambda(2n) + 3 = \lambda(n) + 4 = l(n) = m + 8$.

More details of the proofs of the Propositions and Theorem 1 are available in [12] and in private manuscripts.

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