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O-2-TRANSITIVE ORDERED PERMUTATION GROUPS

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The group of all automorphisms of a chain Ω forms a lattice-ordered group $A(\Omega)$ under the pointwise order. Let G be an l-subgroup of $A(\Omega)$ which is o-2-transitive, i.e., for any $\beta < \gamma$ and $\sigma < \tau$, there exists $g \in G$ such that $\beta g = \sigma$ and $\gamma g = \tau$. It is shown that G is a complete subgroup of $A(\Omega)$ if and only if G is completely distributive if and only if G contains an element $\neq 1$ of bounded support. There is a discussion of the pathological groups in which these conditions are absent.

1. The dichotomy among o-2-transitive groups. The group $A(\Omega)$ or order-preserving permutations (automorphisms) of a chain Ω becomes a lattice-ordered group (l-group) when ordered pointwise, i.e., $f \leqq g$ if and only if and only if $\beta f \leqq \beta g$ for all $\beta \in \Omega$. We assume throughout this paper that (G, Ω) is an l-permutation group, i.e., that G is an l-subgroup of $A(\Omega)$ (simultaneously a subgroup and a sublattice).

Let $\bar{\Omega}$ be the completion by Dedekind cuts (without end points) of Ω. Each $g \in G$ can be extended uniquely to an order-preserving permutation (o-permutation) of $\bar{\Omega}$, which will also be denoted by g. For $\bar{\omega} \in \bar{\Omega}$, let $G_{\bar{\omega}}$ be the *stabilizer subgroup* $\{g \in G \mid \bar{\omega} g = \bar{\omega}\}$. $G_{\bar{\omega}}$ is a *prime* subgroup of G (i.e., a convex l-subgroup of G such that $g_1 \wedge g_2 = 1$, with $g_1, g_2 \in G$, implies $g_1 \in G_{\bar{\omega}}$ or $g_2 \in G_{\bar{\omega}}$). If G is transitive on Ω, then of course all G_α's ($\alpha \in \Omega$) are conjugate in G.

The author showed in [5, Theorem 7] that for a transitive l-subgroup G of $A(\Omega)$, the following are equivalent:

(1) G_α is a *closed* subgroup of G for one (hence every) $\alpha \in \Omega$, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G_\alpha$, then $g \in G_\alpha$.

(2) G is a *complete* subgroup of $A(\Omega)$, i.e., if in G, $g = \bigvee_{i \in I} g_i$, then also in $A(\Omega)$, $g = \bigvee_{i \in I} g_i$.

(3) Sups *in G are pointwise*, i.e., if $g = \bigvee_{i \in I} g_i$ with each $g_i \in G$, then for each $\beta \in \Omega$, βg is the sup in Ω of $\{\beta g_i \mid i \in I\}$.

Moreover, it was shown in [5, Corollary 15] that in the presence of these conditions, we have

(4) G is a *completely distributive l-group*, i.e., $\bigwedge_{i \in I} \bigvee_{k \in K} g_{ik} = \bigvee_{f \in K^I} \bigwedge_{i \in I} g_{if(i)}$ for any collection $\{g_{ik} \mid i \in I, k \in K\}$ of G for which the indicated sups and infs exist.

The *distributive radical* $D(G)$ is the intersection of the closed prime subgroups of G [1, Theorem 3.4]. $D(G) = \{1\}$ iff G is completely distributive [1, Corollary 3.8]; and at the opposite extreme, $D(G) = G$ iff G has no closed prime subgroups $\neq G$.

The *support* Supp (k) of $k \in G$ means $\{\omega \in \Omega \mid \omega k \neq \omega\}$; it is *bounded* if there exist $\beta, \gamma \in \Omega$ such that $\beta < \mathrm{Supp}(k) < \gamma$ (i.e., $\beta < \sigma < \gamma$ for all $\sigma \in \mathrm{Supp}(k)$). It is well known (see, for example, the proof of [2, Theorem 6]) that the elements of bounded support in an o-2-transitive l-permutation group G form an l-ideal L of G which is contained in all l-ideals $\neq \{1\}$. If G contains no element $\neq 1$ of bounded support, so that $L = \{1\}$, then with an eye on the next theorem, we shall say that G is a *pathologically* o-2-transitive group.

MAIN THEOREM 1. *Suppose that (G, Ω) is an o-2-transitive l-permutation group. Then conditions* (1), (2), (3), *and* (4) *are all equivalent, and they fail if and only if G is pathological. Moreover, in the pathological case, G has no proper closed prime subgroups, so that the distributive radical $D(G) = G$.*

Proof. First, assume that G has an element $\neq 1$ of bounded support. Then since G is o-2-transitive, given any nondegenerate interval Δ of Ω, G has an element $\neq 1$ with support a subset of Δ. Now suppose $g = \bigvee_{i \in I} g_i$, with $1 < g \in G \backslash G_\alpha$ and each $g_i \in G_\alpha$. Pick $1 > h \in G$ such that $\mathrm{Supp}(h) \subseteqq (\alpha g^{-1}, \alpha)$, where the usual notation is used for intervals of Ω. Then for each $i \in I$, $g_i \leqq hg < g$ (since when $\eta \in \mathrm{Supp}(h)$, $\eta g_i < \alpha < \alpha hg$), a contradiction. Therefore, G_α is closed, and the other conditions follow.

Now assume that G lacks elements $\neq 1$ of bounded support. We can express an arbitrary $1 < g \in G$ as $\bigvee_{i \in I} g_i$ with each $g_i \in G_\alpha$, as follows: For each $\beta \notin [\alpha g^{-1}, \alpha]$, we have either $\alpha < \beta \leqq \beta g$, or else $\beta \leqq \beta g < \alpha$, so we may use o-2-transitivity to pick $g_\beta \in G_\alpha$ such that $\beta g_\beta = \beta g$. Now $g = \bigvee (g_\beta \wedge g)$, for if $g_\beta \wedge g \leqq h < g$ for each β, then $\mathrm{Supp}(h^{-1}g) \subseteqq [\alpha g^{-1}, \alpha]$, violating the hypothesis, since for $\beta \notin [\alpha g^{-1}, \alpha]$, we have $\beta g = \beta(g_\beta \wedge g) \leqq \beta h \leqq \beta g$. Since each $g_\beta \wedge g \in G_\alpha$, G_α is not closed in G.

It remains only to show that in the pathological case, G contains no proper closed prime subgroup, for then $D(G) = G$ and G is not completely distributive. Suppose P is such a subgroup. In [6, Corollary 4], it is shown that every closed convex l-subgroup of an l-permutation group (G, Ω) must be $\bigcap \{G_{\bar{\omega}} \mid \bar{\omega} \in \bar{\Delta}\}$ for some $\bar{\Delta} \subseteqq \bar{\Omega}$. (In [6], it is assumed that G is a complete subgroup of $A(\Omega)$, so that the $G_{\bar{\omega}}$'s will be closed, but no other use is made of completeness.) But in fact the $G_{\bar{\omega}}$'s are closed, for P is closed, and in any l-group, a prime subgroup containing a closed prime is itself closed [1, Lemma 3.3]. But it was shown above that no G_α, $\alpha \in \Omega$, is closed; and in view of the following lemma, the proof also applies to $G_{\bar{\omega}}$, $\bar{\omega} \in \Omega$.

LEMMA 2. *Let (G, Ω) be an o-2-transitive l-permutation group.*

Let $\bar{\omega} \in \bar{\Omega}$, *and let* $\beta, \gamma \in \Omega$ *with either* $\bar{\omega} < \beta < \gamma$, *or* $\beta < \gamma < \bar{\omega}$. *Then there exists* $g \in G_{\bar{\omega}}$ *such that* $\beta g = \gamma$.

Proof. Suppose that $\bar{\omega} < \beta < \gamma$, and pick $\alpha \in \Omega$ such that $\bar{\omega} < \alpha < \beta < \gamma$. Use o-2-transitivity to pick $k \in G$ such that $\alpha k < \bar{\omega}$ and $\beta k = \gamma$, and take g to be $k \vee 1$. The other case is similar. This concludes the proofs of the lemma and the theorem.

Incidentally, conditions (1), (2), (3), and (4) still make sense when G is any subgroup (not necessarily an l-subgroup) of $A(\Omega)$; and if G is o-2-transitive and contains an element $\neq 1$ of bounded support, these conditions hold. (The first paragraph of the proof of Theorem 1 can easily be adapted to show that sups are pointwise. From this (2) and (1) follow as in [5], and (2) implies (4).)

If $\omega g \neq \omega$, $\{\gamma \in \Omega \mid \omega g^{-n} < \gamma < \omega g^{n}$ for some integer $n\}$ is called an *interval of support* of g; and in [5], G is said to be *depressible* if for every $g \in G$ and every interval of support Δ of g, there exists $k \in G$ such that $\omega k = \omega g$ if $\omega \in \Delta$, but $\omega k = \omega$ if $\omega \notin \Delta$. Convex l-subgroups of $A(\Omega)$ are automatically depressible.

PROPOSITION 3. *Depressible o-2-transitive l-permutation groups are never pathological.*

Proof. The following lemma establishes the existence of an element having a bounded interval of support, and depressibility does the rest.

LEMMA 4. *Let* (G, Ω) *be an o-2-transitive l-permutation group. Then, for every positive integer* n, (G, Ω) *is o-n-transitive*, i.e., *if* $\beta_1 < \cdots < \beta_n$ *and* $\gamma_1 < \cdots < \gamma_n$, *there exists* $g \in G$ *such that* $\beta_i g = \gamma_i$, $i = 1, \cdots, n$.

Proof. Given $\beta_1 < \cdots < \beta_n$ and $\gamma_1 < \cdots < \gamma_n$, we may suppose by induction that there exists $h \in G$ such that $\beta_i h = \gamma_i$, $i = 1, \cdots, n-1$. If $\beta_n h \geqq \gamma_n$, we use o-2-transitivity to pick $k \in G$ such that $\beta_1 k = \gamma_{n-1}$ and $\beta_n h = \gamma_n$. Now $\beta_i(h \wedge k) = \gamma_i$, $i = 1, \cdots, n$. If $\beta_n h < \gamma_n$, a similar argument works.

2. Pathologically o-2-transitive groups. The following example of a pathological group was given by Holland in [3, p. 433]. Let Ω be the reals and let G be the l-subgroup of $A(\Omega)$ consisting of those o-permutations g of Ω for which there exists a positive integer $n = n_g$ such that $(\omega + n)g = \omega g + n$ for all $\omega \in \Omega$. Lloyd [4, p. 399] used very special properties of this example to show that G is not completely distributive (cf. Theorem 1), but is l-simple (has no proper l-ideals). Are all pathological groups l-simple? The author has been unable to settle this question, but attempts to construct additional examples of pathological groups seem to lead inevitably to some sort of periodicity sufficient to guarantee l-simplicity, as in the following modification of Holland's example.[1]

As in that example, let Ω be the reals. Now let H be the l-subgroup of $A(\Omega)$ consisting of those o-permutations h of Ω having the property that for all $\omega \in \Omega$, there exists a positive integer $n = n_{h,\omega}$ such that $(\omega + qn)h = \omega h + qn$ for all integers q. (For definiteness, let $n_{h,\omega}$ be the least positive integer having this property.) H contains the previous group G and is also pathologically o-2-transitive. H and G are not isomorphic as l-groups, for there exists $1 < z \in G$ (defined by $\omega z = \omega + 1$) such that every $g \in G$ commutes with some z^p, $p > 0$, whereas it can be shown that H contains no such element z.

Of course, still other examples can be obtained by letting Ω be the rationals (or other appropriate subgroup of the reals) and proceeding in either of the two ways already mentioned.

Lloyd's proof of the l-simplicity of the first example does not apply to the modification. However, all of the above examples can be shown to be l-simple by the following fairly similar argument, which will be phrased in terms of the modified group H. Let $\{1\} \neq L$ be an l-ideal of H. Since for every $1 < h \in H$, $\{\omega h - \omega \mid \omega \in \Omega\}$ has an upper bound, namely $0h - 0 + n_{h,0}$, we shall have $L = H$ provided there exists $\varepsilon > 0$ such that the translation $\omega \to \omega + \varepsilon$ is exceeded by some $g \in L$. Now pick $1 < k \in L$ and $\alpha \in \Omega$ such that $\alpha k > \alpha$. Let t be the translation $\omega \to \omega + (\alpha k - \alpha)$, and let p be an integer such that $p(\alpha k - \alpha) > n_{k,\alpha}$. Let $k_1 = k$, let $k_i = t^{-1}k_{i-1}t$ $(i = 2, \cdots, p)$, and let $g = k_1 \cdots k_p \in L$. The reader can verify that $(\alpha + qn_{k,\alpha})g = \alpha g + qn_{k,\alpha}$ for all integers q, so that $n_{g,\alpha} \leqq n_{k,\alpha}$. Now $\alpha + n_{g,\alpha} \leqq \alpha + n_{k,\alpha} < \alpha g$. Hence g exceeds the translation $\omega \to \omega + \varepsilon$, where $\varepsilon = \alpha g - (\alpha + n_{g,\alpha})$.

As mentioned above, it may be the case that all pathologically o-2-transitive groups are l-simple. At any rate, any proper l-ideal must itself be a pathologically o-2-transitive group, as we proceed to prove.

An *o-block* of a transitive l-permutation group (G, Ω) is a nonempty convex subset Δ of Ω having the property that for any $g \in G$, either $\Delta g = \Delta$ or Δg does not meet Δ. The *o-block system* $\tilde{\Delta}$ determined by Δ is the set of translates Δg $(g \in G)$, a partition of Ω. If Δ contains more than one point and $\Delta \neq \Omega$, $\tilde{\Delta}$ is *proper*. G is *o-primitive* if it has no proper o-block systems.

LEMMA 5. *Let (G, Ω) be a transitive l-permutation group, and let L be a proper l-ideal of G which is intransitive on Ω. Then the*

[1] Andrew Glass has recently shown that pathological groups need not be l-simple, and need not be periodic in any sense.

orbits of L form a proper o-block system of G.

Proof. The analogous statement for nonordered permutation groups is precisely Proposition 7.1 of [7]. Here the fact that L is an l-ideal forces the blocks to be convex.

THEOREM 6. *Let (G, Ω) be a (pathologically) o-2-transitive l-permutation group, and let $\{1\} \neq L$ be an l-ideal of G. Then (L, Ω) is also (pathologically) o-2-transitive.*

Proof. o-2-transitive groups are certainly o-primitive, so by the lemma, L must be transitive on Ω. Let $\alpha < \beta < \gamma$, all in Ω. Pick $1 \leqq g \in G_\alpha$ such that $\beta g = \gamma$, and pick $1 \leqq k \in L$ such that $\beta k \geqq \gamma$. Then $1 \leqq k \wedge g \leqq k \in L$, so $k \wedge g \in L_\alpha$, and $\beta(k \wedge g) = \gamma$. Hence $\{\beta \in \Omega \mid \beta > \alpha\}$ is all one orbit of G_α, from which it follows easily that L is o-2-transitive. Certainly if G contains no element $\neq 1$ of bounded support, neither does L.

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