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ON THE STRUCTURE OF FINITE RINGS II

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In this paper we develop a structure theory for modules and bimodules over complete matrix rings over Galois rings, and we use this module theory to study the additive structure of the components of a Peirce decomposition of a general finite ring.

We recall that any finite ring is the direct sum of rings of prime power characteristic. This follows from noticing that when one decomposes the additive group of a finite ring into its primary components, the components are ideals of prime power characteristic (cf. [4]). We thus restrict ourselves to considering rings of prime power characteristic without loss of generality up to direct sum formation.

We next recall the definition of a Galois ring. Let k, r be positive integers and p be a prime integer. The *Galois ring of characteristic* p^k *and order* p^{kr} is defined to be $Z[x]/(p^k, f(x))$ [8], [10] where Z denotes the rational integers and $f(x) \in Z[x]$ is monic of degree r and irreducible. A Galois ring is uniquely determined up to isomorphism by the integers p, k, and r, and we shall denote the Galois ring of characteristic p^k and order p^{kr} by $G(k, r)$. The prime p will generally be clear from context. Note that $G(1, r) \cong GF(p^r)$ and $G(k, 1) \cong Z/(p^k)$.

If R is a finite ring of characteristic p^k which contains a 1 then R contains a Galois ring $G(k, r)$ for some r which contains the 1 of R. Indeed $Z/(p^k)\cdot 1$ will always be such a ring. Therefore, any finite ring of characteristic p^k is thus a faithful left and right $G(k, r)$-module for some r.

We now seek to develop a module theory for matrix rings over Galois rings. In a sense, the theory is already developed in that a matrix ring over a Galois ring is Morita equivalent to a Galois ring and hence the categories of modules will be category isomorphic, and a module and bimodule theory already is known for modules over Galois rings [11]. However, we seek slightly more information than is given by the category isomorphism from Morita theory. In what follows Q will denote the matrix ring $M_n(G(k, r))$.

PROPOSITION 1. *Let M be a finitely generated left Q-module. Then M is a direct sum of cyclic left Q-modules.*

Proof. Every finitely generated left $G(k, r)$-module is a direct sum of cyclic left $G(k, r)$-modules by Corollary 2 to Proposition 1.1 of [11].

Therefore, by Proposition 2.7 of [9] every finitely generated left Q-module is a direct sum of cyclic Q-modules.

We have thus reduced the study of Q-modules to the study of cyclic Q-modules. Let Qx be a cyclic left Q-module. Consider the map $q \mapsto qx$ from Q to Qx. This map is clearly a Q-module homomorphism and thus has a kernel L which is a left ideal of Q. We are thus led to consider the left ideals of quasi-simple rings.

PROPOSITION 2. *For $j = 1, \cdots, n$, let e_j denote the matrix whose only nonzero entry is a 1 in the jjth position. Let L be a left ideal in Q. Then L is isomorphic to $\sum_{j=1}^{t} p^{i_j} Qe_j$ for some choice of integers $0 \leqq i_1, \cdots, i_t < k$ and some $t \leqq n$.*

Proof. The proposition boils down to showing that L is isomorphic to a sum of p^ith multiples of columns of $M_n(G(k, r))$. Let M denote the set of all top rows of matrices in L. M is then, in a natural way a left $G(k, r)$-module and is thus isomorphic to a direct sum of cyclic left $G(k, r)$-modules. Say $M \cong \sum_{j=1}^{t} G(k, r)x_j$ where the x_j's are n-tuples over $G(k, r)$. In fact they are the top rows of certain matrices in L. Note that, since M is contained in a $G(k, r)$-module which is free on n generators, we must conclude that $t \leqq n$. Let a_j be the smallest positive integer such that $p^{a_j} x_j = 0$. Note that $0 < a_j \leqq k$ for all $j = 1, \cdots, t$. Now any left ideal of $M_n(G(k, r))$ is completely determined by its set of top rows, because to multiply on the left by elements of $M_n(G(k, r))$ is to perform operations on the rows of matrices in L. Thus it follows that $L \cong \sum_{j=1}^{t} p^{k-a_j} Qe_j$, since the set of top rows of the ideal on the right is isomorphic to the set of top rows of L.

PROPOSITION 3. *Any finitely generated left Q-module is isomorphic to a direct sum of p^jth multiples of columns of $M_n(G(k, r))$. Moreover, any finitely generated indecomposable left Q-module is isomorphic to a p^jth multiple of a column of $M_n(G(k, r))$.*

Proof. From Proposition 1 it suffices to prove the result for cyclic left modules. As noted above a cyclic module is isomorphic to Q/L for some left ideal L. Apply Proposition 2 and let $L \cong \sum_{i=1}^{n} p^{j_i} Qe_i$ where e_i is the element of Q corresponding to the matrix in $M_n(G(k, r))$ which has a 1 in the iith position and 0's elsewhere. Now $0 \leqq j_1, \cdots, j_n \leqq k$ so define $M = \sum_{i=1}^{n} p^{k-j_i} Qe_i$. It is easy to see that $M \cong Q/L$, and Qe_i is isomorphic to a column in $M_n(G(k, r))$.

To see that any finitely generated indecomposable left module is isomorphic to a p^jth multiple of a column of $M_n(G(k, r))$, let M be a finitely generated indecomposable left Q-module. Then being finitely generated it is the sum of a finite number of modules isomorphic to p^jth multiples of columns of $M_n(G(k, r))$. But clearly any column of $M_n(G(k, r))$ is indecomposable. Therefore, applying the Krull-Schmidt theorem we conclude that the decomposition of M as a sum of p^jth multiples of columns of $M_n(G(k, r))$ consists of one p^jth column of $M_n(G(k, r))$ and we are done.

We next turn our attention to bimodules over matrix rings over Galois rings. Let Q_1, Q_2 be two such rings. If M is a (Q_1, Q_2)-module then it is a left $Q_1 \otimes_Z Q_2^{op}$-module where Q_2^{op} is a ring which has the same additive group as Q_2 but in which multiplication is defined by $a \cdot b = ba$, the product on the right being taken in Q_2. But Q_2 is a matrix ring over a commutative ring and matrix rings over commutative rings are anti-isomorphic to themselves via the transpose map.

We now consider the tensor product of matrix rings over Galois rings.

PROPOSITION 4. *Let* $Q_1 = M_{n_1}(G(k_1, r_1))$, $Q_2 = M_{n_2}(G(k_2, r_2))$. *Let* $d = \gcd\{r_1, r_2\}$, $k = \min\{k_1, k_2\}$, $m = \operatorname{lcm}\{r_1, r_2\}$.
Then

$$Q_1 \otimes_Z Q_2 \cong \sum_1^d{}^{\cdot}\, M_{n_1 n_2}(G(k, m)) .$$

Proof. In order to prune the hanging gardens of subscripts in what follows we shall denote $Z/(p^k)$ by K. We first note that

$$M_{n_1}(G(k_1, r_1)) \otimes_Z M_{n_2}(G(k_2, r_2)) \cong M_{n_1}(G(k, r_1)) \otimes_K M_{n_2}(G(k, r_2)) .$$

Thus

$$\begin{aligned} Q_1 \otimes_Z Q_2 &\cong M_{n_1}(G(k, r_1)) \otimes_K M_{n_2}(G(k, r_2)) \\ &\cong M_{n_1}(K) \otimes_K G(k, r_1) \otimes_K G(k, r_2) \otimes_K M_{n_2}(K) \\ &\cong M_{n_1}(K) \otimes_K \sum_1^d{}^{\cdot}\, G(k, m) \otimes_K M_{n_2}(K) \end{aligned}$$

(by Proposition 1.2 of [11])

$$\begin{aligned} &\cong \sum_1^d{}^{\cdot}\, (G(k, m) \otimes_K M_{n_1}(K) \otimes_K M_{n_2}(K)) \\ &\cong \sum_1^d{}^{\cdot}\, (G(k, m) \otimes_K M_{n_1 n_2}(K)) \\ &\cong \sum_1^d{}^{\cdot}\, M_{n_1 n_2}(G(k, m)) . \end{aligned}$$

We are now able to obtain a description of (Q_1, Q_2)-modules where Q_1 and Q_2 are matrix rings over Galois rings.

PROPOSITION 5. *Let* $Q_1 = M_{n_1}(G(k_1, r_1))$, $Q_2 = M_{n_2}(G(k_2, r_2))$ *and* $k = \min\{k_1, k_2\}$, $m = \operatorname{lcm}\{r_1, r_2\}$. M *is a* (Q_1, Q_2)*-module. Then* M *is of the form*

$$M \cong \sum_{i=1}^{n} p^{j_i} M_{n_1,n_2}(G(k, m))$$

where $M_{n_1,n_2}(G(k, m))$ *denotes the set of* $n_1 \times n_2$ *matrices over* $(G(k, m))$.

Proof. It is instructive to first ask how $M_{n_1}(G(k_1, r_1))$ acts as a ring of left operators on $M_{n_1,n_2}(G(k, m))$ and how $M_{n_2}(G(k_2, r_2))$ acts as a ring of right operators on $M_{n_1,n_2}(G(k, m))$. Well, since

$$G(k_1, r_1) \otimes_Z G(k_2, r_2) \cong \sum_{1}^{d}{}^{\cdot}\, G(k, m)$$

where $d = \gcd\{r_1, r_2\}$ by Proposition 1.2 of [11] it follows that $G(k_1, r_1)$ acts as a ring of left operators and that $G(k_2, r_2)$ acts as a ring of right operators on $G(k, m)$. We can thus impose a (Q_1, Q_2)-module structure on $M_{n_1,n_2}(G(k, m))$ by defining

$$[a_{ij}][b_{ij}] = \left[\sum_{q=1}^{n_1} a_{iq} b_{qj}\right]$$

if $[a_{ij}] \in M_{n_1}(G(k_1, r_1))$, $[b_{ij}] \in M_{n_1,n_2}(G(k, m))$ and

$$[b_{ij}][c_{ij}] = \left[\sum_{q=1}^{n_2} b_{iq} c_{qj}\right]$$

if $[b_{ij}] \in M_{n_1,n_2}(G(k, m))$ and $[c_{ij}] \in M_{n_2}(G(k_2, r_2))$.

Now let M be a (Q_1, Q_2)-module. Then M can be considered as a $Q_1 \otimes_Z Q_2^{op}$-module and as $Q_2^{op} \cong Q_2$ it can be considered as a left $Q_1 \otimes_Z Q_2$-module. Let $e_1, \cdots, e_d$ be a complete set of orthogonal primitive central idempotents for $\sum^{\cdot d}_{1} M_{n_1 n_2}(G(k, m))$. Then since $e_1 + \cdots + e_d = 1$, $M = 1M = (e_1 + \cdots + e_d)M = e_1 M + \cdots + e_d M$ and this sum is direct since the e_i are orthogonal idempotents. Moreover, each $e_i M$ is a left $M_{n_1 n_2}(G(k, m))$-module. We then conclude that M is isomorphic to a direct sum of p^jth multiples of columns of the component matrices $\sum^{\cdot d}_{1} M_{n_1 n_2}(G(k, m))$.

It thus suffices to show that a column of a component matrix in $\sum^{\cdot d}_{1} M_{n_1 n_2}(G(k, m)) = Q_1 \otimes_Z Q_2^{op}$ is isomorphic to $M_{n_1,n_2}(G(k, m))$ as a (Q_1, Q_2)-module. We first note the isomorphism from $M_{n_1}(G(k_1, r_1)) \otimes_Z M_{n_2}(G(k_2, r_2))$ into

$$\sum_{1}^{d}{}^{\cdot}\, M_{n_1 n_2}(G(k, m)) = M_{n_1 n_2}\Big(\sum_{1}^{d}{}^{\cdot}\, G(k, m)\Big) = M_{n_1 n_2}(G(k_1, r_1) \otimes_Z (G(k_2, r_2))$$

is defined by $[a_{ij}] \otimes [b_{pq}] \rightarrow$

$$\begin{pmatrix}
a_{11}\otimes b_{11} & \cdots & a_{11}\otimes b_{n_2 1} & & a_{1n_1}\otimes b_{11} & \cdots & a_{1n_1}\otimes b_{n_2 1} \\
\vdots & & \vdots & \cdots & \vdots & & \\
a_{11}\otimes b_{1n_2} & \cdots & a_{11}\otimes b_{n_2 n_2} & \cdots & a_{1n_1}\otimes b_{1n_2} & \cdots & a_{1n_1}\otimes b_{n_2 n_2} \\
& \vdots & & & & \vdots & \\
a_{n_1 1}\otimes b_{11} & \cdots & a_{n_1 1}\otimes b_{n_2 1} & & a_{n_1 n_1}\otimes b_{11} & \cdots & a_{n_1 n_1}\otimes b_{n_2 1} \\
\vdots & & \vdots & \cdots & \vdots & & \\
a_{n_1 1}\otimes b_{1n_1} & \cdots & a_{n_1 1}\otimes b_{n_2 n_2} & & a_{n_1 n_1}\otimes b_{1n_2} & \cdots & a_{n_1 n_1}\otimes b_{n_2 n_2}
\end{pmatrix}.$$

One can check by straightforward computation that a column in this matrix ring, i.e., something of the form

$$\begin{pmatrix}
a_{1j}\otimes b_{p1} \\ \vdots \\ a_{1j}\otimes b_{pn_2} \\ \vdots \\ a_{n_1 j}\otimes b_{p1} \\ \vdots \\ a_{n_1 j}\otimes b_{pn_2}
\end{pmatrix}$$

is isomorphic to the matrix whose iqth entry is $a_{ij}\otimes b_{pq}$ as a (Q_1, Q_2)-module. But

$$A = \left\{ \begin{pmatrix}
a_{1j}\otimes b_{p1} \\ \vdots \\ a_{1j}\otimes b_{pn_2} \\ \vdots \\ a_{n_1 j}\otimes b_{p1} \\ \vdots \\ a_{n_1 j}\otimes b_{pn_2}
\end{pmatrix} \,\middle|\, a_{\cdot j}\in G(k_1, r_1),\ b_{pq}\in G(k_2, r_2) \right\}$$

has a decomposition as a sum of indecomposable modules as a direct sum of d columns of $M_{n_1,n_2}(G(k, m))$ and the (Q_1, Q_2)-module $M_{n_1,n_2}(G(k_1, r_1)) \otimes_Z (G(k_2, r_2))$ which is isomorphic to A has a decomposition as a sum of indecomposable (Q_1, Q_2)-modules as a direct sum of d copies of $M_{n_1,n_2}(G(k, r))$. The Krull-Schmidt theorem tells us then that a column of a component matrix in $\sum\cdot_1^d M_{n_1 n_2}(G(k, m))$ is isomorphic as a (Q_1, Q_2)-module to $M_{n_1,n_2}(G(k, m))$.

We now apply these results to the study of the additive structure of an arbitrary finite ring. But in order to do this we must first obtain the existence of a subring of our ring, which is a direct sum of matrix rings over Galois rings and which contains all of the idempotents. The existence of such a subring and its uniqueness up to inner automorphism follows directly from Theorem 33 of [1] which was viewed by Azumaya as a generalization of the Wedderburn-Malcev theorem [3; § 72.19]. In addition Clark [2] recently proved the existence of such a subring of a finite ring using elementary methods. However, in the case of a finite ring more can be said about this subring than existence and uniqueness up to inner automorphism. Specifically we have:

PROPOSITION 6. *Let R be a finite ring with 1 of characteristic p^k and radical J. Then R contains a subring Q isomorphic to a direct sum of matrix rings over Galois rings such that $Q/pQ \cong R/J$ and a (Q, Q)-submodule M of J such that $R = Q + M$ with $Q \cap M = \{0\}$.*

REMARK. Once we have the existence of Q it is immediate that Q is a direct summand of R when R is considered either as a left or a right Q-module because Q is quasi-Frobenius. However, it does not seem immediately obvious that a complementary left Q direct summand will be a right Q-module or that any complementary module can be chosen to be contained in J.

Proof of Proposition 6. Suppose $R/J \cong \sum^{\cdot}{}_{i=1}^{m} M_{n_i}(GF(p^{r_i}))$ and let $\overline{e_i}$ be the multiplicative identity of the simple component of R/J isomorphic to $M_{n_i}(GF(p^{r_i}))$. Then $\overline{e_1}, \cdots, \overline{e_m}$ is a finite set of orthogonal idempotents in R/J. Let $e_1, \cdots, e_m$ be orthogonal idempotents of R such that $e_i + J = \overline{e_i}$, and such that $e_1 + \cdots + e_m = 1$ (Proposition 5 on p. 54 of [7]). Consider the Peirce decomposition of R with respect to this set of orthogonal idempotents.

$$R = \sum_{i=1}^{m} e_i R e_i + \sum_{i \neq j} e_i R e_j .$$

As is easy to check each $e_i R e_j$ is a left $\sum^{\cdot}{}_{i=1}^{m} e_i R e_i$-module and a right $\sum_{i=1}^{m} e_i R e_i$-module so this is a $(\sum_{i=1}^{m} e_i R e_i, \sum_{i=1}^{m} e_i R e_i)$-module direct sum decomposition of R. Now, as in the proof of Theorem 2 on p. 56 of [7] the $e_i R e_i$ are primary rings which annihilate each other in pairs and for all $i \neq j$ $e_i R e_j \subset J$. Since each $e_i R e_i$ is primary, again using Theorem 1 of p. 56 of [7] we have that each $e_i R e_i$ is isomorphic to a complete matrix ring over a completely primary ring C_i, $e_i R e_i / e_i J e_i \cong M_{n_i}(GF(p^{r_i}))$ so by lifting idempotents again we conclude that $e_i R e_i \cong M_{n_i}(C_i)$. Let J_i be the radical of C_i with $C_i/J_i \cong GF(p^{r_i})$, and the characteristic of C_i be p^{k_i}. Then by Theorem 8 of [10] we have that C_i contains a subring isomorphic to $G(k_i, r_i)$, we define $Q = \sum^{\cdot}{}_{i=1}^{m} M_{n_i}(G(k_i, r_i))$. Now by Proposition 2.2 of [11] each C_i contains a $(G(k_i, r_i)), (G(k_i, r_i))$-submodule N_i with $N_i \subset J_i$ such that $C_i = G(k_i, r_i) + N_i$. Thus $e_i R e_i = M_{n_i}(C_i) = M_{n_i}(G(k_i, r_i)) + M_{n_i}(N_i)$ with

$M_{n_i}(N_i)$ a $(M_{n_i}(G(k_i, r_i)))$, $M_{n_i}(G(k_i, r_i))$-submodule of $M_{n_i}(C_i)$. Since the e_iRe_i annihilate each other in pairs we conclude that each $M_{n_i}(N_i)$ is a (Q, Q)-submodule. Now each e_iRe_j is a $(\sum_{i=1}^m e_iRe_i, \sum_{i=1}^m e_iRe_i)$-submodule so it is *a fortiori* a (Q, Q)-submodule, and we have the following (Q, Q)-module direct sum decomposition.

$$R = Q \dotplus \sum_{i=1}^{m}{}^{\cdot}\, M_{n_i}(N_i) \dotplus \sum_{i\neq i}{}^{\cdot}\, e_iRe_j$$

with $M = \sum^{\cdot m}_{i=1} M_{n_i}(N_i) \dotplus \sum_{i\neq j} e_iRe_j \subset J$. Moreover, $Q = \sum_{i=1}^m M_{n_i}(G(k_i, r_i))$ and $pQ = \sum_{i=1} M_{n_i}(pG(k_i, r_i))$. Hence

$$\begin{aligned} Q/pQ &= \sum_{i=1}^{m} M_{n_i}(G(k_i, r_i)) \,/\, \sum_{i=1}^{m} M_{n_i}(pG(k_i, r_i)) \\ &\cong \sum_{i=1}^{m} M_{n_i}\,(G(k_i, r_i)/pG(k_i, r_i)) \\ &\cong \sum_{i=1}^{m} M_{n_i}(GF(p^{r_i}) \cong R/J\,. \end{aligned}$$

In the classical Wedderburn-Malcev theorem we have $R = S + J$ where S is semi-simple and $S \cap J = \{0\}$. The question arises: in the decomposition we obtained, $R = Q + M$ can we take $M = J$? Well $M \subset J$ so surely $R = Q + J$. However, one can see that $Q \cap J = pQ$ and so if $Q \cap J = (0)$ then the characteristic of Q, hence of R is p, since Q contains the multiplicative identity of R. So we ask instead, can we assume that M is an ideal of R, or at least a subring? First we note that since $R = Q + M$ and M is a (Q, Q)-submodule of R, that M will be a two-sided ideal of R if and only if it is a subring of R. If the characteristic of R is p then R is an algebra over the field $Z/(p)$, and since any finite extension of a finite field is a separable extension, the hypotheses of the classical Wedderburn-Malcev theorem are satisfied and the answer is yes. However, in general the answer is no, as is shown by the following counterexample. Let

$$R = \left\{\begin{bmatrix} a & b \\ 2c & d \end{bmatrix} \in M_2(Z/(4)) \mid a, b, c, d \in Z/(4)\right\}.$$

One can check that R is a completely primary finite ring with radical

$$J = \left\{\begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix} \in M_2(Z/(4)) \mid a, b, c, d \in Z/(4)\right\}.$$

In this ring we can take

$$Q = \left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in M_2(Z/(4)) \mid a \in Z/(4)\right\}$$

and for all invertible $x \in R$ $x^{-1}Qx = Q$. So M is a direct complement of Q, $M \subset J$ so every element of M is of the form

$$\begin{bmatrix} 2a & b \\ 2c & 2d \end{bmatrix}.$$

Now

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \in R$$

so there must be some element of M of the form

$$\begin{bmatrix} 2a & 1 \\ 2 & 2a \end{bmatrix}$$

for some $a \in Z/(4)$. But then

$$\begin{bmatrix} 2a & 1 \\ 2 & 2a \end{bmatrix}^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \in Q$$

and we conclude that M is not a subring.

Finally, we conclude with remarks on the additive structure of general finite rings of characteristic p^k. Let R be a finite ring of characteristic p^k.

Let J be the radical of R. Lift the multiplicative identity of R/J to an idempotent $e \in R$. (If $R = J$ then we take $e = 0$.) Then eRe is a finite ring of characteristic p^j for some $j \leqq k$ and with a multiplicative identity. So we apply Proposition 6 to eRe and obtain a subring Q and a (Q, Q)-submodule $N \subset eJe$ which satisfy the properties of Proposition 6. We let $M_1 = \{ea - eae \in R \mid a \in R\}$, $M_2 = \{ae - eae \in R \mid a \in R\}$ and $R_0 = \{a - ae - ea + eae \in R \mid a \in R\}$. Then $R = eRe \dotplus M_1 \dotplus M_2 \dotplus R_0$ is an (eRe, eRe)-module direct sum decomposition hence *a fortiori* a (Q, Q)-module decomposition of R. We consider R_0 as a $(Z/(p^k), Z/(p^k))$-module, M_1 as a right $Z/(p^k)$-module and M_2 as a left $Z/(p^k)$-module via the module structures they inherit as additive subgroups of a ring of characteristic p^k. We then let $\bar{Q} = Q \dotplus Z/(p^k)$ and define a $(\bar{Q}, \bar{Q})$-module structure on R by $(q, z)(r_1 + m_1 + m_2 + r_0) = qr_1 + qm_1 + zm_2 + zr_0$ and $(r_1 + m_1 + m_2 + r_0)(q, z) = r_1q + m_1z + m_2q + r_0z$ where $q \in Q$, $z \in Z/(p^k)$, $r_1 \in eRe$, $m_1 \in M_1$, $m_2 \in M_2$, and $r_0 \in R_0$. Then the decomposition $R = eRe + M_1 + M_2 + R_0$ is a $(\bar{Q}, \bar{Q})$-module direct sum decomposition, M_1 is a $(Q, Z/(p^k))$-module M_2 is a $(Z/(p^k), Q)$-module and R_0 is a nilpotent subring which is also a $(Z/(p^k), Z/(p^k))$-submodule. If $eRe = Q + N$ is the decomposition given by Proposition 6 then if we define $\bar{N} = N + M_1 + M_2 + R_0$ then $R = Q + \bar{N}$ is a $(\bar{Q}, \bar{Q})$-module direct sum decomposition of R into a quasi-semi-simple ring and a $(\bar{Q}, \bar{Q})$-submodule of J.

We thus have a Peirce decomposition of a general finite ring of characteristic p^k

$$R = \sum_{i=1}^{m} e_i R e_i + \sum_{i\neq j}^{m} e_i R e_j + \sum_{i=1}^{m} (1-e) R e_i + \sum_{i=1}^{m} e_i R(1-e) + (1-e)R(1-e)\ ,$$

where $e_1, \cdots, e_m$ are a complete set of orthogonal idempotents which are central modulo the radical, and where $e_1 + \cdots + e_m = e$. The $e_i R e_i$ are matrix rings over completely primary finite rings and completely primary finite rings were studied in § 3 of [11]. If the completely primary finite rings of $e_i R e_i$ is C_i and its radical is J_i with $C_i/J_i \cong GF(p^{r_i})$ and the characteristic of C_i is p^{k_i} and $e_i R e_i \cong M_{n_i}(C_i)$, then $e_i R e_j$ is a $(M_{n_i}(G)(k_i, r_i))$, $M_{n_j}(G(k_j, r_j))$-module, and the structure of such modules was studied in Propositions 1–5. Each $(1-e)Re_i$ is a right $M_{n_i}(G(k_i, r_i))$-module and each $e_i R(1-e)$ is a left $M_{n_i}(G(k_i, r_i))$-module and a structure theory for such modules was also developed in Propositions 1–5. Finally $(1-e)R(1-e)$ is a nilpotent finite ring and nilpotent finite rings were also studied in § 3 of [11].

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