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AXIOMS FOR THE ČECH COHOMOLOGY OF PARACOMPACTA

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The Čech cohomology of paracompact Hausdorff spaces is characterized (up to isomorphism) by supplementing the Eilenberg-Steenrod axioms for cohomology.

1. The theorem. Let $\boldsymbol{C}$ be an admissible category for homology theory, as defined by Eilenberg-Steenrod [2, p. 5]. We say H is a *cohomology theory* on $\boldsymbol{C}$ if H satisfies the Eilenberg-Steenrod axioms [2, p. 14] and (1.1) and (1.2) below.

(1.1) Additivity axiom. If a space X is the union of a collection $\mathscr{U}$ of pairwise disjoint open sets and for each U in $\mathscr{U}$ the inclusion map $j_U: U \subset X$ is in $\boldsymbol{C}$, then $\{j_U^*: H^q(X) \to H^q(U) \mid U \in \mathscr{U}\}$ is a representation of $H^q(X)$ as a direct product.

(1.2) Nonnegativity axiom. $H^q(X, A) = 0$ if $(X, A) \in \boldsymbol{C}$ and $q < 0$.

We say a cohomology theory H is *point reductive* if (1.3) holds. The Čech and Alexander-Spanier cohomology theories are examples of point reductive cohomology theories.

(1.3) If $X \in \boldsymbol{C}$, if S is a singleton (one-point) subset of X, if $h \in H^q(X)$ and if $h \mid S = 0$, then there is a neighborhood N of S such that the inclusion map $N \subset X$ is in $\boldsymbol{C}$ and $h \mid N = 0$.

A *homomorphism* $t: H \to J$ of cohomology theories H and J on $\boldsymbol{C}$ is a natural transformation from H to J that commutes with coboundary homomorphisms. A pair (X, A) is a *paracompact pair* if X is a paracompact Hausdorff space and A is a closed subset of X. The category $\boldsymbol{P}$ of paracompact pairs and all maps among them is an admissible category for homology theory. We shall prove—

THEOREM 1. *Suppose H and L are cohomology theories on $\boldsymbol{P}$, S is a singleton, $k^0(S): H^0(S) \to L^0(S)$ is a homomorphism and H is point reductive. There is a unique extension of $k^0(S)$ to a homomorphism $k: H \to L$ of cohomology theories. If L is point reductive and $k^0(S)$ is an isomorphism, then k is an isomorphism.*

For related theorems see [1], [2, p. 287, Theorem 12.1], and [6].

2. *The proof*. As an immediate consequence of a theorem due to Lawson [3] we have the following.

LEMMA 2.1. *Suppose J and H are point reductive cohomology theories on $\boldsymbol{P}$ and $m: J \to H$ is a homomorphism of cohomology theories such that $m^0(S): J^0(S) \to H^0(S)$ is an isomorphism for some singleton S. Then m is an isomorphism of cohomology theories.*

A *polyhedron* is the union of the simplexes of a geometric simplicial complex with the metric topology [2, p. 75]. A polyhedron and its underlying simplicial complex will be denoted by the same symbol. If L is a subcomplex of a simplicial complex K, (K, L) is a *polyhedral pair*. The category $\boldsymbol{K}$ of polyhedral pairs and all maps among them is an admissible category for homology theory. It is a subcategory of $\boldsymbol{P}$.

If H is a cohomology theory on $\boldsymbol{K}$, the Čech method may be applied to H to define a cohomology theory J on $\boldsymbol{P}$. J is called the *Čech extension* of H. We briefly recall the method (see [4]). Let (X, A) be a paracompact pair and let $\Lambda(X)$ be the collection of all locally finite open covers of X. If $\alpha \in \Lambda(X)$, let (X_α, A_α) be the polyhedral pair determined by the nerve of α. If $\beta \in \Lambda(X)$ and β refines α, there is a simplicial map $r_{\alpha\beta}: (X_\beta, A_\beta) \to (X_\alpha, A_\alpha)$ that maps each vertex V of X_β to a vertex U of X_α such that $V \subset U$. If in addition $\gamma \in \Lambda(X)$ and γ refines β, $r_{\alpha\beta} r_{\beta\gamma}$ and $r_{\alpha\gamma}$ are homotopic, which implies that $r^*_{\beta\gamma} r^*_{\alpha\beta} = r^*_{\alpha\gamma}$. Hence there is a direct system of groups$\{ H^q(X_\alpha, A_\alpha) \mid \alpha \in \Lambda(X)\}$ and homomorphisms $\{r^*_{\alpha\beta} \mid \beta$ refines $\alpha\}$, whose direct limit we shall denote by $J^q(X, A)$. The coboundary homomorphism $\delta: J^q(A) \to J^{q+1}(X, A)$ for a paracompact pair (X, A) and the homomorphism $J^q(f)$: $J^q(Y, B) \to J^q(X, A)$ induced by a map $f: (X, A) \to (Y, B)$ in $\boldsymbol{P}$ are suitable limit homomorphisms.

LEMMA 2.2. *If H is a cohomology theory on $\boldsymbol{K}$, then the Čech extension of H is a point reductive cohomology theory on $\boldsymbol{P}$.*

The proof is left to the reader.

Suppose H is a cohomology theory on $\boldsymbol{P}$, L is the restriction of H to $\boldsymbol{K}$ and J is the Čech extension of L. We shall construct a homomorphism $m: J \to H$ called the *canonical homomorphism*.

Let (X, A) be a paracompact pair and let $\Lambda(X)$ be the collection of locally finite open covers of X. If $\alpha \in \Lambda(X)$, there is a map $r_\alpha: (X, A) \to (X_\alpha, A_\alpha)$ defined by a partition of unity subordinate to α [5, p. 833, Proposition 2]. Any two choices of r_α are homotopic. If $\beta \in \Lambda(X)$ and β refines α, then $r_{\alpha\beta} r_\beta$ and r_α are homotopic, which implies that $r^*_\beta r^*_{\alpha\beta} = r^*_\alpha$. Hence the homomorphisms $\{H^q(r_\alpha) \mid \alpha \in \Lambda(X)\}$ induce a homomorphism $m^q(X, A)\colon J^q(X, A) \to H^q(X, A)$. It follows from the way m is defined that m is a homomorphism from J to H.

If S is a singleton, $m^0(S)\colon J^0(S) \to H^0(S)$ is an isomorphism because $\{r_\alpha \mid \alpha \in \Lambda(S)\}$ consists of just one map, a homeomorphism. Hence by Lemmas 2.1 and 2.2 we have—

LEMMA 2.3. *If H is a point reductive cohomology theory on $\boldsymbol{P}$ and J is the Čech extension of the restriction of H to $\boldsymbol{K}$, then the canonical homomorphism $m\colon J \to H$ is an isomorphism.*

Lemma 2.3 implies Lemma 2.4, which in turn implies Lemma 2.5.

LEMMA 2.4. *If H is a point reductive cohomology theory on $\boldsymbol{P}$ and $(X, A) \in \boldsymbol{P}$, then $(H^q(X, A), \{H^q(r_\alpha) \mid \alpha \in \Lambda(X)\})$ is a direct limit of the direct system $(\{H^q(X_\alpha, A_\alpha)\}, \{H^q(r_{\alpha\beta})\})$.*

LEMMA 2.5. *Suppose H and L are cohomology theories on $\boldsymbol{P}$ and $t\colon H|\boldsymbol{K} \to L|\boldsymbol{K}$ is a homomorphism of cohomology theories. If H is point reductive, there is a unique extension of t to a homomorphism $k\colon H \to L$.*

Lemma 2.6 was essentially proved by Milnor in [6], although the uniqueness part of it was not stated there.

LEMMA 2.6. *Suppose H and J are cohomology theories on $\boldsymbol{K}$. If S is a singleton and $t^0(S)\colon H^0(S) \to J^0(S)$ is a homomorphism, there is a unique extension of $t^0(S)$ to a homomorphism $t\colon H \to J$ of cohomology theories on $\boldsymbol{K}$.*

Theorem 1 follows from Lemmas 2.5 and 2.6.

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