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ON THE MAXIMUM AND MINIMUM OF PARTIAL SUMS OF RANDOM VARIABLES

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Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variable with $S_k = X_1 + X_2 + \cdots + X_k$ and $S_k^+ = \max[0, S_k]$. We shall derive formulas for the computation of $E\left[\min_{1\leq k\leq n} S_k^+\right]$, $E\left[\max_{1\leq k\leq n} S_k\right]$, and $E\left[\min_{1\leq k\leq n} S_k\right]$. The formulas are then applied to the case of standard normal random variables.

2. Notation, definitions, and preliminary lemmas. Let $x = (x_1, \cdots, x_n)$ be a vector with real components and $-x = (-x_1, \cdots, -x_n)$. In some instances we shall also assume that the components x_i, $i = 1, 2, \cdots, n$ are rationally independent. (For rational r_i, $r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0$ if and only if each $r_i = 0$.) Let $x_{k+n} = x_k$, and $x(k) = (x_k, x_{k+1}, \cdots, x_{k+n-1})$, $k = 1, 2, \cdots, n$. Let $s_k = x_1 + x_2 + \cdots + x_k$. Call the polygon connecting the points $(0, 0), (1, s_1), \cdots, (k, s_k), \cdots, (n, s_n)$ the sum polygon of the vector x, and the line connecting $(0, 0)$ with (n, s_n) the chord of the sum polygon. The sum polygon for the cyclically permuted vector $x(k)$ is defined the same way.

F. Spitzer proved in [2] (see especially page 325, lines 8-12) the following lemma.

LEMMA 1. *Let $x = (x_1, \cdots, x_n)$ be a vector such that the components x_i, $i = 1, \cdots, n$, are rationally independent. Consider the sum polygons of the n cyclic permutations of x and prescribe an integer r between 0 and $n - 1$. The sum polygon of exactly one of the cyclic permutations of x has the property that exactly r of its vertices lie strictly above its chord.*

We adopt the following notations and definitions. For any real a,

$$a^+ = \max[0, a]\,, \qquad a^- = \min[0, a] \tag{2.1}$$

σ is the permutation on n symbols, so that

$$\sigma x = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix} x = (x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n})\,. \tag{2.2}$$

We write

$$
\begin{aligned}
s_k(\sigma x) &= x_{\sigma_1} + x_{\sigma_2} \cdots + x_{\sigma_k} \\
S(\sigma x) &= \max_{1 \leq k \leq n} s_k^+(\sigma x) = \max_{1 \leq k \leq n} \left(\sum_{i=1}^{k} x_{\sigma_i} \right)^+ \\
S^*(\sigma x) &= \min_{1 \leq k \leq n} s_k^+(\sigma x) = \min_{1 \leq k \leq n} \left(\sum_{i=1}^{k} x_{\sigma_i} \right)^+ .
\end{aligned} \tag{2.3}
$$

τ is the permutation on n symbols represented as a product of cycles, including the one-cycles, and with each index contained in exactly one cycle. For example, with $n = 7$, we could have

$$\tau = (14)(2)(3756) . \tag{2.4}$$

For each x and τ we define $T(\tau x)$ as follows. Suppose τ is the example given (2.4) above, then

$$T(\tau x) = (x_1 + x_4)^+ + x_2^+ + (x_3 + x_7 + x_5 + x_6)^+ . \tag{2.5}$$

In formal notations we write

$$\tau = (\alpha_1(\tau))(\alpha_2(\tau)) \cdots (\alpha_{n(\tau)}(\tau)) \tag{2.6}$$

where the $\alpha_i(\tau)$, $i = 1, 2, \cdots, n(\tau)$, are disjoint sets of integers whose union is the set $[1, 2, \cdots, n]$. Then $T(\tau x)$ is defined as

$$T(\tau x) = \sum_{i=1}^{n(\tau)} \left(\sum_{k \in \alpha_i(\tau)} x_k \right)^+ . \tag{2.7}$$

If $X_1, X_2, \cdots, X_n$ are random variables and we write $X = (X_1, \cdots, X_n)$ then it is understood that

$$
\begin{aligned}
S_k &= X_1 + X_2 + \cdots + X_k \\
S_k^+ &= \max [0, S_k] \\
S(\sigma X) &= \max_{1 \leq k \leq n} \left(\sum_{i=1}^{k} X_{\sigma_i} \right)^+ \\
T(\tau X) &= \sum_{i=1}^{n(\tau)} \left(\sum_{k \in \alpha_i} X_k \right)^+ .
\end{aligned} \tag{2.8}
$$

The following lemma is also due to F. Spitzer [2].

LEMMA 2. *Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables. Then,*

$$
\begin{aligned}
E\left[\max_{1 \leq k \leq n} S_k^+\right] &= \frac{1}{n!} \sum_{\sigma} E[S(\sigma X)] \\
&= \frac{1}{n!} \sum_{\tau} E[T(\tau X)] = \sum_{k=1}^{n} \frac{1}{k} E[S_k^+] ,
\end{aligned} \tag{2.9}
$$

where it is understood that the second sum is over equivalence classes of τ with the definition that if $\tau = (\alpha_1(\tau))(\alpha_2(\tau)) \cdots (\alpha_{n(\tau)}(\tau))$ and $\tau' = (\alpha'_1(\tau'))(\alpha'_2(\tau')) \cdots (\alpha'_{n(\tau')}(\tau'))$ then $\tau \sim \tau'$ if $n(\tau) = n(\tau')$ and for each i there is exactly one j and for each j there is exactly one i for which $\alpha_i(\tau)$ is some cyclic permutation of $\alpha'_j(\tau')$.

F. Spitzer proved Lemma 2 by using Lemma 1 and the following fundamental principle. Let $X = (X_1, \cdots, X_n)$ be an n-dimensional vector valued random variable, and let $\mu(x) = \mu(x_1, \cdots, x_n)$ be its probability measure (defined on Euclidean n-space E_n). Suppose that X has the property that $\mu(x) = \mu(gx)$ for every element g of a group G of order h of transformations of E_n into itself. Let $f(x) = f(x_1, \cdots, x_n)$ be a μ-integrable complex valued function on E_n. Then the expected value of $f(X)$ is

$$Ef(X) = \int f(x)d\mu(x) = \int \bar{f}(x)d\mu(x) \tag{2.10}$$

where

$$\bar{f}(x) = \frac{1}{h}\sum_{g \in G} f(gx) .$$

To make use of this fundamental principle and Lemmas 1 and 2 in the proof of our main results we now define the permutation μ (not the measure μ above) as follows. (It is similar but not identical to the permutation τ defined above.) Let μ be represented as a product of cycles, including the one cycle, with each index $1, \cdots, n$ contained in exactly one cycle, and beginning always with a one cycle. For example, with $n = 8$, we could have

$$\mu = (2)(14)(8)(3756) . \tag{2.11}$$

For each x and μ we define $U(\mu x)$ as follows. Suppose μ is the example given in (2.11) above, then

$$U(\mu x) = (x_1 + x_4)^+ + (x_8)^+ + (x_3 + x_7 + x_5 + x_6)^+ . \tag{2.12}$$

Observe that x_2, the single element in the beginning one cycle, is not included in the sum. In formal notation, let

$$\mu = (\beta_1(\mu))(\beta_2(\mu)) \cdots (\beta_{n(\mu)}(\mu)) \tag{2.13}$$

where the $\beta_i(\mu)$, $i = 1, \cdots, n(\mu)$, are disjoint sets of integers whose union is the set $[1, 2, \cdots, n]$, and $\beta_1(\mu))$ consists of exactly one of the integers $1, \cdots, n$. Then,

$$U(\mu x) = \sum_{i=2}^{n(\mu)} \Big(\sum_{k \in \beta_i(\mu)} x_k\Big)^+ . \tag{2.14}$$

We next define for each fixed vector x a map $\mu_x(\sigma)$ as follows. Given

$$\sigma = \begin{pmatrix} 1\ 2 \cdots n \\ i_1 i_2 \cdots i_n \end{pmatrix} \tag{2.15}$$

consider the sum polygon through the points $(1, x_{i_1})$, $(2, x_{i_1} + x_{i_2})$, $\cdots$, $(k, x_{i_1} + \cdots + x_{i_k})$, $\cdots$, (n, s_n). Define the highest concave majorant of the sum polygon as that unique concave polygon which goes through $(1, x_{i_1})$ and (n, s_n) in such a way that all its vertices are also vertices of the sum polygon and that it always lies below or coincides with the sum polygon. Suppose now that the highest concave majorant constructed for the permutation σ has the vertices $(1, x_{i_1})$, $(k_1, x_{i_1} + \cdots + x_{i_{k_1}})$, $\cdots$, $(k_\nu, x_{i_1} + \cdots + x_{i_{k_\nu}})$, (n, s_n) where $1 < k_1 < \cdots < k_\nu < n$. Then we define

$$\mu_x(\sigma) = (i_1)(i_2, \cdots, i_{k_1})(i_{k_1+1}, \cdots, i_{k_2}) \cdots (i_{k_\nu+1}, \cdots, i_n) . \tag{2.16}$$

Now observe that

$$\begin{aligned} S^*(\sigma x) &= \min_{1 \leq k \leq n} \left(\sum_{j=1}^{k} x_{i_j} \right)^+ = [s_n - U(\mu_x(\sigma)x)]^+ \\ &= S(\sigma(-x)) + s_n - U(\mu_x(\sigma)x) \end{aligned} \tag{2.17}$$

or

$$- U(\mu_x(\sigma)x) = S^*(\sigma x) - S(\sigma(-x)) - s_n . \tag{2.18}$$

Going in the other direction, we define for each fixed vector x with rationally independent components, a map $\sigma_x(\mu)$ as follows. Suppose that

$$\begin{aligned} \mu &= (\beta_1(\mu))(\beta_2(\mu)) \cdots (\beta_{n(\mu)}(\mu)) \\ &= (j_1)(\beta_2(\mu)) \cdots (\beta_{n(\mu)}(\mu)) . \end{aligned} \tag{2.19}$$

(See the paragraph following equation (2.12) for the definition of μ, and notice in particular that the leading cycle $\beta_1(\mu)$ must be a one cycle.) Rewrite each set $\beta_2(\mu)$, $\beta_3(\mu)$, $\cdots$, $\beta_{n(\mu)}(\mu)$ as follows. Suppose $\beta_k(\mu) = (3, 6, 9, 2)$. Consider the sum polygon of the vector (x_3, x_6, x_9, x_2). By Lemma 1 there is a unique cyclic permutation of (x_3, x_6, x_9, x_2) such that all of the vertices of the sum polygon lie strictly above its chord. Suppose that unique permutation is (x_9, x_2, x_3, x_6) then rewrite $\beta_k(\mu)$ as $(9, 2, 3, 6)$ and call it $\beta'_k(\mu)$. Define μ' as

$$\begin{aligned} \mu' &= (\beta_1(\mu))(\beta'_2(\mu) \cdots (\beta'_{n(\mu)}(\mu) \\ &= (j_1)(\beta'_2(\mu)) \cdots (\beta'_{n(\mu)}(\mu)) . \end{aligned} \tag{2.20}$$

Observe that

$$U(\mu x) = U(\mu' x) . \tag{2.21}$$

Now consider the vector

$$y = (y_2, y_3, \cdots, y_n) = \left(\sum_{k \in \beta_2'(\mu)} x_k, \sum_{k \in \beta_3'(\mu)} x_k, \cdots, \sum_{k \in \beta'_{n(\mu)}(\mu)} x_k \right). \tag{2.22}$$

There is a unique permutation of the components of y, say

$$y' = (y_2', y_3', \cdots, y_n') = \left(\sum_{k \in \beta_2''(\mu)} x_k, \sum_{k \in \beta_3''(\mu)} x_k, \cdots, \sum_{k \in \beta''_{n(\mu)}(\mu)} x_k \right) \tag{2.23}$$

so that

$$\frac{\sum_{k \in \beta_2''(\mu)} x_k}{\sum_{k \in \beta_2''(\mu)} 1} < \frac{\sum_{k \in \beta_3''(\mu)} x_k}{\sum_{k \in \beta_3''(\mu)} 1} < \cdots < \frac{\sum_{k \in \beta''_{n(\mu)}(\mu)} x_k}{\sum_{k \in \beta'_{n(\mu)}} 1}. \tag{2.24}$$

Now define

$$\mu'' = (j_1)(\beta_2''(\mu))(\beta_3''(\mu)) \cdots (\beta''_{n(\mu)}(\mu)). \tag{2.25}$$

Observe that

$$U(\mu''x) = U(\mu'x) = U(\mu x). \tag{2.26}$$

Suppose μ'' can be written as

$$\mu'' = (j_1)(j_2, \cdots, j_{h_1})(j_{h_1+1}, \cdots, j_{h_2}) \cdots (j_{h_\nu+1}, \cdots, j_n) \tag{2.27}$$

we then define

$$\sigma_x(\mu) = \begin{pmatrix} 1 \; 2 \cdots n \\ j_1 j_2 \cdots j_n \end{pmatrix} \tag{2.28}$$

observe that with respect to the sum polygon through the points $(1, x_{j_1})$, $(2, x_{j_1} + x_{j_2})$, $\cdots$, $(k, x_{j_1} + \cdots + x_{j_k})$, $\cdots$, (n, s_n) the vertices of the highest concave majorant are at the points $(1, x_{j_1})$, $(h_1, x_{j_1} + \cdots + x_{j_{h_1}})$, $(h_2, x_{j_1} + \cdots + x_{j_{h_2}})$, $\cdots$, $(h_{j_\nu}, x_{j_1} + \cdots + x_{j_{h_\nu}})$, (n, s_n). Thus

$$\mu_x(\sigma_x(\mu)) = \mu''. \tag{2.29}$$

Now use equations (2.17), (2.29), and (2.26) to obtain

$$\begin{aligned} S^*(\sigma_x(\mu)x) &= S(\sigma_x(\mu)(-x)) + s_n - U(\mu_x(\sigma_x(\mu))x) \\ &= S(\sigma_x(\mu)(-x)) + s_n - U(\mu''x) \\ &= S(\sigma_x(\mu)(-x)) + s_n - U(\mu x) \end{aligned} \tag{2.30}$$

which is equivalent to

$$-U(\mu x) = S^*(\sigma_x(\mu)x) - S(\sigma_x(\mu)(-x)) - s_n. \tag{2.31}$$

Now consider the sets $[-U(\mu x)]$ and $[S^*(\sigma x) - S(\sigma(-x)) - s_n]$ generated by letting σ and τ run through all permutations. We have shown above that for vectors x with rationally independent components, the maps $\sigma_x(\mu)$ and $\mu_x(\sigma)$ define a one-to-one map of the set $[-U(\mu x)]$ onto the set $[S^*(\sigma x) - S(\sigma(-x) - s_n]$. Furthermore, since the set x of vectors in E_n with rationally independent component is dense in E_n and the functions $S(\sigma x)$ and $U(\mu x)$ are continuous functions of x, we have proved the following lemma.

LEMMA 3. *For an arbitrary fixed vector $x = (x_1, \cdots, x_n)$ the sets $[-U(\mu x)]$ and $[S^*(\sigma x) - S(\sigma(-x)) - s_n]$ which are generated by letting σ and τ run through all permutations are identical sets.*

3. The main theorems.

THEOREM 1. *Let $X_1, X_2, \cdots, X_n$ be independent, identically distributed random variables.*

$$\begin{aligned} E\Big[\min_{1\leq k\leq n} S_k^+\Big] &= E(S_n) + \sum_{k=1}^{n} \frac{1}{k} E[(-S_k)^+] - \sum_{k=1}^{n-1} \frac{1}{k} E(S_k^+) \\ &= E(S_n) + \frac{1}{n} E(S_n^+) - \sum_{k=1}^{n} \frac{1}{k} E(S_k)\,. \end{aligned} \tag{3.1}$$

Proof. Using the fundamental principle (2.10) we may write

$$E\Big[\min_{1\leq k\leq n} S_k^+\Big] = \frac{1}{n!} \sum_{\sigma} E[S^*(\sigma X)] \tag{3.2}$$

and applying (2.17) to the term on the right we have

$$\begin{aligned} E\Big[\min_{1\leq k\leq n} S_k^+\Big] &= \frac{1}{n!} \sum_{\sigma} E[S(\sigma(-X)) + S_n - U(\mu_x(\sigma)X)] \\ &= \frac{1}{n!} \sum_{\sigma} E[S(\sigma(-X))] + E(S_n) \\ &\quad - \frac{1}{n!} \sum_{\sigma} E[U(\mu_x(\sigma)X)]\,. \end{aligned} \tag{3.3}$$

Now apply Lemma 3 to obtain

$$E\Big[\min_{1\leq k\leq n} S_k^+\Big] = \frac{1}{n!} \sum_{\sigma} E[S(\sigma(-X))] + E(S_n) - \frac{1}{n!} \sum_{\mu} E[U(\mu X)] \tag{3.4}$$

where it is understood that the second sum is over equivalence classes of μ with the definition that if

$$\mu = (\beta_1(\mu))(\beta_2(\mu)) \cdots (\beta_{n(\mu)}(\mu))$$

and

$$\mu' = (\beta_1'(\mu'))(\beta_2'(\mu')) \cdots (\beta_{n(\mu')}'(\mu'))$$

then $\mu \sim \mu'$ if $n(\mu) = n(\mu')$, $\beta_1(\mu) = \beta_1'(\mu')$ and for each i there is exactly one j and for each j there is exactly one i for which $\beta_i(\mu)$ is some cycle permutation of $\beta_j'(\mu')$.

Let us consider the last term in equation (3.4). Define $X' = (X_2, \cdots, X_n)$ and keep in mind that the random variables $X_1, X_2, \cdots, X_n$ are independent and identically distributed. It is then clear from our definition of $T(\tau x)$ and $U(\mu x)$ that

$$\frac{1}{n!}\sum_{\mu} E[U(\mu X)] = \frac{1}{n!}\sum_{\tau} E[T(\tau X')] \,. \tag{3.5}$$

Now apply Lemma 2 to equations (3.4) and (3.5) to obtain

$$E\Big[\min_{1\leq k\leq n} S_k^+\Big] = \sum_{k=1}^{n}\frac{1}{k}E[(-S_k)^+] + E(S_n) - \sum_{k=1}^{n-1}\frac{1}{k}E[S_k^+] \tag{3.6}$$

which is the first part of equation (3.1). Then, using the fact that $E[(-S_k)^+] - E(S_k^+) = -E(S_k)$, we obtain the second part of equation (3.1) as follows.

$$\begin{aligned}
&E(S_n) + \sum_{k=1}^{n}\frac{1}{k}E[(-S_k)^+] - \sum_{k=1}^{n-1}\frac{1}{k}E(S_k^+)\\
&= E(S_n) + \sum_{k=1}^{n}\frac{1}{k}E[(-S_k)^+] - \sum_{k=1}^{n}\frac{1}{k}E(S_k^+) + \frac{1}{n}E(S_n^+)\\
&= E(S_n) + \sum_{k=1}^{n}\frac{1}{k}\{E(-S_k)^+ - E(S_k^+)\} + \frac{1}{n}E(S_n^+)\\
&= E(S_n) + \frac{1}{n}E(S_n^+) - \sum_{k=1}^{n}\frac{1}{k}E(S_k)\,.
\end{aligned}$$

THEOREM 2.

$$\begin{aligned}
E\Big[\max_{1\leq k\leq n} S_k\Big] &= \sum_{k=1}^{n-1}\frac{1}{k}E[(-S_k)^+] + E(S_n)\\
&= E(S_n) + E\Big[\max_{1\leq k\leq n-1}(-S_k)^+\Big]\,.
\end{aligned} \tag{3.7}$$

Proof.

$$\max_{1\leq k\leq n} S_k = \max_{1\leq k\leq n} S_k^+ + \max_{1\leq k\leq n} S_k^-, \text{ and } \max_{1\leq k\leq n} S_k^- = -\min_{1\leq k\leq n}(-S_k)^+ \,. \tag{3.8}$$

Therefore,

$$E\left[\max_{1\leq k\leq n} S_k\right] = E\left[\max_{1\leq k\leq n} S_k^+\right] - E\left[\min_{1\leq k\leq n}(-S_k)^+\right]. \tag{3.9}$$

From Theorem 1, we have

$$\begin{aligned} &E\left[\min_{1\leq k\leq n}(-S_k)^+\right] \\ &= E(-S_n) + \sum_{k=1}^{n}\frac{1}{k}E[(S_k)^+] - \sum_{k=1}^{n-1}\frac{1}{k}E[(-S_k)^+]. \end{aligned} \tag{3.10}$$

From Lemma 2, we have

$$E\left[\max_{1\leq k\leq n} S_k^+\right] = \sum_{k=1}^{n}\frac{1}{k}E[S_k^+]. \tag{3.11}$$

Therefore,

$$\begin{aligned} &E\left[\max_{1\leq k\leq n} S_k\right] \\ &= \sum_{k=1}^{n}\frac{1}{k}E[S_k^+] - E(-S_n) - \sum_{k=1}^{n}\frac{1}{k}E[(S_k)^+] + \sum_{k=1}^{n-1}\frac{1}{k}E[(-S_k)^+] \\ &= \sum_{k=1}^{n-1}\frac{1}{k}E[(-S_k)^+] - E(-S_n) \\ &= E(S_n) + \sum_{k=1}^{n-1}\frac{1}{k}E[(-S_k)^+]. \end{aligned} \tag{3.12}$$

THEOREM 3.

$$E\left[\min_{1\leq k\leq n} S_k\right] = E(S_n) - \sum_{k=1}^{n-1}\frac{1}{k}E(S_k^+) = E(S_n) - E\left[\max_{1\leq k\leq n-1} S_k^+\right]. \tag{3.13}$$

Proof.

$$\min_{1\leq k\leq n} S_k = \min_{1\leq k\leq n} S_k^+ + \min_{1\leq k\leq n} S_k^-, \text{ and } \min_{1\leq k\leq n} S_k^- = -\max_{1\leq k\leq n}(-S_k)^+. \tag{3.14}$$

Therefore,

$$\begin{aligned} E\left[\min_{1\leq k\leq n} S_k\right] &= E\left[\min_{1\leq k\leq n} S_k^+\right] - E\left[\max_{1\leq k\leq n}(-S_k)^+\right] \\ &= E(S_n) + \sum_{k=1}^{n}\frac{1}{k}E[(-S_k)^+] - \sum_{k=1}^{n-1}\frac{1}{k}E(S_k^+) \\ &\quad - \sum_{k=1}^{n}\frac{1}{k}E[(-S_k)^+] \\ &= E(S_n) - \sum_{k=1}^{n-1}\frac{1}{k}E(S_k^+) = E(S_n) - E\left[\max_{1\leq k\leq n-1} S_k^+\right]. \end{aligned} \tag{3.15}$$

4. Application to standard normal random variables. If X_1,

$X_2, \cdots, X_n$ are standard normal random variables, then

$$E\left[\max_{1\le k\le n} S_k^+\right] = \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \tag{4.1}$$

$$E\left[\min_{1\le k\le n} S_k^+\right] = \frac{1}{\sqrt{2\pi n}} \tag{4.2}$$

$$E\left[\operatorname{man}_{1\le k\le n} S_k\right] = \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k}} \tag{4.3}$$

$$E\left[\min_{1\le k\le n} S_k\right] = -\frac{1}{\sqrt{2\pi}}\sum_{k=1}^{n-1}\frac{1}{\sqrt{k}}\ . \tag{4.4}$$

Statement (4.1) was proved by J. A. McFadden and J. L. Lewis in [1]. They applied F. Spitzer's lemma (Lemma 2) to the fact that for standard normal random variables,

$$E(S_k^+) = \sqrt{\frac{k}{2\pi}}\ . \tag{4.5}$$

To obtain (4.2), (4.3), and (4.4), apply (4.5) and the fact that for standard normal random variables, $E(S_k^+) = E[(-S_k)^+]$, to Theorems 1, 2, and 3 respectively.

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