PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 2, 1974

REMARK ON MAPPINGS NOT RAISING DIMENSION OF CURVES

J. KRASINKIEWICZ

The purpose of this note is to prove three theorems on dimension raising ability of certain classes of maps defined on 1-dimensional continua. In particular we obtain a generalization of a recent result of J. Jobe concerning dimension raising ability of inverse arc functions defined on dendrites.

By a continuum we mean a compact connected metric space. A 1-dimensional continuum is called a curve. If each point of a continuum X has arbitrary small neighborhood with finite boundary, then X is said to be regular. X is suslinian provided any collection of mutually disjoint nondegenerate subcontinua of X is at most countable [6]. For a nondegenerate continuum we have the following implications:

(i) (regular) $\Rightarrow$ (suslinian) $\Rightarrow$ (1-dimensional).

Let f be a mapping of a continuum X into a continuum Y. We shall consider the following properties of f:

(α) for every arc $L \subset Y$ there exists an arc $M \subset X$ which is mapped by f onto L, i.e., $f(M) = L$.

(β) for every arc $L \subset Y$ there exists a continuum $M \subset X$ which is mapped by f onto L.

(γ) for every continuum $L \subset Y$ there exists a continuum $M \subset X$ which is mapped by f onto L.

THEOREM 1. *If f is a mapping with property (β) of a suslinian continuum X onto a locally connected continuum Y, then Y is suslinian.*

Proof. Suppose it is not true. Then there is an uncountable collection $\{B\}$ of nondegenerate mutually disjoint subcontinua of Y. Consider a member $B \in \{B\}$. Let a and b be distinct points of B. Let $U_1, U_2, \cdots$ be a decreasing sequence of neighborhoods of B (in Y) which limits on B, i.e.,

$$(1) \qquad \bigcap_n U_n = B\ .$$

For each positive integer n there is a locally connected continuum C_n such that

$$(2) \qquad B \subset C_n \subset U_n \qquad \text{(see [5], p. 260)}\ .$$

Let L_n be an arc in C_n joining a and b. We may assume that $\{L_n\}$ is a convergent sequence (otherwise we take a convergent subsequence). Let B' denote the limit of this sequence. Hence by (1) and (2) we have

(3) B' is a nondegenerate subcontinuum of B (because $a, b \in B'$) .

For each integer n there is a continuum $A_n \subset X$ which is mapped by f onto L_n. Choose a convergent subsequence of $\{A_n\}$ and let A_B be its limit. It is clear that

$$(4) \qquad f(A_B) = B' .$$

According to (3) and (4) we see that for each $B \in \{B\}$ we can construct a nondegenerate continuum $A_B \subset X$ which is mapped by f onto a subcontinuum of B. It follows that $\{A_B : B \in \{B\}\}$ constitute an uncountable collection of nondegenerate mutually disjoint subcontinua of X, contrary to our assumption on X. This proves the theorem.

Mappings with property (α) were considered by J. Jobe in [3] (where they are called inverse arc functions). There was shown that if f is a mapping with property (α) from a dendrite X with countably number of endpoints onto Y, then $\dim Y \leqq 1$ (dendrite = locally connected continuum containing no simple closed curve). J. Jobe asks if the above result can be extended onto all dendrites. Since $(\alpha) \Rightarrow (\beta)$, then the following corollary to Theorem 1 answers this question in the affirmative.

COROLLARY. *If f is a mapping with property (β) defined on a dendrite X, then $f(X)$ is at most 1-dimensional.*

Proof. Clearly, $f(X)$ is a locally connected continuum. Since each dendrite is regular ([5], p. 301), the corollary is an immediate consequence of (i) and Theorem 1.

We are now going to prove two theorems related to the above corollary.

Let D be the unit disk in the complex plane and let S denote the boundary of D. A mapping $f: X \to D$ is called essential in the sense of Alexandroff-Hopff, briefly: AH-essential, provided the partial mapping

$$f \mid f^{-1}(S): f^{-1}(S) \longrightarrow S$$

can not be extended onto X. It is known that

(ii) If X is compact and $\dim X \geqq 2$, then there exists an AH-essential map of X onto D (see [7]).

By a classical result of Mazurkiewicz [7] we have

(iii) An AH-essential map has property (γ).

A space X is said to be contractible with respect to S, briefly: cr S, if each map $f: X \rightarrow S$ is nullhomotopic. It is well known that

(iv) Each closed subset of a cr S curve is cr S ([2], p. 83).

It has been proved by M. K. Fort, Jr. [1] that there exists a continuum $K \subset D$ such that

(v) No continuum cr S can be mapped onto K.

Using these facts we shall prove the following

THEOREM 2. *If X is a cr S curve and $f: X \rightarrow Y$ has property* (γ), *then* $\dim Y \leqq 1$.

Proof. Suppose $\dim Y \geqq 2$. Hence by (ii) there is an AH-essential map $g: Y \rightarrow D$. Since the composition of two maps having property (γ) is a map with property (γ), then by (iii) the map $h = gf$ has property (γ). Let $K \subset D$ be the Fort continuum. There exists a continuum $L \subset X$ such that $h(L) = K$. By (iv), L is cr S. Hence K can be obtained as a continuous image of a cr S continuum, contrary to (v). This contradiction completes the proof.

A continuum X is tree-like if for each $\varepsilon > 0$ there exist a finite tree T and a continuous map $f: X \rightarrow T$ onto T such that $\operatorname{diam} f^{-1}(t) < \varepsilon$ for every $t \in T$. It is known that every tree-like continuum is cr S. Recently the author has proved that if Y is a cr S curve and if there exists a tree-like curve which can be mapped onto Y, then Y is tree-like [4]. Combining these results with Theorem 2 we obtain

THEOREM 3. *Let f be a mapping from a tree-like curve onto a continuum Y. If f has property* (γ) *and Y is cr S, then Y is tree-like.*

REFERENCES

1. M. K. Fort, Jr., *Images of plane continua*, Amer. J. Math., **81** (1959), 541-546.

2. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, 1948.

3. J. Jobe, *Dendrites, dimension, and the inverse arc function*, Pacific J. Math., **45** (1973), 245-256.

4. J. Krasinkiewicz, *Curves which are continuous images of tree-like continua are movable*, *Fund. Math.*, *(to appear)*.

5. K. Kuratowski, *Topology*, vol. 2, Warsaw-New York, 1968.

6. A. Lelek, *On the topology of curves II*, Fund. Math., **70** (1971), 131-138.

7. S. Mazurkiewicz, *Sur l'existence des continus indécomposables*, Fund. Math., **25** (1935), 327-328.

Received January 10, 1974 and in revised form September 18, 1974.

POLISH ACADEMY OF SCIENCES