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DOUBLY STOCHASTIC MATRICES WITH MINIMAL PERMANENTS

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A simple elementary proof is given for a result of D. London on permanental minors of doubly stochastic matrices with minimal permanents.

A matrix with nonnegative entries is called *doubly stochastic* if all its row sums and column sums are equal to 1. A well-known conjecture of van der Waerden [3] asserts that the permanent function attains its minimum in Ω_n, the set of $n \times n$ doubly stochastic matrices, uniquely for the matrix all of whose entries are $1/n$. The conjecture is still unresolved.

A matrix A in Ω_n is said to be *minimizing* if

$$\operatorname{per}(A) = \min_{S \in \Omega_n} \operatorname{per}(S).$$

The properties of minimizing matrices have been studied extensively in the hope of finding a lead to a proof of the van der Waerden conjecture.

Let $A(i\,|\,j)$ denote the submatrix obtained from A by deleting its ith row and its jth colum. Marcus and Newman [3] have obtąined inter alia the following two results.

THEOREM 1. *A minimizing matrix A is fully indecomposable*, i.e.,

$$\operatorname{per}(A(i\,|\,j)) > 0$$

for all i and j.

In other words, if A is a minimizing $n \times n$ matrix then for any (i,j) there exists a permutation σ such that $j = \sigma(i)$ and $a_{s,\sigma(s)} > 0$ for $s = 1,\cdots,i-1,i+1,\cdots,n$.

THEOREM 2. *If $A = (a_{ij})$ is a minimizing matrix then*

$$\operatorname{per}(A(i\,|\,j)) = \operatorname{per}(A) \tag{1}$$

for any (i,j) for which $a_{ij} > 0$.

The result in Theorem 2 is of considerable interest. For, if it could be shown that (1) holds for all permanental minors of A, the van der Waerden conjecture would follow. London [2] obtained the following result.

THEOREM 3. *If A is a minimizing matrix, then*

$$\operatorname{per}(A(i\,|\,j)) \geqq \operatorname{per}(A) \tag{2}$$

for all i and j.

London's proof of Theorem 3 depends on the theory of linear inequalities. Another proof of London's result is due to Hedrick [1]. In this paper I give an elementary proof of the result that is considerably simpler than either of the above noted proofs.

Proof of Theorem 3. Let $A = (a_{ij})$ be an $n \times n$ minimizing matrix. Let σ be a permutation on $\{1,\cdots,n\}$ and $P = (p_{ij})$ be the corresponding permutation matrix. For $0 \leqq \theta \leqq 1$, define

$$f_P(\theta) = \operatorname{per}((1-\theta)A + \theta P).$$

Since A is a minimizing matrix, we have

$$f_P'(0) \geqq 0$$

for any permutation matrix P. Now

$$f_P'(0) = \sum_{s,t=1}^{n} (-a_{st} + p_{st}) \operatorname{per}(A(s\,|\,t))$$

$$= \sum_{s,t=1}^{n} p_{st} \operatorname{per}(A(s\,|\,t)) - n \operatorname{per}(A)$$

$$= \sum_{s=1}^{n} \operatorname{per}(A(s\,|\,\sigma(s))) - n \operatorname{per}(A).$$

Hence,

$$\sum_{s=1}^{n} \operatorname{per}(A(s\,|\,\sigma(s))) \geqq n \operatorname{per}(A) \tag{3}$$

$$\sum_{s=1}^{n} \operatorname{per}(A(s\,|\,\sigma(s))) \geqq n \operatorname{per}(A)$$

for any permutation σ. Since A is a minimizing matrix and thus, by Theorem 1, fully indecomposable, we can find for any given (i,j) a permutation σ such that $j = \sigma(i)$ and $a_{s,\sigma(s)} > 0$ for $s = 1,\cdots,i-1,i+1,\cdots,n$. But then by Theorem 2,

$$\mathrm{per}(A(s\,|\,\sigma(s))) = \mathrm{per}(A)$$

for $s = 1,\cdots,i-1,i+1,\cdots,n$, and it follows from (3) that

$$\mathrm{per}(A(i\,|\,j)) \geqq \mathrm{per}(A).$$

References

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2. David London, *Some notes on the van der Waerden conjecture*, Linear Algebra and Appl., **4** (1971), 155–160.
3. Marvin Marcus and Morris Newman, *On the minimum of the permanent of a doubly stochastic matrix*, Duke Math. J., **26** (1959), 61–72.

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