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ON CONJUGATE BANACH SPACES WITH THE RADON-NIKODÝM PROPERTY

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It is shown that if the unit ball $B_{X^{**}}$ of X^{**} is Eberlein compact in the weak* topology, or if X^* is isomorphic to a subspace of a weakly compactly generated Banach space then X^* possesses the Radon-Nikodým property (RNP). This extends the classical theorem of N. Dunford and B. J. Pettis. If X is a Banach space with X^{**}/X separable then both X^* and X^{**} (and hence X) have the RNP. It is also shown that if a conjugate space X^* possesses the RNP and X is weak* sequentially dense in X^{**} then $B_{X^{**}}$ is weak* sequentially compact. Thus, in particular, if X^{**}/X is separable then $B_{X^{***}}$ is weak* sequentially compact.

1. Introduction. A Banach space X is said to have the *Radon-Nikodým property* (RNP) if for each positive finite measure space $(\Omega, \Sigma, \lambda)$ and every λ-continuous vector measure $\mu: \Sigma \to X$ with finite variation, there exists a Bochner integrable function $f: \Omega \to X$ such that

$$\mu(A) = \textit{Bochner} \int_A f(\omega) d\lambda \quad \textit{for all} \quad A \in \Sigma$$

The classical theorems of Dunford and Pettis [3] and Phillips [6] show that every separable conjugate space and every reflexive Banach space has RNP.

Recent work aimed at extending the Radon-Nikodým theorem to vector measures has yielded more general theorems which characterizes Banach spaces with the Radon-Nikodým property. For the purposes of this paper, we only list those that will be employed and refer to [8] for a more detailed introduction.

The two following theorems are essentially due to Uhl [9].

THEOREM 1.1. *Let X be a Banach space. Then the following statements are equivalent*:

(i) *X possesses RNP*;

(ii) *every subspace (by a subspace, we refer to a closed infinite—dimensional linear submanifold) of X possesses* RNP;

(iii) *every separable subspace of X possesses* RNP.

For a Banach space X, denote by X^* its conjugate space.

THEOREM 1.2. *If for every separable subspace Y of X, Y^* is separable. Then X^* has* RNP.

The converse of Theorem 1.2 is proved by Stegall [8], i.e.,

THEOREM 1.3 *Suppose X^* has* RNP. *Then for every separable subspace Y of X, Y^* is separable.*

We shall use these three theorems to deduce our main results. It seems to be an open question whether a conjugate Banach space X^* has RNP whenever the unit ball $B_{X^{**}}$ of X^{**} is weak* sequentially compact. Our result shows that when $B_{X^{**}}$, in its weak* topology, is homeomorphic to a weakly compact subset of some Banach space, or when X^* is isomorphic to a subspace of a weakly compactly generated Banach space (in either case, $B_{X^{**}}$ is weak* sequentially compact) then X^* possesses the RNP. This result improves the classical Dunford-Pettis-Phillips theorem on RNP.

The possession of RNP by the conjugate spaces of the Banach spaces X with X^{**}/X separable is investigated. For such spaces X, both X^* and X^{**} (and hence X) have the RNP.

It is also shown that if a conjugate space X^* possesses the RNP and X is weak* sequentially dense in X^{**} then $B_{X^{**}}$ is weak* sequentially compact. Thus, in particular, if X^{**}/X is separable then $B_{X^{***}}$ is weak* sequentially compact.

2. The Radon-Nikodým property in X^* and the weak* sequential compactness of the unit ball of X^{**}. In the terminology of [4], a Banach space X is called *quasi-separable* if for each separable subspace Y of X, Y^* is separable; on account of Theorems 1.2 and 1.3, this concept is equivalent to the possession of RNP by X^*. We indicate here that if X is quasi-separable then ever continuous linear closed image of X has the same property. For if Z is a continuous linear image of X then Z^* is isomorphic to a subspace of X^*; Z^* then has RNP. Thus by Theorem 1.3, every separable subspace of Z has a separable conjugate. This solves the question proposed by Lacey and Whitley [4] that whether a quotient space of a quasi-separable space is itself quasi-separable.

It is also not known whether a Banach space X is quasi-separable if $B_{X^{**}}$ is weak* sequentially compact. This can be equivalently translated as whether a conjugate space X^* has RNP if $B_{X^{**}}$ is weak* sequentially compact. Before proceeding to our discussion, recall that a Banach space X is said to be *weakly compactly generated* (WCG) if it is the closed span of some weakly compact subset of itself. As a result of Amir and Lindenstrauss [1], X is WCG if and only if B_{X^*} in its weak* topology, is affine homeomorphic to a weakly compact subset of some Banach space. A compact Hausdorff space S is *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space. In view of Eberlein's theorem, S is sequentially compact if it is Eberlein compact. Our result shows that if $B_{X^{**}}$ is Eberlein compact in its weak* topology, or if X^* is isomorphic to a subspace of a WCG space then X^* has RNP.

For a subspace $Y \subset X$, set

$$Y^{\perp} = \{f \in X^*: f(y) = 0 \text{ for all } y \in Y\} .$$

THEOREM 2.1. *Let X be a Banach space. Suppose $B_{X^{**}}$ is Eberlein compact in the weak* topology; then X^* possesses the* RNP.

Proof. In view of Theorem 1.2, it suffices to show that every separable subspace of X has a separable conjugate space.

Let Y be a separable subspace of X. By Goldstine's theorem, B_Y is weak*-dense in $B_{Y^{**}}$; thus $B_{Y^{**}}$ is weak*-separable. Let $J: Y \to X$ be the inclusion map. Observe that $J^{**}: Y^{**} \to X^{**}$ is a weak* isomorphism of Y^{**} onto $Y^{\perp\perp}$ with $J^{**}(B_{Y^{**}}) = B_{Y^{\perp\perp}}$. Hence $B_{Y^{\perp\perp}}$ is weak*-separable. Moreover, $B_{Y^{\perp\perp}}$ is weak* closed in $B_{X^{**}}$, which is Eberlein compact by hypothesis, whence $B_{Y^{\perp\perp}}$ is itself Eberlein compact.

It is well known that a separable Eberlein compact space is metrizable. We have then that $B_{Y^{\perp\perp}}$ is metrizable. This then implies that $B_{Y^{**}}$ is metrizable. Therefore, Y^* is separable; which completes the proof.

THEOREM 2.2. *Suppose X^* is isomorphic to a subspace of a* WCG *Banach space Z; then X^* possesses* RNP.

Proof. Again, it suffices to show that every separable subspace of X. has a separable conjugate space. Let Y be a separable subspace of X Apply the same argument as in the proof of Theorem 2.1, we see that $B_{Y^{**}}$ is weak*-separable.

Let (x_n^{**}) be a weak*-dense sequence in $B_{Y^{\perp\perp}}$ and $J: X^* \to Z$ be an isomorphism. $J^*: Z^* \to X^{**}$ is then surjective. By the Open Mapping Theorem, there exists a bounded sequence (z_n^*) in Z^* such that $J^* z_n^* = x_n^{**}$. Denote by W the weak*-closure of $\{z_n^*\}$. By the hypothesis that Z is WCG, B_{Z^*} is then Eberlein compact in the weak* topology and hence W is also Eberlein compact. This together with the separability of W implies that W is a compact metric space in the weak* topology. $J^*(W)$ is then weak* compact and contains $\{x_n^{**}\} \subset B_{Y^{\perp\perp}}$. Hence $J^*(W) = B_{Y^{\perp\perp}}$. Moreover, being a continuous image of a compact metric space, $B_{Y^{\perp\perp}}$ is compact metrizable. Therefore, $B_{Y^{**}}$ is metrizable and Y^* is separable.

It follows immediately from either Theorem 2.1 or Theorem 2.2 that

COROLLARY 2.3. *If X^* is* WCG *then X^* has* RNP.

REMARK. Corollary 2.3 can be proved by use of Theorem 1.2 and the fact that if a Banach space Y is separable and Y^* is WCG then Y^* is also separable. This result improves the classical Dunford-Pettis-Phillips Theorem on RNP, and is well known at present. However, recently H. P. Rosenthal [7] has given a counter-example to the heredity problem for WCG Banach space. Indeed, the Banach space X_R he exhibited has the following properties: (i) X_R is a subspace of a WCG space $L^1(\mu)$ and X_R is not WCG; (ii) X_R is isomorphic to a conjugate Banach space; (iii) the unit ball of X_R^* is Eberlein compact in its weak* topology. Thus our independent proof appears necessary.

Observe that those conjugate Banach spaces X^* with RNP discussed in the above theorems have the property that $B_{X^{**}}$ is weak* sequentially compact. For the converse, we have obtained sufficient conditions to ensure that $B_{X^{**}}$ is weak* sequentially compact whenever X^* has the RNP. In the following theorem, we set for each $A \subset X^{**}$

$$A^{\top} = \{f \in X^*: x^{**}(f) = 0 \text{ for all } x^{**} \in A\}$$

and write "$\approx$" whenever two Banach spaces are isometrically isomorphic.

THEOREM 2.4. *If X^* possesses the* RNP *and X is weak* sequentially dense in X^{**}, then $B_{X^{**}}$ is weak* sequentially compact.*

Proof. Let (x_n^{**}) be a sequence in $B_{X^{**}}$. By assumption, X is weak* sequentially dense in X^{**}; for each x_n^{**}, there exists a sequence $(x_n^k)_k$ in X such that $(x_n^k)_k$ converges to x_n^{**} in the weak* topology of X^{**}.

Let $\tilde{Y}$ be the weak* closed subspace of X^{**} spanned by $\{x_n^{**}\}$ and $\tilde{Z}$ be the weak* closed subspace of X^{**} spanned by $\{x_n^k\}_{n,k}$. We have then that $\tilde{Y} \subset \tilde{Z}$ and

$$\tilde{Y} = (\{x_n^{**}\}^{\top})^{\perp} \approx (X^*/\{x_n^{**}\}^{\top})^*) ,$$
$$\tilde{Z} = (\{x_n^k\}_{n,k}^{\top})^{\perp} \approx (X^*/\{x_n^k\}_{n,k}^{\perp})^* .$$

Let Z be the closed subspace of X spanned by $\{x_n^k\}_{n,k}$. Observe that Z is weak*-dense in $Z^{\perp\perp}$, whence $Z^{\perp\perp} = \tilde{Z}$. By hypothesis, X^* has

RNP; hence Z^* is separable. But

$$Z^* \approx X^*/\{x_n^k\}_{n,k}^{\top} \text{ and } \widetilde{Y} \subset \widetilde{Z} ;$$

$X^*/\{x_n^{**}\}^{\top}$ is a continuous linear image of $X^*/\{x_n^k\}_{n,k}^{\top}$. Thus $X^*/\{x_n^{**}\}^{\top}$ is separable. It follows then that the unit ball of $(X^*/\{x_n^{**}\}^{\top})^*$ is weak* sequentially compact.

Moreover, since $(X^*/\{x_n^{**}\}^{\top})^*$ is weak* isomorphic to $\widetilde{Y}$, the sequence (x_n^{**}) in $\widetilde{Y}$ has a weak* convergent subsequence. This is equivalent to saying that $B_{X^{**}}$ is weak* sequentially compact.

The Theorem above will be used in § 3 to prove that if X^{**}/X is separable then $B_{X^{***}}$ is weak* sequentially compact.

3. The Banach space X with X^{**}/X separable. In this section, we give examples of Banach space X such that both X^* and X^{**} (and hence X) have RNP. The Banach space X we are considering has the property that X^* is WCG and $B_{X^{***}}$ is weak* sequentially compact.

THEOREM 3.1. *Let X be a Banach space such that X^{**}/X^* is separable. Then both X^* and X^{**} has* RNP.

Proof. In view of Theorem 1.2, it suffices to show that every separable subspace of X (resp. X^*) has a separable conjugate space.

Let Y be a separable subspace of X. Note that Y^{**}/Y is isomorphic to a subspace of X^{**}/X^* [2, p. 908]. By hypothesis, X^{**}/X is separable, so is Y^{**}/Y. It follows then that Y^{**} and hence Y^* is separable.

Assume Z is a separable subspace of X^*. It is known that there exists a separable subspace W of X such that Z is isometrically isomorphic to a subspace of W*. Z^* is then a continuous linear image of the separable space W^{**}. Thus Z^* is separable.

REMARK. It is obvious that if both X^* and X^{**} have RNP then every separable subspace of X has a separable second conjugate. Indeed, if Y is a separable subspace of X, Y^* is then separable since X^* has RNP. But Y^{**} is isometrically isomorphic to a subspace of X^{**}; Y^{**} has RNP. Thus by Theorem 1.3, Y^{**} is separable. Note that the given hypothesis doesn't necessarily imply that X^{**}/X is separable. As a counterexample, we refer to [5, p. 124].

Together with the result of Theorem 2.4, we obtain

COROLLARY 3.2. *Suppose X^{**}/X is separable. Then $B_{X^{**}}$ and $B_{X^{***}}$* (and hence B_{X^*}) *are sequentially compact in their respective weak* topologies.*

Proof. Since X^{**}/X is separable, X^* and X^{**} have RNP by Theorem 3.1. Also a result of [5, p. 123] shows that X^* (resp. X^{**}) is weak* sequentially dense in X^{**} (resp. X^{***}). Thus $B_{X^{**}}$ (resp. $B_{X^{***}}$) is weak* sequentially compact by Theorem 2.4. Moreover, since B_{X^*} is a continuous linear image of $B_{X^{***}}$ in the respective weak* topologies, B_{X^*} is then weak* sequentially compact.

COROLLARY 3.3. *Suppose X is non-reflexive and X^{**}/X is separable. Then neither X nor X^* is weakly sequentially complete.*

Proof. Follows from Theorem 3.1 and Theorem 1.3.

As a final result, we further prove that when X^{**}/X is separable X^* is indeed WCG.

LEMMA 3.4. *Let Z be a* WCG *subspace of a Banach space Y such that Y/Z is separable. Then Y is* WCG.

Proof. Y/Z is separable, hence there exists a separable subspace $W \subset Y$ such that $Z + W$ is dense in Y. But both W and Z are WCG; thus Y is WCG.

THEOREM 3.5. *Suppose X^{**}/X is separable. Then X^* is* WCG.

Proof. It is known that, under the given hypothesis, there exists a separable subspace Z such that X/Z is reflexive [5, p. 121]. We have then that $Z^\perp$ is reflexive and $X^*/Z^\perp$ is separable. It follows from Lemma 3.4 that X^* is (WCG)

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