THE PRESERVERS OF ANY ORTHOGONAL GROUP

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Let L be an invertible linear map on the space $M(n, k)$ of n-square matrices over a field k of characteristic not 2. In this paper we classify all such L which preserve a particular orthogonal group of a nonsingular symmetric bilinear form. We use some elementary facts about algebraic groups and an idea of Dieudonné's. There is some indication that our use of an algebraic geometric setting is the proper one for many problems of this type.

There are a considerable number of results which may be paraphrased as follows: "let $L\colon M(n, k) \to M(n, k)$ be a linear transformation that preserves some property related to matrix multiplication. Then L is almost an inner automorphism of $M(n, k)$." While the statement of these results exhibits a large degree of similarity, an examination of the proofs reveals almost no similarity. The property in question could be determinant, nonsingularity, or orthogonality.

For examples of such results, see [4, 5, 6, 7, 8, 10]. In particular, an excellent survey is in [6].

We remark that we must assume L is invertible, since Wei's results [10] show that if singular maps are allowed, then pathological cases can occur.

1. Notation. Let K be an algebraically closed field of characteristic not 2. If n is a positive integer, K^n will be the vector space of n-tuples of elements of K, $M(n, k)$ the algebra of n-square matrices over K, $GL(n, K)$ the group of matrices in $M(n, K)$ with nonzero determinant, and $K[x_1, \cdots, x_n]$ the algebra of polynomials in n variables with coefficients in K.

2. Algebraic groups. A subset V of K^n is called an algebraic set if there exists an ideal $j(V) \subseteq K[x_1 \cdots x_n]$ such that

$$V = \{p \in K^n \colon f(p) = 0 \quad \text{for all } f \in j(V)\}.$$

The ideal $j(V)$ is called the ideal of V. If I is an ideal in $K[x_1, \cdots, x_n]$ then the ideal $\operatorname{rad}(I) = \{f \in K[x_1, \cdots, x_n] \colon f^r \in I$ for some non-negative integer $r\}$ is called the radical of I. We say V is an irreducible algebraic set if $\operatorname{rad} j(G) = j(G)$. If $p \in V$ then $M_p = \{f \in K[x_1 \cdots x_n] \colon f(p) = 0\}$ is a maximal ideal and if $p = (a_1, \cdots, a_n)$ we have $M_p =$

$\langle x_1 - a_1, \cdots, x_n - a_n \rangle$. Conversely, given a maximal ideal M in $K[x_1 \cdots x_n]$, we have that there exists a unique point $p \in K^n$ such that $M = M_p$.

If $V \subseteq K^n$ and $W \subseteq K^m$ are algebraic sets then a function $F: V \to W$ is called a polynomial function if there exist $F_1, \cdots, F_m \in K[x_1, \cdots, x_n]$ such that for all $v \in V$ $F(v) = (F_1(v), \cdots, F_m(v))$. The algebra of all polynomial functions $f: V \to K$ is called the co-ordinate ring of V and is denoted by $K[V]$. It is easy to see that

$$K[V] \cong K[x_1, \cdots, x_n]/j(V)$$

and the isomorphism is given by $f \leftrightarrow F + j(V)$ where $f \in K[V]$ and $F \in K[x_1 \cdots x_n]$ is such that $f(v) = F(v)$ for all $v \in V$.

If $V \subseteq K^n$ and $W \subseteq K^m$ are algebraic sets and $F: V \to W$ a polynomial function define $\bar{F}: K[W] \to K[V]$ by $\bar{F}(f) = f \circ F$. It is easily verified that $\bar{F}$ is an algebra homomorphism. If $\phi: K[W] \to K[V]$ is an algebra homomorphism define $\hat{\phi}: V \to W$ as follows: If $p \in V$ let M_p be the maximal ideal associated with M_p and $\mathcal{M}_p$ the image of M_p in $K[V]$. Then $\mathcal{M}_p$ is a maximal ideal in $K[V]$ and $\phi^{-1}(\mathcal{M}_p)$ is a maximal ideal in $K[W]$ so $\phi^{-1}(\mathcal{M}_p) = \mathcal{M}_q$ for some $q \in \mathcal{W}$. Set $\hat{\phi}(p) = q$. One verifies that $\hat{\phi}$ is a polynomial function. Further $\hat{\hat{\phi}} = \phi$ and $\bar{\bar{F}} = F$.

If $G \leqq \mathrm{GL}(n, K)$ we shall say G is an algebraic group if G is a subgroup of $\mathrm{GL}(n, K)$ and G is an algebraic set. If $g \in G$ and $x \in M(n, K)$ we say that x is tangent to G at g if

$$\frac{d}{dt} f(g + tx) \bigg|_{t=0} = 0 \quad \text{for all } f \in j(G).$$

The set of all tangents to G at g is called the tangent space to G at g and is denoted by $T_g(G)$ or simply T_g if G is understood. If 1 is the identity matrix we write $T_1(G) = \Omega$ and call Ω the tangent space to G. As is well known Ω is a Lie algebra and $T_g = g\Omega$.

3. Linear mappings on algebraic groups.

Let G be an algebraic group and suppose $L: M(n, K) \to M(n, K)$ is a linear mapping with the properties that L is nonsingular, $L(1) = 1$ and $L(G) \leqq G$.

PROPOSITION 3.1. *If $g \in G$ then $L(T_g) = T_{L(g)}$. Hence $L(g\Omega) = L(g)\Omega$.*

Proof. Clearly L is a polynomial function and since $L(G) \leqq G$ clearly $F \in j(G)$ implies $F \circ L \in j(G)$. Now suppose $x \in T_g$. Then

$$\frac{d}{dt} f(g + tx) \bigg|_{t=0} = 0 \quad \text{for all } f \in j(G).$$

Now, if $F \in j(G)$ we have

$$\frac{d}{dt}F((L(g)+tL(x)))\bigg|_{t=0} = \frac{d}{dt}(F(L(g+tx)))|_{t=0}$$

$$= \frac{d}{dt}(F \circ L(g+tx))\bigg|_{t=0}$$

$$= 0 \quad \text{since } F \circ L \in j(G), x \in T_g.$$

Hence $L(x) \in T_{L(g)}$. Since g and $L(g)$ are nonsingular, we may conclude that $\dim(\Omega) = \dim(g\Omega) = \dim(L(g)\Omega)$; and then since L is nonsingular, we may conclude that $L(g\Omega) = L(g)\Omega$.

PROPOSITION 3.2.   *If $g \in G$ then $L(g\Omega \cap \Omega) = L(g)\Omega \cap \Omega$.*

Proof.   Clearly $L(g\Omega \cap \Omega) \subseteq L(g\Omega) \cap L(\Omega) = $ (by 3.1) $L(g)\Omega \cap \Omega$. Conversely, suppose $u \in L(g)\Omega \cap \Omega$. Then $L^{-1}(u) \in L^{-1}(L(g)\Omega \cap \Omega)$ $= $ (by 3.1) $L^{-1}(L(g\Omega) \cap L(\Omega)) = $ (since L is a bijection) $g\Omega \cap \Omega$.

DEFINITION 3.3.   *Let r be a nonnegative integer and define*

$$G(r) = \{g \in G: \dim(g\Omega \cap \Omega) = r\}.$$

PROPOSITION 3.4. $L(G(r)) \subseteq G(r)$.

Proof.   If $g \in G(r)$, $\dim(g\Omega \cap \Omega) = r$; however we have $L(g\Omega \cap \Omega) = L(g)\Omega \cap \Omega$ and since L is nonsingular it follows that $r = \dim(g\Omega \cap \Omega) = \dim L(g\Omega \cap \Omega) = \dim(L(g)\Omega \cap \Omega)$.

4.   Extensions of fields and related algebraic groups.

PROPOSITION 4.1.   *Let $k \subseteq K$ be fields.   Suppose $G \leqq GL(n, k)$ is an algebraic group and $L: M(n, k) \to M(n, k)$ a nonsingular linear mapping with the property that $L(G) \subseteq G$.   Let G^K be the smallest algebraic subgroup of $\mathrm{GL}(n, K)$ containing G.   Then*

(a) $G^K \cap M(n, k) = G$
(b)   *there   exists   a   nonsingular   linear   mapping $L^K: M(n, K) \to M(n, K)$ with   the   properties   that $L^K(G^K) \subseteq G^K$ and $L^K | M(n, k) = L$.*

Proof.   Part (a) follows from Chevalley [2], Ch. II, Théorèm 3.   To establish part (b) choose a basis for $M(n, K)$ out of $M(n, k)$ and let L^K be the linear mapping of $M(n, K)$ defined by the matrix of L with respect to

this basis. Since the determinants of L and L^K are the same L^K is nonsingular and clearly $L^K | M(n, k) = L$. It only remains to show that $L^K(G^K) \subseteq G^K$. It is known [Chevalley (loc. cit.)] that the ideal associated with G^K is the ideal formed by all linear combinations of elements of $j(G)$ with coefficients from K. Now suppose f belongs to $j(G^K)$ and g belongs to G^K. Then $f = \sum_{i=1}^m a_i f_i$ for some a_i in K and f_i in $j(G)$; hence

$$f(L^K(g)) = \sum_{i=1}^m a_i f_i(L^K(g))$$

$$= \sum_{i=1}^m a_i \bar{L}^K(f_i)(g)$$

$$= \sum_{i=1}^m a_i \bar{L}(f_i)(g) = 0$$

since $L(f_i) \in j(G) \subseteq j(G^K)$ and $\bar{L}^K(f_i) = L(f_i)$ since $f_i \in j(G)$. Therefore $L^K(g)$ belongs to G^K.

5. The orthogonal groups

DEFINITION 5.1. *Let A be an invertible symmetric matrix in $M(n, k)$. Then G is an orthogonal group if $G = \{X \mid X'AX = A\}$.*

PROPOSITION 5.2. *Let $L: M(n, k) \to M(n, k)$ be linear and let $L(G) = G$ where G is an orthogonal group. If K is the algebraic closure of k, then $L^K(G^K) = G^K$.*

Proof. Clear from Proposition 4.1.

DEFINITION 5.3. *Let K be algebraically closed and G an orthogonal group in $M(n, K)$. Let $O(n, K) = \{X \mid X^T X = I_n\}$. If $L(G) = G$, define a linear map $\hat{L}: O(n, K) \to O(n, K)$ by the diagram*

$$O(n, K) \xrightarrow{\tilde{B}} G \xrightarrow{L} G \xrightarrow{\tilde{B}^{-1}} O(n, K).$$

In the diagram, $\tilde{B}$ is conjugation by a matrix B such that $BGB^{-1} = O(n, K)$. Such a matrix B exists because K is algebraically closed, and we may write $A = B^T B$.

PROPOSITION 5.4. *L is a G-conjugation if and only if $\hat{L}$ is an $O(n, K)$-conjugation. If $\hat{L}$ is the transpose map, then L is the transpose map followed by similarity by A.*

Proof. The first part is clear from the diagram. In the second case, let A be the matrix of Definition 5.1. Write $A = B^T B$. Then B is the matrix we need. *Let $U \in G$.* If $\hat{L}$ is the transpose map,

$$L(U) = \check{B}\hat{L}\check{B}^{-1}(U) = B^{-1}B^{-1^T}U^T B^T B = A^{-1}U^T A.$$

Clearly, since $U \in G$, $A^{-1}U^T A \in G$.

6. The main theorem. *Let K be any field of characteristic $\neq 2$, and let G be any orthogonal group in $M(n, K)$. If L is an invertible linear map on $M(n, K)$ such that $L(I_n) = I_n$, and $L(G) = G$, then L is conjugation by a matrix in $M(n, \tilde{K})$ where $\tilde{K}$ is the algebraic closure of K, or L is such a conjugation followed by the transpose map.*

REMARK 6.1. By §5, we will assume henceforth that K is algebraically closed and that $G = O(n, K)$.

REMARK 6.2. The assumption $L(I_n) = I_n$ is just a normalization of the problem. The Main Theorem can be modified slightly and then $L(I_n) = I_n$ can be dropped.

7. The characterization of symmetries. Recall that a symmetry in G is an involution sending a nonisotropic vector v to $-v$ and fixing pointwise the orthogonal compliment of v.

We wish to characterize symmetries and their negatives in a manner which will be invariant under L. Let Ω be the tangent space of G; in this case, since $G = O(n, K)$, Ω is all skew symmetric matrices. Let $V = K^n$ be the underlying vector space.

PROPOSITION 7.1. *Let $A \in G$, $A \neq \pm I_n$ $n \geq 3$. Then*

(1) $$\dim(A\Omega \cap \Omega) \leq (n-1)(n-2)/2$$

with equality if and only if $A = \pm$ a symmetry.

REMARK 7.2. In view of 3.4 and 7.1, any L satisfying $L(G) \subseteq G$ also sends symmetries to $\pm$ symmetries. This fact enables us to use the fundamental theorem of projective geometry in proving the main theorem for $n \geq 3$. The main theorem has to be proved separately for $n = 2$. Our proof of 7.1 will be by induction on n.

Now suppose equality holds in (1). We consider three cases.

Case 1. A has a non-isotropic eigenvector v. Let $Av = \lambda v$. Clearly $\lambda = \pm 1$. We are trying to show that $\pm A$ is a symmetry, so assume $\lambda = -1$. Now v is nonisotropic; thus A is G-conjugate to a matrix $B = -1 \dotplus B_1$. If $B_1 = I_{n-1}$, we are done; if not, we show B_1 is the

negative of a symmetry. Let Ω_1 be the $(n - 1) \times (n - 1)$ skew-symmetric matrices. By induction

$$\dim(B_1\Omega_1 \cup \Omega_1) \leqq (n - 2)(n - 3)/2.$$

Put

$$W = \left[\begin{array}{c|c} 0 & x \\ \hline -x' & W_1 \end{array} \right] \in \Omega.$$

Then $BW \in \Omega$ if and only if $B_1 W_1 \in \Omega_1$ and $B_1 x' = -x'$. The set of all x' such that $B_1 x' = -x'$ has dimension no larger than $n - 2$ and hence

$$\dim(B\Omega \cap \Omega) \leqq (n - 2)(n - 3)/2 + (n - 2)$$
$$= (n - 1)(n - 2)/2$$

with equality holding if and only if B_1 is $-$ a symmetry. This establishes Case 1.

In the final two cases we assume all eigenvectors of A are isotropic.

Case 2. A has two isotropic eigenvectors v, w which generate a hyperbolic plane. Let $Av = \lambda v$, $Aw = \mu w$, with $(w, w) = (v, v) = 0$, and $(w, v) = 1$. Thus, $\lambda\mu = 1$. Since G acts transitively on hyperbolic planes, A is G-conjugate to a matrix B of the form $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \dotplus B_1$ with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ having eigenvalues λ, μ.

If b or $c = 0$, B has non-isotropic eigenvectors. Thus, $bc \neq 0$.
Let $W \in \Omega$ and write

$$W = \left[\begin{array}{cc|c} 0 & w & \\ & & x \\ -w & 0 & \\ \hline & -x' & W_1 \end{array} \right]$$

where $x = \begin{bmatrix} x_3 \cdots x_n \\ y_3 \cdots y_n \end{bmatrix}$. If $BW \in \Omega$, we must have $w = 0$ and $B_1 W_1$ skew. The set of all W_1 for which this is true has dimension $< (n - 3)(n - 4)/2$ because B_1 has no non-isotropic eigenvectors. Since $bc \neq 0$, there is, for each $i = 3, \cdots, n$, a dependence relation between y_i and all x_i. It follows that

$$\dim(B\Omega \cap \Omega) < \frac{(n-3)(n-4)}{2} + n - 2$$

$$= \frac{(n-1)(n-2)+6-2n}{2}$$

$$\leqq (n-1)(n-2)/2$$

if $n \geqq 3$. This finishes Case 2.

Case 3. All eigenvectors of A are isotropic and any two eigenvectors of A are orthogonal. It follows that the subspace U of V spanned by the eigenvectors of A is totally isotropic and hence its dimension is $\leqq [n/2]$.

Now if W and $AW \in \Omega$, then $(AW)' = -AW$, i.e., $AW - WA' = 0$. Thus the map $H = I_n \otimes A - A \otimes I_n$ has W in its null space. Therefore, we must show that the restriction of H to $\overset{2}{\wedge} V$ has nullity $< (n-1)(n-2)/2$. Call this restriction $\hat{H}$. Clearly this nullity is maximized if all eigenvalues of H are the same, say all are 1.

Let $(\lambda - 1)^{m_i}$, $i = 1, \cdots, r$, be the elementary divisors of H. For $v, w \in V$, put $v \circ w = (v \otimes w + w \otimes v)/2$. The span of all $v \circ w$ is $V^{(2)}$, the symmetric tensors. Clearly $\hat{H}; \overset{2}{\wedge} V \to V^{(2)}$. If $(\lambda - 1)^m$ is an elementary divisor of A, pick $v_1, \cdots, v_m \in V$ such that $Av_1 = v_1$, $Av_i = v_i + v_{i-1}$, $i = 2, \cdots, m$. By looking at $H(v_1 \wedge v_i)$, we see that $v_1 \circ v_1, \cdots, v_1 \circ v_{m-1} \in \text{rng } \hat{H}$. Suppose $(\lambda - 1)^p$ is another elementary divisor of A and pick $w_1, \cdots, w_p \in V$ as we did the v_i. By looking at $\hat{H}(v_1 \wedge w_i)$ and $\hat{H}(v_i \wedge w_1)$, we note that $v_1 \circ w_i$ and $v_j \circ w_1 \in \text{rng } \hat{H}$ for $i = 1, \cdots, p-1, j = 1, \cdots, m-1$. It follows that the rank of $\hat{H}$ is at least

$$\sum_{i<j} (m_i - 1) + (m_j - 2) + \sum_{i=1}^{r} (m_i - 1) = (r-1)n - 3\binom{r}{2} + n - r$$

$$= r(n - (3r-1)/2).$$

A little calculus shows that this value is minimized when r is at its extremes, i.e., when $r = 1$ or $[n/2]$. When $r = 1$, we get rank $\hat{H} \geqq n - 1$. When $r = [n/2]$, we get rank $\hat{H} > n^2/8 + n/4$.

If $r = 1$, let $Av_1 = v_1$, $Av_i = v_{i-1}$, $i = 2, \cdots, n$. Look at $\hat{H}(v_2 \wedge v_3)$ ($n \geqq 3$) to see that rank $\hat{H} \geqq n$. Then nullity $\hat{H} \leqq (n^2 - n)/2 - n < (n-1)(n-2)/2$.

If $r = [n/2]$, then nullity $\hat{H} \leqq (n^2 - n)/2 - n^2/8 - n/4 < (n-1)(n-2)/2$, unless $n = 3$ or 4. If $n = 3$, then $r = 1$, and we are done. Let $n = 4$. If $(\lambda - 1)$, $(\lambda - 1)^3$ are the elementary divisors of A, pick a basis v, w, x, y of V with $Av = v$, $Aw = w$, $Ax = x + w$,

$Ay = y + x$. It is easy to show that $\langle v, w, x \rangle$ is totally isotropic. If $(\lambda - 1)^2, (\lambda - 1)^2$ are the elementary divisors of A, pick a basis v, w, x, y of V with $Av = v$, $Aw = w + v$, $Ax = x$, $Ay = y + x$. Then direct computation shows that $\operatorname{rng} \hat{H} = \langle v \circ v, x \circ x, v \circ x, w \circ x - v \circ y \rangle$. Thus, rank $\hat{H} = 4$ and nullity $\hat{H} = 2 < (4 - 1)(4 - 2)/2$.

We still have to do cases 1 and 2 when $n = 3$. Then these two cases coincide. Let v be a nonisotropic eigenvector and assume $Av = v$. If $A \neq \pm I_3$ nor $\pm$ a symmetry, there are two possibilities for $\langle v \rangle^\perp$: $\langle v \rangle^\perp = \langle u, w \rangle$ and $Au = \lambda u$, $Aw = \lambda^{-1} w$, $\lambda \neq \lambda^{-1}$, or $Au = \pm u$, $Aw = \pm w + u$. In the first case, $\hat{H}$ has nullity 0. In the second case, V is degenerate. This concludes the proof of 7.1.

8. The fundamental theorem of projective geometry. We now know that L maps every symmetry to another symmetry or its negative. Thus, since L maps G onto itself, L sets up a $1 - 1$ correspondence ϕ between the nonisotropic lines of V. We will extend ϕ to all lines of V.

PROPOSITION 8.1. *There is an extension of ϕ to all lines of V such that ϕ satisfies the hypothesis of the Fundamental Theorem of Projective Geometry.*

REMARK. Recall the Fundamental Theorem of Projective Geometry. If $P(V)$ is projective space, and ϕ is a bijection of the points of $P(V)$ preserving collinearitry, then ϕ is induced by a semilinear map on V.

Let L_1, L_2, L_3 be nonisotropic lines and let $M_i = \phi(L_i)$. We need to show that ϕ preserves orthogonality and coplanarity among L_1, L_2, L_3. If $n \geq 4$, the proof will then follow from the same procedure as in [3, p. 48]. If $n = 3$, we do things a little differently.

Let σ_i be the symmetry of L_i and τ_i the symmetry of M_i, $i = 1, 2, 3$.

We now prove the following.

PROPOSITION 8.2. *If σ_1, σ_2, σ_3 and L_1, L_2, L_3 are as above, then*

$$\dim(\sigma_1 \Omega \cap \sigma_2 \Omega \cap \sigma_3 \Omega \cap \Omega) \leqq (n - 2)(n - 3)/2$$

with equality if and only if L_1, L_2, and L_3 lie in the same plane.

Proof. Pick an orthonormal basis $v_1, \cdots, v_n$ such that $\langle v_1 \rangle = L_1$, and $L_1, L_2 \in \langle v_1, v_2, v_3 \rangle$ ($\langle L_1, L_2 \rangle$ may be degenerate). Represent σ_1 with $A_1 = -1 \dotplus I_{n-1}$, σ_2 with $A_2 = B \dotplus I_{n-3}$ and σ_3 with $A_3 = \begin{bmatrix} R & S \\ S' & U \end{bmatrix}$, where R is 3×3. If

$$W = \begin{bmatrix} W_1 & W_2 \\ -W_2' & W_4 \end{bmatrix}$$

is a skew matrix such that A_1W and A_2W are skew, then an easy computation shows that the first row and column of W are 0 and that $BW_2 = W_2$. Thus, $\dim(\Omega \cap \sigma_1\Omega \cap \sigma_2\Omega) = (n-2)(n-3)/2$ and we are done in one direction.

Now if A_3W is also skew, for all $W \in \Omega \cap \sigma_1\Omega \cap \sigma_2\Omega$, then we obtain $U = I_{n-3}$, $S = 0$ and $RW_2 = W_2$. It follows that σ_1, σ_2, σ_3 have a common fixed $(n-2)$-dimensional subspace and thus L_1, L_2, L_3 are in the same plane. This proves our result.

We now need to show that ϕ preserves orthogonality of nonisotropic lines.

PROPOSITION 8.3. *Suppose* $\dim V = 2$ *and* σ_1, σ_2 *are distinct symmetries. If for every symmetry* $\sigma_3 \neq \sigma_1$ *or* σ_2, $\sigma_1 + \sigma_2 - \sigma_3$ *is also a symmetry, then* σ_1 *and* σ_2 *reflect in orthogonal lines.*

Proof. Let $\sigma_1v_1 = -v_1$ and pick v_2 so that $v_1 \perp v_2$. Let $\sigma_2v_1 = av_1 + bv_2$, $\sigma_2v_2 = bv_1 - av_2$, and $a^2 + b^2 = 1$. If σ_3 is any other symmetry, we have $\sigma_3v_1 = cv_1 + dv_2$, $\sigma_3v_2 = dv_1 - cv_2$. Thus, for all c and d such that $c^2 + d^2 = 1$, we have

$$\det \begin{pmatrix} 1 + a + c & b + d \\ b + d & -1 - a - c \end{pmatrix} = -1.$$

This can happen only if $a = -1$.

COROLLARY 8.4. *Let* σ_1, σ_2 *be two symmetries in an* n-*dimensional nondegenerate space* V, $n \geq 3$. *Let* L_1, L_2 *be the corresponding lines. If, for every symmetry* $\sigma_3 \neq \sigma_1$ *or* σ_2, *whose line of reflection is in the plane determined by* L_1, L_2, $\sigma_1 + \sigma_2 - \sigma_3$ *is a symmetry, then* L_1 *is orthogonal to* L_2.

The next result finishes Proposition 8.1.

PROPOSITION 8.5. *The map* ϕ *preserves orthogonality.*

Proof. Let σ_1, σ_2 be orthogonal symmetries. Let σ_3 be any symmetry whose line is in the plane corresponding to σ_1, σ_2. Let τ_i be the images. By Proposition 8.2, the lines of τ_1, τ_2, τ_3 lie in a plane. We now apply Corollary 8.4 to finish.

We can now follow the procedures of Dieudonné to extend ϕ to all lines of V. This extension satisfies the hypotheses of the Fundamental Theorem of Projective Geometry and also preserves orthogonality. To ϕ, we associate a map ψ in the projective space $P(V)$, and conclude that ψ is induced by a semilinear map $g : V \to V$. Let ρ be the automorphism of K corresponding to g. We can also show that for all $u, v \in V$,

$$(g(u), g(v)) = \lambda(u, v)^\rho$$

where λ is independent of u and v. It follows that if A is a symmetry, then

(3) $$L(A) = \pm g A g^{-1}.$$

Now L is linear; thus $\rho = $ identity and g is linear.

PROPOSITION 8.6. *The $\pm$ sign is always $+$, that is, symmetries are mapped to symmetries.*

Proof. Let σ_1 be a symmetry. Let $\sigma_2, \cdots, \sigma_n$ be symmetries such that the n associated lines are pairwise orthogonal. Then $\Sigma \sigma_i = (n - 2)I_n$, and since $L(I_n) = I_n$, we are done.

Fix the standard basis $E = \{e_1, \cdots, e_n\}$ and represent the symmetries as matrices with respect to E. The linear span of the symmetries is the space S of all symmetric matrices in $M(n, K)$. Thus, on S, the map L is conjugation. Hence, in Chapter 10, we will assume that L fixes S pointwise.

9. The cases $n = 2, 3$. We must handle $n = 2, 3$ separately.

PROPOSITION 9.1. *The main Theorem is valid when $n = 2$.*

Proof. Every member of G is in one of two subspaces of $M(2, K)$

$$V_1 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$$V_2 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Now $\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$ is the tangent space and hence is invariant under L. Thus, $L(V_1) = V_1$ and hence $L(V_2) = V_2$. Since skews are mapped to skews, $L\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Conjugate with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ if necessary to assume that V_1 is pointwise fixed.

Let $L\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Since $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ is a symmetry, it is

G-conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and hence we may assume $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is fixed

and V_1 is still pointwise fixed, by taking the transpose if necessary.

Put $L\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. Then $\begin{pmatrix} a+1 & b \\ b & -a-1 \end{pmatrix} \in$

$\sqrt{2}\,G$. Thus, $a = 0$, $b = \pm 1$. If $b = 1$, we are done. If $b = -1$,

conjugate with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and then apply the transpose map.

PROPOSITION 9.2. *If $n = 3$, the restriction of L to the symmetric matrices is a G-conjugation.*

Proof. By Proposition 8.6, every symmetry σ is mapped to a symmetry. Let $\langle v_1 \rangle$ be the line of reflection of σ and let v_1, v_2, v_3 be an orthonormal basis of V. We may assume $L(\sigma) = \sigma$. Since L preserves orthogonality, (Proposition 8.5 works when $n = 3$) any symmetry reflecting in $\langle v_2, v_3 \rangle$ is mapped to a symmetry reflecting in $\langle v_2, v_3 \rangle$. Thus, from Proposition 9.1, we have that the space

$$\begin{pmatrix} c & 0 & 0 \\ 0 & a & b \\ 0 & b & d \end{pmatrix}$$

is acted upon by L by G-conjugation. Hence, we assume this space is fixed pointwise.

Consider the image B of the symmetry

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since orthogonality of nonisotropic lines is preserved, B commutes with $\operatorname{diag}(c, c, -c)$ for all c. Thus

$$L(B) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A similar argument on

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

finishes the proof.

10. The tangent space.   We have $\Omega =$ the tangent space of G, i.e., Ω is the space of skew-symmetric matrices.   We need to show that $L \mid \Omega = \pm$ identity.   For $i < j$, let F_{ij} be the $n \times n$ matrix whose (i, j) entry is 1, whose (j, i) entry is -1, and all other entries are 0.

PROPOSITION 10.1.   *If for all* $i < j$, $L(F_{ij}) = \pm F_{ij}$, *then* $L \mid \Omega$ *is* $\pm$ *identity.*

Proof. Use induction on n. We are done if $n = 2$. Let $A_1 \in M(n-1, K)$, A_1 skew. Then $0 \dotplus A_1 \to 0 \dotplus B_1$, $B_1 \in M(n-1, K)$. By induction, $B_1 = A_1$ for all skew A_1. Suppose $L(F_{12}) = -F_{12}$.   Note that

$$
\begin{bmatrix}
0 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\[2mm]
-\dfrac{1}{\sqrt{2}} & \dfrac{1}{2} & -\dfrac{1}{2} \\[2mm]
\dfrac{1}{\sqrt{2}} & \dfrac{1}{2} & -\dfrac{1}{2}
\end{bmatrix} \in G.
$$

However, all symmetric matrices are fixed, and hence the above matrix is mapped to

$$
\begin{bmatrix}
0 & -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\[2mm]
\dfrac{1}{\sqrt{2}} & \dfrac{1}{2} & -\dfrac{1}{2} \\[2mm]
\dfrac{1}{\sqrt{2}} & \dfrac{1}{2} & -\dfrac{1}{2}
\end{bmatrix} \notin G.
$$

The above argument may be repeated for all F_{ij}.

PROPOSITION 10.2.   *For all* $i < j$, $L(F_{ij}) = \pm F_{ij}$.

Proof.   Let $A = \text{diag}(\epsilon_1, \cdots, \epsilon_{i-1}, 0, \epsilon_{i+1}, \cdots, \epsilon_{j-1}, 0, \epsilon_{j+1}, \cdots, \epsilon_n)$ where the ϵ_k run through ± 1.   Put $B = L(F_{ij})$.   Now $A \pm B \in G$.   Thus,

$$AA' + BB' = I_n$$

$$AB' + BA' = 0.$$

It follows that B has the appropriate form and the proposition is proved. This also concludes the proof of the main Theorem.

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