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SUFFICIENCY OF JETS

JEAN-JACQUES GERVAIS

We give a necessary and sufficient condition for the C^∞-sufficiency of a jet: this generalizes and improves some results of J. N. Mather and J. C. Tougeron. Our result, given in terms of G-sufficiency which is a generalization of the ordinary sufficiency, can be applied to many cases.

NOTATIONS. Let G be a q-dimensional Lie subgroup of $Gl_p(\boldsymbol{R})$. Let $G(n) = C^\infty_{0,e}(\boldsymbol{R}^n, G)$ be the group of germs at 0 of smooth mappings g from $\boldsymbol{R}^n$ to G such that $g(0) = e$ (where e is the identity of G) and Diff(n) the group of germs at 0 of smooth diffeomorphisms τ from a neighborhood of 0 in $\boldsymbol{R}^n$ on a neighborhood of 0 in $\boldsymbol{R}^n$ such that $\tau(0) = 0$. Let $\mathscr{E}_n$ be the ring of germs at 0 of smooth functions from $\boldsymbol{R}^n$ to $\boldsymbol{R}$ and m its maximal ideal. For $f \in \bigoplus_p m$, $j^r(f)$ will denote the r-jet of f at 0. The set $\mathscr{G}(n) = G(n) \times$ Diff(n) is a group with the following multiplication: $(g_1, \tau_1)\cdot(g_2, \tau_2) = (g_1\cdot(g_2\circ\tau_1^{-1}), \tau_1\circ\tau_2)$. Then we may define an action of $\mathscr{G}(n)$ on $\bigoplus_p m$ by the formula: for $(g, \tau) \in \mathscr{G}(n)$ and $f \in \bigoplus_p m$, $(g, \tau)\cdot f$ is the germ at 0 of the mapping $x \mapsto \tilde{g}(x)\cdot(\tilde{f}\circ\tilde{\tau}^{-1}(X))$ where $\tilde{g}, \tilde{f}$, and $\tilde{\tau}$ are representatives of g, f, and τ respectively.

DEFINITION 1. An r-jet z of an element of $\bigoplus_p m$ is G-sufficient if for any $f \in \bigoplus_p m$ such that $j^r(f) = z$ there exists $(g, \tau) \in \mathscr{G}(n)$ such that $(g, \tau)\cdot f = z$.

REMARK. When $G = \{e\}$ and $p = 1$ the G-sufficiency is the ordinary C^∞-sufficiency of jets.

We will use the well known:

NAKAYAMA'S LEMMA. *Let A be a commutative ring with identity and let I be an ideal in A such that $1 + a$ is invertible for any $a \in I$. Let M and N be submodules of an A-module P such that M is finitely generated and $M \subset N + I.\ M$. Then $M \subset N$.*

Jets G-sufficient. Let $\{A_1, \cdots, A_q\}$ be a base over $\boldsymbol{R}$ of the Lie algebra T_eG of G. For every $g \in G(n)$ there exists $u = (u_1, \cdots, u_q) \in \bigoplus_q m$ such that

$$g(x) = e^{\sum_{i=1}^{q} u_i(x).A_i}\ .$$

Hence we may identify $G(n)$ with $\bigoplus_q m$.

Let $\mathscr{G}^r$ be the analytic Lie group of the r-jets of the elements of $\mathscr{G}(n)$ and let X^r be the space of r-jets of the elements of $\bigoplus_p m$. The group action of $\mathscr{G}(n)$ on $\bigoplus_p m$ induces, for each r, a well defined group action of $\mathscr{G}^r$ on X^r. One easily sees that this group action is analytic for each r.

For $f \in \bigoplus_p m$, let M_f be the $\mathscr{E}_n$-linear mapping:

$$M_f\colon \mathscr{E}_n^{p+q} \longrightarrow \mathscr{E}_n^p ,$$

where M_f is given by the $p \times (q+n)$-matrix with $A_1 \cdot f, \cdots, A_q \cdot f$, $\partial f/\partial x_1, \cdots, \partial f/\partial x_n$ as columns. It is easily seen that for $f \in \bigoplus_p m$ the mapping

$$\bar{M}_f^r\colon \bigoplus_{q+n}\left(\frac{m}{m^{r+1}}\right) \longrightarrow \bigoplus_p\left(\frac{m}{m^{r+1}}\right),$$

derived from M_f, is the tangent mapping at the idendity of the mapping

$$\mathscr{G}^r \ni \gamma \longrightarrow \gamma \cdot j^r(f) \in X^r .$$

THEOREM 1. *Let $z \in X^r$. The following statements are equivalent:*

(i) *z is G-sufficient.*

(ii) *For any homogeneous jet w of degree $r+1$ we have* $m \cdot \operatorname{Im} M_{z+w} \supset m^{r+1} \cdot \mathscr{E}_n^p$ (where $\operatorname{Im} M_{z+w}$ is the range of M_{z+w}).

Proof.

(i) $\Rightarrow$ (ii) Let w and w' be two homogeneous jets of degree $r+1$. Since z is G-sufficient, there exist (g, τ) and $(g', \tau') \in \mathscr{G}(n)$ such that $(g, \tau)\cdot z = z + w$ and $(g', \tau')\cdot z = z + w'$; hence $(g', \tau')\cdot(g, \tau)^{-1}\cdot(z + w) = z + w'$.

Consequently, if we put $\gamma = j^{r+1}((g', \tau')\cdot(g, \tau)^{-1})$, we have $\gamma\cdot(z + w) = z + w'$. We have thus shown that for any homogeneous jet w of degree $r+1$ the $\mathscr{G}^{r+1}$-orbit of $z + w$ in X^{r+1} contains $\{z + w' \mid w' \text{ is a homogeneous jet of degree } r + 1\}$. Since the tangent mapping at the identity of the mapping $\mathscr{G}^{r+1} \ni \gamma \mapsto \gamma\cdot(z+w) \in X^{r+1}$ is

$$\overline{M_{z+w}^{r+1}}\colon \bigoplus_{q+n}\left(\frac{m}{m^{r+2}}\right) \longrightarrow \bigoplus_p\left(\frac{m}{m^{r+2}}\right),$$

derived from M_{z+w}, we have $\operatorname{Im} \overline{M_{z+w}^{r+1}} \supset \bigoplus_p (m^{r+1}/m^{r+2})$, i.e., $m \cdot \operatorname{Im} M_{z+w} + m^{r+2}\cdot \mathscr{E}_n^p \supset m^{r+1}\cdot \mathscr{E}_n^p$. From the Nakayama's lemma, we conclude that $m\cdot\mathrm{Im}\, M_{z+w}\supset m^{r+1}\cdot\mathscr{E}^p_n$.

(ii) $\Rightarrow$ (i)

(a) Let $w_1, \cdots, w_k$ be homogeneous jets of degree $r+1, \cdots, r+k$ respectively and put $z'=\sum_{i=1}^k w_i$. Let $t_0\in[0,1]$. By hypothesis,

$$m\cdot\mathrm{Im}\, M_{z+t_0w_1}\supset m^{r+1}\cdot\mathscr{E}^p_n\ .$$

Hence we have

$$m^{r+1}\cdot\mathscr{E}^p_n\subset m\cdot\mathrm{Im}\, M_{z+t_0w_1}\subset m\cdot\mathrm{Im}\, M_{z+t_0z'}+m^{r+2}\cdot\mathscr{E}^p_n\ .$$

Nakayama's lemma implies

$$m^{r+1}\cdot\mathscr{E}^p_n\subset m\cdot\mathrm{Im}\, M_{z+t_0z'}\ .$$

Then the range of the mapping $\mathscr{G}^{r+k}\ni\gamma\mapsto\gamma\cdot(z+t_0z')$ contains all $r+k$-jets $z+z''$, where z'' is an $r+k$-jet in a neighborhood of t_0z' such that $j^r(z'')=0$. In particular, there exist $t_1<t_0<t_2$ such that for all t' and $t''\in[t_1,t_2]$, there exists $(g,\tau)\in\mathscr{G}(n)$ such that $j^{s+k}((g,\tau)\cdot(z+t'z'))=z+t''z'$. Since $[0,1]$ is compact, it follows that there exists $(g,\tau)\in\mathscr{G}(n)$ such that $j^{s+k}((g,\tau)\cdot(z+z'))=z+0\cdot z'=z$.

(b) Let $f\in\bigoplus_p m$ such that $j^r(f)=z$, we must prove that there exists $(g,\tau)\in\mathscr{G}(n)$ such that $(g,\tau)\cdot f=z$. We have

$$m^{r+1}\cdot\mathscr{E}^p_n\subset m\cdot\mathrm{Im}\, M_{j^{r+1}(f)}\ .$$

Hence

$$m^{r+1}\cdot\mathscr{E}^p_n\subset m\cdot\mathrm{Im}\, M_{j^{r+1}(f)}\subset m\cdot\mathrm{Im}\, M_f+m^{r+2}\cdot\mathscr{E}^p_n\ .$$

Nakayama's lemma implies

$$m^{r+1}\cdot\mathscr{E}^p_n\subset m\cdot\mathrm{Im}\, M_f\ .$$

It follows from a result of J. C. Tougeron [2, Théorème VIII 3.6] that there exists $N\in\boldsymbol{N}$ such that $j^N(f)$ is G-sufficient. If $N\leqq r$ the proof is finished. Suppose $N>r$. By(a), there exist $(g_1,\tau_1)\in\mathscr{G}(n)$ and $\phi\in m^{N+1}\cdot\mathscr{E}^p_n$ such that

$$z=(g_1,\tau_1)\cdot j^N(f)+\phi;\ \text{hence}$$
$$z=(g_1,\tau_1)\cdot[j^N(f)+(g_1,\tau_1)^{-1}\cdot\phi]\ .$$

Since $\phi\in m^{N+1}\cdot\mathscr{E}^p_n$, $(g_1,\tau_1)^{-1}\cdot\phi\in m^{N+1}\cdot\mathscr{E}^p_n$. But $j^N(f)$ is G-sufficient, consequently there exists $(g_2,\tau_2)\in\mathscr{G}(n)$ such that

$$j^N(f)+(g_1,\tau_1)^{-1}\cdot\phi=(g_2,\tau_2)\cdot f\ .$$

Hence

$$z = (g_1, \tau_1)\cdot(g_2, \tau_2)\cdot f \ .$$

DEFINITION 2. Let $f \in m$. We say that f is r-determined if $j^r(f)$ is C^∞-sufficient (i.e., G-sufficient with $G = \{e\}$).

From Theorem 1 we deduce the following two results of J. N. Mather [1], stated as follows in [3, Theorem 2.6 and Corollary 2.10]:

THEOREM 2. *Let $f \in m$ and I_f be the ideal generated in $\mathscr{E}_n$ by the partial derivatives of f. If*

$$m^r \subset m\cdot I_f + m^{r+1} \ ,$$

then f is r-determined.

THEOREM 3. *Let $f \in m$ be r-determined. Then*

$$m^{r+1} \subset m\cdot I_f \ .$$

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UNIVERSITÉ LAVAL
QUEBEC, G1K 7P4, CANADA