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CHARACTERS OF P'-DEGREE IN SOLVABLE GROUPS

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We prove that $|I_p(G)| = |I_p(N(P))|$ for $P \in \mathrm{Syl}\,(G)$, for solvable G. Here p is a prime and $I_p(G)$ is the set of irreducible characters ψ such that $(\psi(1), p) = 1$.

1. Introduction. The groups considered are finite and the group characters are defined over the complex numbers. McKay conjectured $|I_p(G)| = |I_p(N(P))|$ where $P \in \mathrm{Syl}\,(G)$ for simple G and $p = 2$ [6]. I. M. Isaacs has proven the result when $|G|$ is odd and p is any prime (Theorem 10.9 of [4]). We prove the result for solvable G. In fact we generalize this slightly to sets of primes and normalizers of Hall subgroups.

For characters χ and ψ of G, we let $[\chi, \psi]$ denote the inner product of χ and ψ. Let $N \trianglelefteq G$ and $\theta \in IRR(N)$. We write $I_G(\theta)$ to denote the inertia group $\{g \in G \mid \theta^g = \theta\}$. We also write $IRR(G \mid \theta) = \{\chi \in IRR(G) \mid [\chi_N, \theta] \neq 0\}$. Of course, character induction yields a one-to-one map from $IRR(I_G(\theta) \mid \theta)$ onto $IRR(G \mid \theta)$. If $\chi \in IRR(G \mid \theta)$; we say χ (or θ) is fully ramified with respect to G/N if $\chi_N = e\theta$ and $e^2 = |G: N|$. This will occur if $I_G(\theta) = G$ and χ vanishes off N.

Suppose that K/L is an abelian chief factor of G; $\gamma \in IRR(K)$; $\phi \in IRR(L)$; and $[\gamma_L, \phi] \neq 0$. If $K \cdot I_G(\phi) = G$, then one of the following occur:

(a) $\gamma_L = \phi$;

(b) γ and ϕ are fully ramified with respect to K/L, or

(c) $\phi^K = \gamma$.

We note that $K \cdot I_G(\phi) = G$ whenever $I_G(\gamma) = G$. The results of these last two paragraphs are well known (e.g. see Chapter 6 of [5]); and we will use them without reference. In Theorem 3.3, we use known results about character triple isomorphisms (see §8 of [4] or Chapter 11 of [5]); otherwise, everything should be self-explanatory.

I would like to thank E. C. Dade for his preprint [1].

2. Extendability. A straightforward proof of Lemma 2.1 may be found in Lemma 10.5 of [4].

LEMMA 2.1. *Assume $N \trianglelefteq G$, $H \leqq G$, $NH = G$, and $N \cap H = M$. Assume $\phi \in IRR(N)$ is invariant in G and $\phi_M \in IRR(M)$. Then $\chi \leftrightarrow \chi_H$ defines a one-to-one correspondence between $IRR(G \mid \phi)$ and $IRR(H \mid \phi_M)$.*

Theorem 2.2 is a generalization of a result of Dade. He proves the theorem when E is an extra-special p-group and when $p \nmid |L|$ (see Theorems 1.2 and 1.4 of [1]). We use his result to prove this.

THEOREM 2.2. *Assume* (i) G *is the semi-direct product* EH, $E \trianglelefteq G$.

(ii) $1 < \boldsymbol{Z}(E) \leqq \boldsymbol{Z}(G)$ *and* $\boldsymbol{Z}(E)$ *is cyclic;*

(iii) $E/\boldsymbol{Z}(E)$ *is an elementary abelian p-group for some prime* p;

(iv) $[L, E/\boldsymbol{Z}(E)] = E/\boldsymbol{Z}(E)$ *for some* $L/C_H(E) \trianglelefteq H/C_H(E)$ *such that* $p \nmid |L/\boldsymbol{C}_H(E)|$; *and*

(v) $\Lambda \in IRR(E)$ *is faithful.*

Then Λ *extends to an irreducible character* ψ *of* G *such that* $C_G(H) \leqq \ker(\psi)$.

Proof. We may extend Λ to an irreducible character of $E \times \boldsymbol{C}_H(E)$ with kernel $\boldsymbol{C}_H(E)$. It is no loss to assume $\boldsymbol{C}_H(E) = 1$. If $E' = \boldsymbol{Z}(E)$, we finish by Dade's result. We assume $E' < \boldsymbol{Z}(E)$.

Fittings lemma (Theorem 5.2.3 of [3]) implies $E/E' = F/E' \times \boldsymbol{C}_{E/E'}(L)$ where $F/E' = [E/E', L]$. As $p \nmid |L|$, the hypotheses yield $\boldsymbol{Z}(E)/E' = \boldsymbol{C}_{E/E'}(L)$. Note $E' = \boldsymbol{Z}(F)$ and $E/\boldsymbol{Z}(E)$ is isomorphic to F/E'.

Let ϕ be the irreducible constituent of $\Lambda_{Z(E)}$. As $\phi_{E'} \in IRR(E')$, Lemma 2.1 yields $\Lambda_F \in IRR(F)$. By induction on $|G|$, Λ_F extends to some $\beta \in IRR(FH)$. If $I_G(\Lambda) = G$, we have by Lemma 2.1 that $\beta = \psi_{FH}$ for some $\psi \in IRR(G/\Lambda)$. Furthermore, $\psi(1) = \Lambda(1)$. We are done as long as $I_G(\Lambda) = G$. Note that Λ_F and ϕ are H-invariant. So, if $h \in H$, $\Lambda^h = \alpha\Lambda$ for a linear $\alpha \in IRR(E/F)$. This implies $\phi^h = \alpha_{Z(E)}\phi$ and $\alpha_{\boldsymbol{Z}(E)} = 1_{\boldsymbol{Z}(E)}$. So $\alpha = 1_E$, completing the proof.

The following theorem also generalizes a result of Dade (see Theorem 5.10 of [1]).

THEOREM 2.3. *Assume* (i) $G = EH$, $E \triangleleft G$, $E \cap H = Z(E)$ *is in* $\boldsymbol{Z}(G)$;

(ii) $1 \neq \boldsymbol{Z}(E)$ *is cyclic;*

(iii) $E/\boldsymbol{Z}(E)$ *is an elementary abelian p-group for a prime* p;

(iv) $[L, E/\boldsymbol{Z}(E)] = E/\boldsymbol{Z}(E)$ *for some* $\boldsymbol{C}_H(E) \leqq L \trianglelefteq H$ *such that* $p \nmid |L/\boldsymbol{C}_H(E)|$; *and*

(v) λ *is a faithful character of* $\boldsymbol{Z}(E)$.

Then there exists a one-to-one correspondence $T: IRR(G|\lambda) \to IRR(H|\lambda)$ *such that for* $\chi \in IRR(G|\lambda)$, $\chi(1) = e[(\chi T)(1)]$ *where* $e = |E: Z(E)|^{1/2} \in Z$.

Proof. Let $\Lambda \in IRR(E|\lambda)$. As E is nilpotent and λ is faithful, Λ is faithful. If $\boldsymbol{Z}(E) < T < E$ with $|T: \boldsymbol{Z}(E)|$ prime, Λ_T has each extension of λ to T as a costituent. It follows that Λ vanishes on

$E - \boldsymbol{Z}(E)$. So Λ and λ are fully ramified with respect to $E/\boldsymbol{Z}(E)$ and $I_G(\Lambda) = G$.

Let H_1 be an isomorphic copy of H; say $\sigma: H \to H_1$ is an isomorphism. Say $\boldsymbol{Z}(E) = \langle x \rangle$ and $\sigma(x) = x_1$. From the semidirect product $G_1 = E \cdot H_1$. Note, by Theorem 2.2, Λ extends to $\psi \in IRR(G_1)$.

Let $Z_0 = \langle x \rangle \times \langle x_1 \rangle \leqq G_1$. Define $\lambda_1 \in IRR(\langle x_1 \rangle)$ by $\lambda_1(x_1) = \lambda(x)$. Define $\tau: G_1 \to G$ by $\tau(tg) = t \cdot \sigma^{-1}(g)$ for $t \in E$, $g \in H_1$. Then τ is a homomorphism onto G with kernel $Z_1 < Z_0$. So $\tau: G/Z_1 \to G$ is an isomorphism, $\tau(\langle x \rangle \times \langle x_1 \rangle) = \boldsymbol{Z}(E)$, and $(\lambda \times \lambda_1)^\tau = \lambda$, viewing τ as mapping $IRR(Z_0/Z_1)$ to $IRR(\boldsymbol{Z}(E))$.

Hence, we need just show there is a one-to-one correspondence $T: IRR(G_1 | \lambda \times \lambda_1) \to IRR(H_1 | \lambda_1)$ such that $\chi(1) = e[(\chi T)(1)]$.

If $\beta \in IRR(H_1)$, then β is β^* restricted to H_1 for a unique $\beta^* \in IRR(G_1/E)$. Now $\beta \to \beta^* \psi$ defines a one-to-one correspondence from $IRR(H_1)$ onto $IRR(G_1 | \Lambda) = IRR(G_1 | \lambda)$. As $\psi(1) = e$, it suffices to show for $\beta \in IRR(H_1)$ that $\beta \in IRR(H_1 | \lambda_1)$ if and only if $\boldsymbol{Z}_1 \leqq \ker(\beta^* \psi)$. If μ is the irreducible constituent of β restricted to $\langle x_1 \rangle$, then $\beta^* \psi(x, x_1^{-1}) = e\beta(1)\lambda(x)\mu^{-1}(x)$. So $\boldsymbol{Z}_1 \leqq \ker(\beta^* \psi)$ if and only if $\mu = \lambda_1$, completing the proof.

3. The McKay conjecture. If π is a set of primes, let $I_\pi(G) = \{\chi \in IRR(G) | (p, \chi(1)) = 1$ for all $p \in \pi\}$. Now G is π-solvable if G has a normal series where each factor is either a π'-group or a solvable π-group. If G is π-solvable or π'-solvable, the Schur-Zassenhaus theorem implies G has a Hall-π-subgroup and that any two Hall-π-subgroups are conjugate in G (see 6.3.5 and 6.3.6 in [3]). Proof of the following lemma, due to Glauberman [2], requires the conjugacy part of the Schur-Zassenhaus theorem and thus uses the Odd-Order theorem to ensure the solvability of either A or G.

LEMMA 3.1. *Assume A acts on G by automorphisms and $(|A|, |G|) = 1$. Assume A and G act on a set T such that G is transitive on T and $(t \cdot g) \cdot a = (t \cdot a) \cdot g^a$ for all $t \in T$, $g \in G$, $a \in A$. Then*

(a) *A fixes an element of T, and*

(b) *$C_G(A)$ acts transitively on the fixed points in T of A.*

Proof. See [2] or 13.8 and 13.9 of [5].

COROLLARY 3.2. *Assume A acts on G by automorphisms, $N \trianglelefteq G$ is A-invariant, $(|G: N|, |A|) = 1$, and $C_{G/N}(A) = 1$. Let $\chi \in IRR(G)$ and $\phi \in IRR(N)$ be A-invariant. Then*

(a) *χ_N has a unique A-invariant irreducible constituent; and*

(b) *If G/N is abelian, ϕ^G has a unique A-invariant irreducible constituent.*

Proof. Now A and G/N act on the irreducible constituents of χ_N and G/N is transitive. Thus, part (a) follows form Lemma 3.1.

For (b), note A and $IRR(G/N)$ act on the irreducible constituents of ϕ^G and $IRR(G/N)$ is transitive in this action. We are done by Lemma 3.1 if A acts fix point free on $IRR(G/N)$. If $\psi \in IRR(G/N)$ is A-fixed, then A centralizes $G/\mathrm{Ker}(\psi)$ and $\mathrm{Ker}(\psi) = G$. This completes the proof.

THEOREM 3.3. *Assume that G is π'-solvable with a Hall-π-subgroup S; $N = N_G(S)$; $K, L \trianglelefteq G$; $H = LN$; K/L is an abelian π'-group; $KH = G$; and $K \cap H = L$. Let $\theta \in IRR(K)$ such tha $S \leqq I_G(\theta)$. Then*

(a) *θ_L has a unique S-invariant irreducible constituent ϕ; and*

(b) *There is a one-to-one and onto map $T: IRR(G|\theta) \to IRR(H|\phi)$ such that $\chi(1)/(\chi T)(1)$ is an integer dividing $|G: H|$.*

Proof. As $C_{K/L}(S) = 1$, part (a) is a consequence of Corollary 3.2. To prove (b), induct on $|G|$. By induction, it is no loss to assume K/L is chief in G and H is maximal in G. Note $KN = G$. For $n \in N$, θ^n and ϕ^n are S-invariant. If $R = I_G(\theta)$, it then follows from Corollary 3.2 that $R \cap H = I_H(\phi)$. Now character induction yields one-to-one maps from $IRR(R|\theta)$ onto $IRR(G|\theta)$ and from $IRR(R \cap H|\phi)$ onto $IRR(H|\phi)$. As $|G: R| = |H: H \cap R|$, we finish by induction on $|G|$ if $R < G$.

So, we assume $I_G(\theta) = G$ and $I_H(\phi) = H$. If $I_G(\phi) = H$, $\phi^K = \theta$ and character induction defines a one-to-one map from $IRR(H|\phi)$ onto $IRR(G|\phi) = IRR(G|\theta)$. As H is maximal in G; we assume $I_G(\phi) = G$.

If $\theta_L = \phi$, we are done by Lemma 2.1. With no loss, we assume $\theta_L = e\phi$ and $e^2 = |K: L|$. Replace (G, L, ϕ) by an isomorphic character triple (G^*, L^*, ϕ^*) where ϕ^* is faithful and linear (8.2 of [4]). Now θ^* is fully ramified with respect to K^*/L^* and consequently vanishes off L^*. So $\mathbf{Z}(K^*) = L^* \leqq \mathbf{Z}(G^*)$. Note $SL \trianglelefteq H$ and that Fitting's lemma (5.2.3 of [3]) implies $[K/L, S] = K/L$. Also, $G^*/L^* \cong G/L$. For $\chi \in IRR(G|\phi)$ and $\psi \in IRR(H|\phi)$; $\chi^*(1)/\psi^*(1) = (\chi^*(1)/\phi^*(1)) \times (\phi^*(1)/\psi^*(1)) = \chi(1)/\psi(1)$. As $IRR(G|\theta) = IRR(G|\phi)$; the character triple isomorphism and Lemma 2.3 yield here a one-to-one and onto map $F: IRR(G|\theta) \to IRR(H|\phi)$ such that $\chi(1) = e(\chi F)(1)$. This completes the proof.

THEOREM 3.4. *Let G be π'-solvable and let P be a Hall-π-subgroup of G. Then $|I_\pi(G)| = |I_\pi(N_G(P))|$.*

Proof. Induct on $|G|$. Let $N = N_G(P)$ and $K = O^{\pi'\pi}(G)$. We assume $K \neq 1$, else $N = G$. The Frattini argument yields $KN = G$. Let K/L be a chief factor, so that K/L is an elementary abelian q-group for a prime $q \in \pi'$. Let $H = LN$, so that $G = KH$. By definition of K, $C_{K/L}(P) = 1$. So $H \cap K = L$. It suffices via induction to show $|I_\pi(G)| = |I_\pi(H)|$.

Corollary 3.2 gives us a one-to-one correspondence between all P-invariant irreducible characters θ of K and all P-invariant irreducible characters ϕ of L, in which θ and ϕ correspond if and only if $[\theta_L, \phi] \neq 0$ or, equivalently $[\theta, \phi^K] \neq 0$. Furthermore, this correspondence is invariant under conjugation by N. Since $G = KN$ and $H = LN$, we conclude that this correspondence carries G-conjugacy classes of θ's one-to-one and onto the H-conjugacy classes of ϕ's.

Let $S_1 = \{\chi \in IRR(G) \mid \chi_K$ has a P-invariant irreducible constituent$\}$ and $S_2 = \{\psi \in IRR(H) \mid \psi_L$ has a P-invariant irreducible constituent$\}$. The last paragraph and Theorem 3.3 yield a one-to-one and onto map $F\colon S_1 \to S_2$ such that $\chi(1)/(\chi F)(1)$ is an integer dividing $|G\colon H| = |K{:}L|$. If $\chi \in IRR(G)$ (or $\chi \in IRR(H)$) and $p\chi(1)$ for all $p \in \pi$; then $\chi \in S_1$ (respectively, $\chi \in S_2$). Hence $\chi \in I_\pi(G)$ if and only if $\chi \in S_1$ and $(\chi F) \in I_\pi(H)$. The proof is complete.

Actually the above results yield a one-to-one map $T\colon I_\pi(G) \to I_\pi(N)$ such that $\chi(1)/(\chi T)(1)$ divides $|G\colon N|$. In the case $\pi = \{p\}$, the above theorem states precisely that $|I_p(G)| = |I_p(N(P))|$ for G solvable, where $P \in \mathrm{Syl}_p(G)$.

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