

METABELIAN GROUPS WITH AN IRREDUCIBLE
PROJECTIVE REPRESENTATION OF
LARGE DEGREE

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The structure of a metabelian group admitting an irreducible projective complex representation whose degree is the square root of the order of the group studied.

In [7] a class of metabelian groups admitting a faithful irreducible projective representation was characterized. If \bar{G} is a finite group with irreducible projective representation T^* so that $T^*(1)^2 = [\bar{G}:1]$ then T^* is a faithful irreducible projective representation. Conversely, if G is abelian and T^* is a faithful irreducible projective representation then $T^*(1)^2 = [\bar{G}:1]$. We study metabelian groups \bar{G} admitting an irreducible projective representation T^* with $T^*(1)^2 = [\bar{G}:1]$. Given such a group one can construct, using Schur's theory, a central extension G of \bar{G} with center Z and an ordinary faithful irreducible representation T of G with $T(1)^2 = [G:Z]$ and $T^*(gZ) = T(g)$ for all $g \in G$ (see p. 358 of [1]). A group G with an irreducible representation T so that $T(1)^2 = [G:Z]$ is called a group of central type.

Groups of central type have been studied in several places (see for example [3] and [5]). We study metabelian groups \bar{G} admitting an irreducible projective complex representation whose degree is $\sqrt{[\bar{G}:1]}$ by studying the associated central extension G of central type using the methods of ordinary representation theory. In this note all groups are finite and all representations and characters are assumed to be taken over the complex numbers. All unexplained terminology and notation is as in [1].

I. An abelian group G is said to be of symmetric type in case $G = A \times A$ for some abelian group A . It has been shown that an abelian group G is a group of symmetric type if and only if G admits a faithful irreducible projective representation [4] if and only if G admits an irreducible projective representation T^* with $T^*(1)^2 = [G:1]$ ([2]). For any group H let $Z(H)$ denote the center of H .

THEOREM 1. *Let G be a group of central type with center Z . Assume there is a faithful irreducible character ζ on G with $\zeta(1)^2 = [G:Z]$. If there is a normal subgroup H of G with $Z \subseteq H \subseteq G$*

and both G/H and H/Z abelian then

(1) H is a group of central type so $H/Z(H)$ is of symmetric type.

(2) There is a normal subgroup K of G with $H \subseteq K \subseteq G$ so that K/H is of symmetric type and $[G:K] = [Z(H):Z]$.

Proof. First assume G is a p -group, p a prime. By Schur's lemma $\zeta|_Z = \zeta(1)\phi$ for a faithful linear character ϕ on Z . Proceed by induction on $[Z(H):Z]$ to prove: If ψ is an extension of ϕ to $Z(H)$ then $\psi^H = \lambda(1)\lambda$ where $\lambda(1)^2 = [H:Z(H)]$, and thus H is a group of central type with center $Z(H)$. If $[Z(H):Z] = 1$ then H is a group with cyclic center Z such that H/Z is abelian. In this case it is easy to check that the commutator map induces a non-degenerate bilinear function from $H/Z \times H/Z$ to Z . It follows by Proposition 3 of [2] that H is of central type and H/Z is of symmetric type. Moreover $\phi^H = \lambda(e)\lambda$ for an irreducible character λ on H with $\lambda(1)^2 = [H:Z]$. If $Z(H) \neq Z$ let $x \in Z(H) - Z$ with $x^p \in Z$ and $x \cdot Z$ in the center of G/Z . We can find such an x because G is a p -group. Let $H_1 = \{y \in G \mid [x, y] = e\}$. By Theorem 3 of [3], $[G:H_1] = p$ and H_1 is a group of central type with center $Z(H_1) = Z\langle x \rangle$. If ψ is an extension of ϕ to $Z(H_1)$ then $\psi^{H_1} = \rho(e)\rho$ where $\rho(e)^2 = [H_1:Z(H_1)]$. If $X = \ker \psi = \ker \rho$ then X is a subgroup of $Z(H_1)$ so that $X \cap Z = \{e\}$ and H_1/X is a group of central type with a faithful irreducible character ρ . Clearly $Z(H)/X \subseteq Z(H/X)$. If $hX \in Z(H/X)$ then for any $k \in H$, $[h, k] \in X$. But H/Z is abelian so $[h, k] \in Z$, and $Z \cap X = \{e\}$. Thus $h \in Z(H)$ and $Z(H)/X = Z(H/X)$. We apply induction to the group H_1/X , and obtain H/X is a group of central type with center $Z(H)/X$. Thus H is a group of central type with center $Z(H)$ and $H/Z(H)$ is of symmetric type. Moreover, if ψ is an extension of ψ to $Z(H)$ then $\psi^H = \lambda(1)\lambda$ where $\lambda(1)^2 = [H:Z(H)]$.

To prove (2) let $K = \{y \in G \mid \lambda^y = \lambda\}$. Since G/H is abelian, $K \triangleleft G$. We proceed by induction on $[Z(H):Z]$ as in the proof of (1) to show (K, H, λ) is a fully ramified triple [5] or [6], i.e., there is an irreducible character ρ on K so that $\rho|_H = e\lambda$ where $e^2 = [K:H]$. This suffices by Theorem 1.1 of [5] to show K/H is of symmetric type. If $Z(H) = Z$ then $K = G$ and $\rho = \zeta$ works. If $Z(H) \neq Z$ let H_1, X, ψ, λ be as in the proof of (1). If $y \in G - H_1$ then $\lambda^y(x) = \lambda(yxy^{-1}) = \lambda(xz)$ where $z \in Z - \{e\}$. But $\lambda|_{Z(H_1)} = \lambda(e)\psi$ so $\lambda(xz) = [\lambda(e)\psi(x)]\psi(z) = \lambda(x)\phi(z) = \lambda(x)\phi(z)$. Since ϕ is faithful, $y \notin K$ and thus $K \subseteq H_1$. Since $X \cap Z = \{e\}$ we can apply the same argument in G/X to see K/X is the inertia group of λ in G/H . Apply induction to H_1/X . Then $(K/X, H/X, \lambda)$ is a fully ramified triple so (K, H, λ) is fully ramified. To prove $[G:K] = [Z(H):Z]$ we have from

Clifford's theory (p. 342 of [1]) $\rho|_H = e(\lambda_1 + \dots + \lambda_n)$ where by relabeling we can let $\lambda_1 = \lambda$ where the λ_i are conjugate irreducible characters, and $n = [G:K]$. Thus $\rho(1)^2 = [G:Z] = e^2\lambda(1)^2n^2 = [G:K][K:H][H:Z(H)][Z(H):Z]$. Since $e^2 = [K:H]$, $\lambda(e)^2 = [H:Z(H)]$, and $n = [G:K]$ so we have $[G:K] = [Z(H):Z]$.

Now we show Theorem 1 is true without requiring G to be a p -group. By the hypothesis H is nilpotent so $H = \prod_p H_p$ where p are the primes such that $p|[G:1]$ and the H_p are the p -Sylow subgroups of H . Let G_p be a Sylow p -subgroup of G , then $Z(G_p) = G_p \cap Z$ and $Z(G_p) \subseteq H_p \subseteq G_p$. Moreover, G_p is a group of central type whose irreducible character of large degree is faithful. (Theorem 2 of [3].) Thus H_p is a group of central type with center $Z(H_p)$ and there is a normal subgroup K_p of G_p containing H_p with K_p/H_p of symmetric type and $[G_p:K_p] = [Z(H_p):Z(G_p)]$. By Theorem 2 of [3], H is a group of central type with center $Z(H)$, and if K is the normal subgroup of G which is the inverse image under the natural homomorphism of $\prod_p K_p/H_p$ in G/H then K/H is of symmetric type since each Sylow subgroup is and $[G:K] = \prod_p [G_p:H_p] = \prod_p [Z(H_p):Z(G_p)] = [Z(H):Z]$. This proves the theorem.

There is a central extension G of $A_4 \times Z_3$ of order $2^3 \cdot 3^3$ which is of central type, G is not nilpotent, yet G satisfies the hypothesis of Theorem 1.

Theorem 1 does not hold if ζ is not faithful because the center of H does not behave properly. However, if G is a group of central type with center Z and irreducible character ζ with $\zeta(e)^2 = [G:Z]$ then the kernel X of ζ is contained in Z so if one is only interested in the structure of G/Z it is always possible to replace G by G/X where X is the kernel of ζ and apply Theorem 1 to G/X . In Theorem 1 it is not necessarily the case that $G/K \cong Z(H)/Z$ as the following example shows. Let H be the elementary abelian group of order 8 generated by x, y, z . Let J be the cyclic group of order 4 generated by w and let J act as a group of automorphisms of H by $w(x) = y, w(y) = xz, w(z) = z$. Let $G = [H]J$ be the semidirect product. Then the center of G is generated by z and has order 2. Let ϕ be the faithful linear character on Z and extend ϕ to ψ on H by $\psi(y) = \psi(x) = 1$. Then ψ has 4 conjugates under J so $\psi^G = \zeta$ has degree = 4 and is irreducible. Thus G is a group of central type, $H \triangleleft G$, and both H and $G/H = J$ are abelian. Thus H is of central type with center H , the group K of Theorem 1 is H and $G/H \cong J \not\cong H/Z$. Notice, however, that in the group G one can find an abelian normal subgroup N such that $G/N \cong N/Z$, namely $N = \langle xy, w^2z \rangle$. Then $G/N \cong N/Z$ is the Klein 4-group.

Using the results in [7] it is easy to characterize the metacyclic groups G admitting an irreducible projective representation of degree

$n = [\bar{G}:1]^{1/2}$. By 1.6 and 4.1 of [7] such a group must be the semi-direct product of two cyclic groups of order n . If $\bar{G} = \langle a, b \mid a^n = b^n = e, ba = a^r b \rangle$ is the semidirect product of a cyclic group of order n with itself then by 4.4 of [7] \bar{G} admits an irreducible projective representation of degree n if and only if $n - 1$ is the least positive integer exponent so that $1 + r + \dots + r^{n-1} = 0 \pmod{n}$. This was pointed out to me by the referee.

We conclude by showing how to construct a group G of central type with G/Z metabelian. This construction will characterize metabelian groups \bar{G} with an irreducible projective representation of degree $[\bar{G}:1]^{1/2}$.

First it is necessary to recall some terminology and results from § 2 of [6]. Let G be any finite group with normal subgroup N so that G/N is abelian. Let θ be an irreducible character on N with $\theta^g = \theta$ for all $g \in G$. Then (G, N, θ) is called a character triple. Let $x, y \in G$. The character θ can be extended to a character ψ on the subgroup $\langle N, y \rangle$ generated by N and y . The character ψ^x is another extension of ψ to $\langle N, y \rangle$ so $\psi^x = \lambda\psi$ for some linear character λ on $\langle N, y \rangle$. Define a complex valued bilinear function on $G \times G$ by the equation $\langle\langle x, y \rangle\rangle = \lambda(xy x^{-1} y^{-1})$. It is shown in [6] that this inner product is well defined. The bilinear function is nondegenerate if $\{x \in G \mid \langle\langle x, y \rangle\rangle = 1 \text{ for all } y \in G\} = N$. By 2.7 of [6] the triple (G, N, θ) is fully ramified if and only if the bilinear function defined above is nondegenerate.

Begin with an abelian group Z_1 and a cyclic subgroup Z . Let ϕ be a linear character on Z_1 such that $\phi|_Z$ is faithful. Let H be an abelian group of symmetric type and let $\gamma: H \times H \rightarrow Z$ be a nondegenerate bilinear function. Such a function exists if and only if $\exp(H)$ divides $[Z:1]$. Let H be any central extension of Z by H with factor set γ . Then (H, Z_1, ϕ) is a fully ramified triple and thus is a group of central type with center $Z(H) = Z_1$. Also $\phi^H = \lambda(e)\lambda$ for an irreducible character λ on H with $\lambda(e)^2 = [H:Z(H)]$. Next let \bar{K} be an abelian group of symmetric type. Let $\alpha: \bar{K} \rightarrow \text{Aut}(H)$ so that $\phi^{\alpha(x)} = \phi$ for all $x \in \bar{K}$ and $Z \subseteq \{y \in Z_1 \mid \alpha(x)(y) = y \text{ for all } x \in \bar{K}\}$. Note that $\phi^{\alpha(x)} = \phi$ if and only if $\lambda^{\alpha(x)} = \lambda$ since λ vanishes off Z_1 and Z_1 is characteristic in H . Let K be one of the normal extensions of H by \bar{K} determined by α . Then (K, H, λ) is a character triple and (K, H, λ) is fully ramified if and only if the form $\langle\langle \cdot, \cdot \rangle\rangle$ defined above is nondegenerate. In this case $\lambda^K = \rho(e)\rho$ where ρ is irreducible on K and $\rho(e)^2 = [K:Z]$. We choose a K so that (K, H, λ) is fully ramified. Finally let C be an abelian group with $[C:1] = [Z_1:Z]$ and let $\beta: C \rightarrow \text{Aut}(K)$ such that $\phi^{\beta(c)} \neq \phi$ for all $c \neq e$ and $Z \subseteq \{y \in K \mid \beta(c)(y) = y \text{ for all } c \in C\}$. Let G be any normal extension of K by C determined by β with G/H abelian.

Then $\rho^\sigma = \zeta$ is irreducible on G since ρ is distinct from all its conjugates under C and $\zeta(e)^2 = [G:C]^2\rho(e)^2 = [G:Z]$. Thus G is a group of central type satisfying the hypothesis of Theorem 1 and an examination of the proof of Theorem 1 shows every group satisfying the hypothesis of Theorem 1 is constructed in this way.

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