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ON THE MULTIPLICATIVE COUSIN PROBLEMS FOR $N^p(D)$

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Let D be a strictly convex domain in C^n with C^2-class boundary. Let $N^p(D), 1 < p < \infty$, be the set of all holomorphic functions f in D such that $(\log^+|f|)^p$ has a harmonic majorant. The purpose of this paper is to show that the multiplicative Cousin problems for $N^p(D), 1 < p < \infty$, are solvable.

1. Introduction. Let D be a domain in C^n. We denote by S_n the class of bounded domains D in C^n with the properties that there exists a real function ρ of class C^2 defined on a neighborhood W of ∂D such that $d\rho \neq 0$ on $\partial D, D \cap W = \{z \in W: \rho(z) < 1\}$ and the real Hessian of ρ is positive definite on W. For $1 \leqq p \leqq \infty$, we denote by $N^p(D)$ the set of all holomorphic functions f in D such that $(\log^+|f|)^p$ has a harmonic majorant in D. When $p = \infty$, we assume that $|f|$ is bounded in D. When $p = 1, N^1(D)$ is the Nevanlinna class. E. L. Stout [5] proved that the multiplicative Cousin problem with bounded data on every domain of class S_n can be solved. In this paper we shall prove that the multiplicative Cousin problems for $N^p(D), 1<p\leqq\infty$, can be solved. The proof depends on the Riesz type theorem concerning conjugate functions and the estimates obtained by E. L. Stout [5], [6]. The required analysis is available on strictly pseudoconvex domains, but the geometric patching constructions in §3 depend on euclidean convexity. Explicitly, the above results are the following:

THEOREM. *Let $D \in S_n$. Let $\{V_\alpha\}_{\alpha \in I}$ be an open covering of $\bar{D}$, and for each $\alpha, f_\alpha \in N^p(V_\alpha \cap D), 1 < p \leqq \infty$. If for all $\alpha, \beta \in I, f_\alpha f_\beta^{-1}$ is an invertible element of $N^p(V_\alpha \cap V_\beta \cap D)$, then there exists a function $F \in N^p(D)$ such that for all $\alpha \in I, Ff_\alpha^{-1}$ is an invertible element of $N^p(V_\alpha \cap D)$.*

In the case when D is an open unit polydisc in C^n, theorem for $p = 1$ was proved by S. E. Zarantonello [7], and theorem for $p = \infty$ was proved by E. L. Stout [4].

Let $A(D)$ be the sheaf of germs of continuous function on $\bar{D}$ that are holomorphic in D. I. Lieb [2] proved that $H^q(\bar{D}, A(D)) = 0$ for $q > 0$, provided D is a strictly pseudoconvex domain with C^5-boundary. Let $D \in S_n$ and let D have a C^5-boundary. Then, from the above Lieb's result and $H^2(D, Z) = 0$, by applying the standard exact sequence of sheaves

$$0 \longrightarrow \boldsymbol{Z} \longrightarrow A(D) \xrightarrow{\exp} A(D)^{-1} \longrightarrow 0$$

one can solve Cousin II-problems with data from the sheaf $A(D)$.

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2. **H^p-functions.** We now state some properties about H^p-functions. If $D \subset C^n$ is a domain, then for $0 < p < \infty$, $H^p(D)$ is the space of all functions f holomorphic in D such that $|f|^p$ admits a harmonic majorant in D. When $p = \infty$, $H^p(D)$ is the space of all functions bounded and holomorphic in D. For a relatively compact domain D in C^n with ∂D a real submanifold of class C^2, we shall say that a C^2-function ρ defined on a neighborhood of $\bar{D}$ is a characterizing function for D provided $\rho(z) < 1$ if and only if $z \in D$, provided $\partial D = \{z: \rho(z) = 1\}$, and provided $\partial\rho/\partial\nu \geqq c > 0$ on ∂D, where $\partial/\partial\nu$ is the derivative with respect to the outward normal. E. L. Stout [5] proved that $D \in S_n$ is strictly convex and that if $0 \in D$, then D can be defined by a globally defined function which has positive definite real Hessian on $C^n - \{0\}$. From now on, when we consider $D \in S_n$, we assume that the defining function of D is globally defined and we take this function as a characterizing function of D. By E. M. Stein [3], the following (1) and (2) are equivalent for holomorphic funtions f in D and $1 \leqq p \leqq \infty$:

$$\sup_{\varepsilon<1} \left(\int_{\partial D_\varepsilon} |f(x)|^p dS_\varepsilon(x) \right)^{1/p} < \infty , \tag{1}$$

where $D_\varepsilon = \{x: \rho(x) < \varepsilon\}$, $\rho(x)$ a characterizing function of D, and dS_ε is the element of surface area on ∂D_ε.

(2) $|f(x)|^p$ has a harmonic majorant if $p < \infty$. When $p = \infty$ we assume that $|f|$ is bounded in D.

By the Cauchy-Fantappiè integral formula, if $f \in H^p(D)$, $1 \leqq p \leqq \infty$, then for $w \in D$,

$$f(w) = c_n \int_{\partial D} f(z) \times \frac{dz_1 \wedge \cdots \wedge dz_n \sum_{k=1}^{n} (-1)^k \xi_k(z) d\xi_1(z) \wedge \cdots \wedge \widehat{d\xi_k}(z) \wedge \cdots \wedge d\xi_n(z)}{\langle w - z, \nabla\rho(z)\rangle^n}$$

where

$$\xi_k(z) = \frac{\partial\rho}{\partial z_k}(z),\ c_n = \frac{(n-1)!}{(2\pi i)^n},\ \langle w - z, \nabla\rho(z)\rangle = \sum_{j=1}^{n} (w_j - z_j) \frac{\partial\rho}{\partial z_j}(z) ,$$

and $\widehat{\ }$ means to be omitted. Since ∂D is of class C^2, the above integral can be written as

$$f(w) = c_n \int_{\partial D} f(z) \frac{k(z)dS(z)}{\langle w - z, \nabla\rho(z)\rangle^n}$$

where k is a continuous function and dS is the element of surface area on ∂D. Next we have the following propositions proved by E. L. Stout [6] for the Ramírez-Henkin integral. The proofs of the propositions are essentially the same as the proof of Theorem II.1 in E. L. Stout [6], so we omit the proofs.

PROPOSITION 1. *If $f \in H^p(D)$, $1 \leqq p \leqq \infty$, and if ϕ is defined and satisfies a Lipschitz condition on C^n, then the function f_ϕ defined by*

$$f_\phi(w) = c_n \int_{\partial D} \frac{f(z)\phi(z)k(z)dS(z)}{\langle w - z, \nabla\rho(z)\rangle^n}$$

belongs to $H^p(D)$.

PROPOSITION 2. *Let $D \in S_n$. Let $f = u + iv \in O(D)$, where $O(D)$ is the space of all holomorphic functions in D. Let $|u|^p$, $1 < p \leqq \infty$, have a harmonic majorant, and let ϕ be a real function of C^n which satisfies a Lipschitz condition on C^n. Let f_ϕ be the function defined in Proposition 1. Then $|\mathrm{Re} f_\phi|^p$ has a harmonic majorant in D.*

3. Proof of theorem. Let $D \in S_n$. Let $M = \max\{x_{2n}$: for some $z \in \bar{D}$, $z = (z_1, \cdots, z_n)$, $x_{2n} = \mathrm{Im}\, z_n\}$, and let m be the corresponding minimum. Let ε_0 satisfy $0 < \varepsilon_0 < (1/12)(M - m)$. Let η_i, $i = 1, 2$, be real valued functions of a real variable such that

$$\eta_i \text{ is of class } C^2,\ i = 1, 2\,, \tag{1}$$

$$\eta_1(t) = 0 \quad \text{if} \quad t \leqq \frac{1}{2}(M + m) + \frac{5}{2}\varepsilon_0\,, \tag{2}$$

$$\eta_2(t) = 0 \quad \text{if} \quad t \geqq \frac{1}{2}(M + m) - \frac{5}{2}\varepsilon_0\,,$$

$$\eta_1(t) \geqq 2 \quad \text{if} \quad t \geqq \frac{1}{2}(M + m) + 3\varepsilon_0\,, \tag{3}$$

$$\eta_2(t) \geqq 2 \quad \text{if} \quad t \leqq \frac{1}{2}(M + m) - 3\varepsilon_0\,,$$

$$\eta_1''(t) > 0 \quad \text{if} \quad t > \frac{1}{2}(M + m) + \frac{5}{2}\varepsilon_0\,, \tag{4}$$

$$\eta_2''(t) > 0 \quad \text{if} \quad t < \frac{1}{2}(M + m) - \frac{5}{2}\varepsilon_0\,.$$

Let ρ be a characterizing function of D, and let

$$D_1 = \{z: \rho(z) + \eta_1(x_{2n}) < 1\},\ D_2 = \{z: \rho(z) + \eta_2(x_{2n}) < 1\}\ .$$

Then it is easily verified that D_1, D_2 and $D_1 \cap D_2$ are elements of S_n.

LEMMA 2. *Let D, D_1, D_2 be as above. If a positive subharmonic function ϕ in D has harmonic majorants in D_1 and D_2, then ϕ has a harmonic majorant in D.*

Proof. To prove Lemma 2, it suffices to show that

$$\sup_{\varepsilon<1} \int_{\partial D_\varepsilon} \phi dS_\varepsilon < \infty\ .$$

Let $D_{1\varepsilon} = \{\rho(z) + \eta_1(x_{2n}) < \varepsilon\}$, $D_{2\varepsilon} = \{\rho(z) + \eta_2(x_{2n}) < \varepsilon\}$. Then $D_{1\varepsilon} \cup D_{2\varepsilon} = D_\varepsilon$, $\partial D_{1\varepsilon} \cup \partial D_{2\varepsilon} \supset \partial D_\varepsilon$, $D_{1\varepsilon} \subset D_1$, $D_{2\varepsilon} \subset D_2$. Hence we have

$$\int_{\partial D_\varepsilon} \phi dS_\varepsilon \leqq \int_{\partial D_{1\varepsilon}} \phi dS_\varepsilon^1 + \int_{\partial D_{2\varepsilon}} \phi dS_\varepsilon^2$$

where dS_ε^1 and dS_ε^2 are the surface area elements of $\partial D_{1\varepsilon}$ and $\partial D_{2\varepsilon}$, respectively. Integrals on the right are bounded uniformly on ε. Therefore Lemma 2 is proved.

We need two definitions.

DEFINITION 1. We say that a positive subharmonic function ϕ in D has local harmonic majorants if there exists an open covering $\{O_\alpha\}_{\alpha\in I}$ of $\bar{D}$ such that for each $\alpha \in I$, ϕ has a harmonic majorant on $O_\alpha \cap D$.

DEFINITION 2. We say that F is locally in $N^p(D)$ if there exists an open covering $\{V_\alpha\}_{\alpha\in I}$ of $\bar{D}$ such that for each $\alpha \in I$, F restricted to $V_\alpha \cap D$ belongs to $N^p(V_\alpha \cap D)$. The class of functions locally in $N^p(D)$ will be denoted by $N^p_{\mathrm{loc}}(D)$. We denote the group of its invertible elements by $\mathrm{inv}\, N^p_{\mathrm{loc}}(D)$.

LEMMA 3. *Let D, D_1 and D_2 be as in Lemma 2. Let $f = u + iv \in O(D_1 \cap D_2)$. If $|u|^p$ has a harmonic majorant in $D_1 \cap D_2$, then there exist functions f_1 and f_2 such that $f = f_1 + f_2$, where f_i, $i = 1, 2$, is holomorphic in D_i and $|\mathrm{Re}\, f_i|^p$ has a harmonic majorant in D_i, respectively.*

Proof. Let ψ be a function on C^n which satisfies a Lipschitz condition and which has the properties that

$$\psi = 0 \quad \text{on} \quad \left\{z \in \partial(D_1 \cap D_2)\colon x_{2n} < \frac{1}{2}(M+m) - \varepsilon_0\right\},$$

$$\psi = 1 \quad \text{on} \quad \left\{z \in \partial(D_1 \cap D_2)\colon x_{2n} > \frac{1}{2}(M+m) + \varepsilon_0\right\},$$

where ε_0 is the constant used in Lemma 2. Let $\tilde{\rho}$ be a characterizing function of $D_1 \cap D_2$. Write f as a Cauchy-Fantappiè integral. For $w \in D_1 \cap D_2$, we have

$$f(w) = c_n \int_{\partial(D_1 \cap D_2)} \frac{f(z)k(z)dS(z)}{\langle w - z, \nabla\rho(z)\rangle^n} = f_1(w) + f_2(w)$$

where

$$f_1(w) = c_n \int_{\partial(D_1 \cap D_2)} \frac{f(z)(1-\psi(z))k(z)dS(z)}{\langle w - z, \nabla\rho(z)\rangle^n},$$

$$f_2(w) = c_n \int_{\partial(D_1 \cap D_2)} \frac{f(z)\psi(z)k(z)dS(z)}{\langle w - z, \nabla\rho(z)\rangle^n}.$$

The functions f_1 and f_2 are holomorphic on $D_1 \cap D_2$ and that $|f_1|^p$ and $|f_2|^p$ have harmonic majorants on $D_1 \cap D_2$. Moreover, we can write

$$f_1(w) = c_n \int_\Gamma \frac{f(z)(1-\psi(z))k(z)dS(z)}{\langle w - z, \nabla\rho(z)\rangle^n}$$

where $\Gamma = \partial(D_1 \cap D_2) \cap \{x_{2n} \leqq (M+m)/2 + \varepsilon_0\}$. If $E = \{z \in D\colon x_{2n} \geqq (M+m)/2 + 2\varepsilon_0\}$, then the distance between E and the tangent plane of $\partial(D_1 \cap D_2)$ at z is positive, where z is contained in $\partial(D_1 \cap D_2) \cap \{x_{2n} \geqq (M+m)/2 + \varepsilon_0\}$. Therefore f_1 is holomorphic in D_1. Let ρ_1 be a characterizing function of D_1. Then we have

$$\int_{\rho_1=\varepsilon} |f_1|^p dS_\varepsilon^1 \leqq \int_{\{\tilde{\rho}=\varepsilon\}\cap(D_1-E)} |f_1|^p d\tilde{S}_\varepsilon + \int_{\{\rho=\varepsilon\}\cap E} |f_1|^p dS_\varepsilon$$

where $d\tilde{S}_\varepsilon$ is the element of surface area of $\partial(D_1 \cap D_2)_\varepsilon$. Integrals on the right are bounded uniformly on ε. Therefore $|f_1|^p$ has a harmonic majorant in D_1. Hence $|\operatorname{Re} f_1|^p$ has a harmonic majorant in D_1. The proof that $|\operatorname{Re} f_2|^p$ has a harmonic majorant is the same as the proof for f_1. Therefore Lemma 3 is proved.

LEMMA 4. *Let $D \in S_n$. Then any positive subharmonic function ϕ in D with local harmonic majorant has a harmonic majorant. A one variable version of this result has been given by P. M. Gauthier and W. Hengartner* [1].

Proof. Suppose ϕ does not have a harmonic majorant in D. Let D_1 and D_2 be subdomains of D constructed in Lemma 2. By Lemma

2, ϕ cannot have harmonic majorants on both D_1 and D_2. Say D_1. The x_{2n}-width of D_1, i.e., the number $\max|x'_{2n} - x''_{2n}|$, the maximum taken over all pairs of points z', z'' in D_1, is not more than three fourths of the x_{2n}-width of D. We now treat D_1 as we treated D, using the coordinate x_{2n-1} rather than x_{2n}, and we find a smaller set $D_{11} \subset D_1$ on which the problem is not solvable and which has the property that the x_{2n-1}-width of D_{11} is not more than three fourths that of D_1. We iterate this process, running cyclically through the real coordinate of C^n, and we obtain a shrinking sequence of sets on which our problem is not solvable. But there is an open covering $\{O_\alpha\}$ of $\bar{D}$ such that on each $O_\alpha \cap D$, ϕ has a harmonic majorant. One of the domains on which ϕ has no harmonic majorant will fall inside some $O_\alpha \cap D$, which is a contradiction. Therefore Lemma 4 is proved.

By using Lemmas 2, 3, 4, we are going to prove our theorem.

Proof of theorem. Suppose theorem does not hold. Let D_1 and D_2 be subdomains of D constructed in Lemma 2. If there were functions $F_i \in N^p(D_i)$ such that for every $\alpha \in I$ and $i = 1, 2$, $F_i f_\alpha^{-1}$ belongs to $\operatorname{inv} N^p(D_i \cap V_\alpha)$. Then $F_1F_2^{-1} = F_1 f_2^{-1} f_\alpha F_2^{-1}$ would be $\operatorname{inv} N^p(D_1 \cap D_2 \cap V_\alpha)$ for every α. Thus, $F_1F_2^{-1}$ would be in

$$\operatorname{inv} N^p_{\mathrm{loc}}(D_1 \cap D_2) = \operatorname{inv} N^p(D_1 \cap D_2) .$$

By Lemma 4, if we set $\tilde{F} = F_1F_2^{-1}$, then $(\log^+|\tilde{F}|)^p$ and $(\log^-|\tilde{F}|)^p$ have harmonic majorants in $D_1 \cap D_2$. So if $\tilde{F} = e^f$, then $|\operatorname{Re} f|^p = (\log^+|\tilde{F}| + \log^-|\tilde{F}|)^p$. Therefore $|\operatorname{Re} f|^p$ has a harmonic majorant. From Lemma 3, we can write $f = f_1 + f_2$, where $f_i \in O(D_i)$, $i = 1, 2$, and $|\operatorname{Re} f_i|^p$ has a harmonic majorant in D_i, respectively. If we set $G_1 = \exp(f_1)$, $G_2 = \exp(-f_2)$, then $(\log^+|G_i|)^p$, $(\log^-|G_i|)^p \leqq |\operatorname{Re} f_i|^p$, $i=1, 2$, respectively. Therefore G_i, $i=1, 2$, is an invertible element of $N^p(D_i)$, respectively. Moreover, $F_1F_2^{-1} = \exp(f_1)\exp(f_2) = \exp(f) = G_1G_2^{-1}$. If we define $F = F_1G_1^{-1}$ on D_1 and $F = F_2G_2^{-1}$ on D_2, then $F \in N^p(D)$ and for each $\alpha \in I$, $Ff_\alpha^{-1} \in \operatorname{inv} N^p_{\mathrm{loc}}(V_\alpha \cap D)$. But this is impossible since we have assumed our theorem not to be true. So we can assume that our problem is not solvable on D_1. We iterate the same process as in the proof of Lemma 4, and we have a contradiction. Therefore theorem is proved.

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