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A COHOMOLOGICAL INTERPRETATION OF BRAUER GROUPS OF RINGS

RAYMOND T. HOOBLER

Dedicated to Gerhard Hochschild on the occasion of his 65th birthday

Quillen's proof of the Serre conjecture introduced a new tool for passing from local to global results on affine schemes. We use this to prove the theorem below characterizing the image of the injection $i\colon Br(X) \to H^2(X_{et}, G_m)$ when $X = \operatorname{Spec} A$, is a regular scheme. A result of M. Artin then allows us to conclude that $Br(X) \cong H^2(X_{et}, G_m)$ if $X = \operatorname{Spec} A$ is a smooth, affine scheme over a field. For such rings, this proves the Auslander Goldman conjecture [2], $Br(A) = \bigcap Br(A_{\mathfrak{p}})$, $\mathfrak{p} \in P(A)$, the set of height one primes of A.

We begin with following theorem.

THEOREM. *Let $X = \operatorname{Spec} A$ be a regular scheme. If $c \in H^2(X_{et}, G_m)$ and $c_y = i([\Lambda_y])$ in $H^2(\operatorname{Spec}(A_{\mathfrak{m}_y})_{et}, G_m)$ for all closed points $y \in X$, then $c = i([\Lambda])$.*

Proof. If $f \in A$, let c_f denote the restriction of c to

$$H^2(\operatorname{Spec}(A_f)_{et}, G_m)\ .$$

Let $S = \{f \in A \mid c_f = i([\Lambda])$ for some Azumaya algebra Λ over $A_f\}$. We will show that S is an ideal. Then $S = A$ since the hypothesis on c prevents S from being contained in any maximal ideal of A.

Suppose $f_1, f_2 \in S$ and $f \in Af_1 + Af_2$. Then $\operatorname{Spec}(A_f) = D_{f_1} \cup D_{f_2}$ where $D_{f_i} = \operatorname{Spec}(A_{ff_1})$. Hence we may assume $A_f = A$ and $\operatorname{Spec}(A)$ is covered by $D_{f_1} \cup D_{f_2}$. Let Λ_1, Λ_2 be Azumaya algebras over A_{f_1}, A_{f_2} with $i([\Lambda_1]) = c_{f_1}$ and $i([\Lambda_2]) = c_{f_2}$. Since i is injective, $[\Lambda_{1f_2}] = [\Lambda_{2f_1}]$; that is, there are locally free coherent $A_{f_1f_2}$ modules P_1, P_2 such that $\Lambda_{1f_2} \otimes \operatorname{End}(P_1) \cong \Lambda_{2f_2} \otimes \operatorname{End}(P_2)$. Since $K^0(A_{f_i}) \to K^0(A_{f_1f_2})$ is onto (A is regular) [3] and we may assume the rank of P_i is large, there are locally free coherent A_{f_i} modules Q_i such that $Q_{if_i} \cong P_i$ [3, Chapter IX, 4.1]. Replacing Λ_i by $\Lambda_i \otimes \operatorname{End}(Q_i)$, we may assume that $\Lambda_{1f_2} \cong \Lambda_{2f_1}$. Using this patching isomorphism we produce an algebra Λ with $\Lambda_{f_1} \cong \Lambda_1$, $\Lambda_{f_2} \cong \Lambda_2$. Since $H^2(X_{et}, G_m) \to H^2(D_{f_i et}, G_m)$ is a monomorphism, $c = i([\Lambda])$.

COROLLARY 1. *Let $X = \operatorname{Spec}(A)$ be a smooth k scheme where k is a field. Then $Br(X) \cong H^2(X_{et}, G_m)$.*

Proof. Since X is regular, $H^2(X_{et}, G_m)$ is torsion [4]. If $c \in$

$H^2(X_{et}, G_m)$ has order n and n is relatively prime to the characteristic of k, then the Kummer sequence

$$o \longrightarrow \mu_n \longrightarrow G_m \longrightarrow G_m \longrightarrow 0$$

shows that c is in the image of $H^2(X_{et}, \mu_u)$. Now the existence of good neighborhoods [1] on $\bar{X} = X \otimes_k \bar{k}$, $\bar{k}$ = algebraic closure of k, shows that elements of $H^2(\bar{X}_{et}, \mu_n)$ are locolly isotrivial, i.e., there is a Zariski covering $\{U_i\}$ of $\bar{X}$ and a finite, etale covering space $V_i \rightarrow U_i$ which splits elements of $H^2(X_{et}, \mu_n)$. Consequently X has a Zariski open covering $\{U_\alpha\}$ and finite, flat coverings $\pi_\alpha : W_\alpha \rightarrow U_\alpha$ such that $\pi_\alpha^*(c|_{U_\alpha}) = 0$. Hence by the criterion in [6], $c|_{U_\alpha}$ is in the image of $Br(U_\alpha)$ and so by the theorem c is in the image of $Br(X)$. If c has order p^n, $p = \operatorname{char} k$, then we know that $F_X^{n*}(c) = 0$ where F_X is the Frobenius map. Since it defines a finite flat covering of X, the same criterion shows that c is in the image of $Br(X)$ (see [7] where this argument is given in more detail.).

COROLLARY 2. *Let A be an algebra of finite type over a field k such that $A \otimes \bar{k}$ is regular,* i.e., Spec A *is smooth over k. Then*

$$Br(A) = \cap Br(A_\mathfrak{p}),\ \mathfrak{p} \in P(A) = \{\mathfrak{p}/height\ \mathfrak{p} = 1\}\ .$$

Proof. We will use induction on $n = \dim A$. If $n = 0, 1$, or 2, the result was proven in [2, 4]. Since A is a regular ring, $\mathrm{Br}(A) \subset \cap \mathrm{Br}(A_\mathfrak{p})$, $\mathfrak{p} \in P(A)$. Hence the argument of the theorem shows that

$$S = \{f \in A/Br(A_f) = \cap Br(A_\mathfrak{p}),\ \mathfrak{p} \in P(A_f)\}$$

is an ideal in A and so is either A or is contained in a maximal ideal of A. Hence we may assume A is a local ring of dimension greater than 2.

Let $c \in \cap Br(A_\mathfrak{p})$, $\mathfrak{p} \in P(A)$, and $X = \operatorname{Spec} A$ and U be the punctured spectrum. Since $Br(A_\mathfrak{p}) = H^2(\operatorname{Spec}(A_\mathfrak{p})_{et}, G_m)$, there is a cohomology class $c_1 \in \cap H^2(\operatorname{Spec}(A_\mathfrak{p})_{et}, G_m) \subseteqq H^2(\operatorname{Spec}(K)_{et}, G_m)$, $\mathfrak{p} \in P(A)$, with $c_1 = i(c)$ where K is the quotient field of A. Now the Mayer-Vietoris sequence, which may be viewed as the Cech spectral sequence for the covering $\{U_1, U_2\}$ of $U_1 \cup U_2$, and the induction hypothesis show that there is a $c' \in H^2(U_{et}, G_m)$ whose restriction to $H^2(\operatorname{Spec}(A_{et}, G_m)$ is $i(c)$. Suppose c' is of order n where $(n, \operatorname{char} k) = 1$. Then the Kummer sequence shows that there is a cohomology class in $H^2(U_{et}, \mu_n)$ whose image in $H^2(U_{et}, G_m)$ is c'. But $H^2(X_{et}, \mu_n) = H^2(U_{et}, \mu_n)$ by relative cohomological purity [1, Expose XVI] and so there is a cohomology class c'' in $H^2(X_{et}, G_m)$ whose restriction to U is c'. By the first corollary c'' is in the image of $\mathrm{Br}(X)$ as desired.

If $n = p^m$, $p = \operatorname{char}(k)$, the same argument will work if we can show that c' is in the image of $H^2(X_{et}, G_m)$. We will then be done since $c' = c_1' + c_2'$ where the order of $c_1' = p^m$ and the order of c_2' is prime to p. The obstruction to c' being in the image of $H^2(X_{et}, G_m)$ lies in the local cohomology group $H_P^3(X_{et}, G_m)$ where P is the closed point of X. Moreover since $F_{X*}^m(c') = p^m c' = 0$ where $F_X\colon X \to X$ is the purely inseparable Galois covering defined by the Frobenius map, the obstruction lies in the kernel of $F_X^{m*}\colon H_P^3(X_{et}, G_m) \to H_P^3(X_{et}, G_m)$.

We have an exact sequence of sheaves on X_{et} [7]

$$0 \longrightarrow G_m \xrightarrow{j} F_{X*}G_m \longrightarrow \mathscr{Z}_X^1 \longrightarrow \Omega_X^1 \longrightarrow 0$$

where $\mathscr{Z}_X^1$ and Ω_X^1 are free A-modules (A is smooth and local) whose definition is unimportant. If C denotes the cokernel of j, then $H_P^2(X_{et}, C)$ is trapped between $H_P^1(X_{et}, \Omega_X^1)$ and $H_P^2(X_{et}, \mathscr{Z}_X^1)$. Since Ω_X^1 and $\mathscr{Z}_X^1$ are coherent sheaves, their local cohomology in the etale and Zariski topology coincide and hence vanish because $H_{\{m\}}^i(\mathrm{Spec}(A), A) = 0$ if A is regular and $i < \dim A$. Since $\dim A > 2$, we conclude that

$$F_X^*\colon H_P^3(X_{et}, G_m) \longrightarrow H_P^3(X_{et}, F_{X*}G_m) = H_P^3(X_{et}, G_m)$$

is injective and so c' is in the image of $H^2(X_{et}, G_m)$.

COROLLARY 3. *Let $\pi\colon X \to Y$ be a proper, smooth morphism of fibre dimension one where X is a smooth scheme over a field. Then $R^2\pi_* G_m = 0$.*

Proof. We may assume that Y is a strictly local k-scheme, k a field, and we must show that $H^2(X_{et}, G_m) = 0$. Since π has fibre dimension one and is proper, X is a union of two affine schemes which are limits of smooth k-schemes. Consequently $H^2(X_{et}, G_m) = Br(X)$. But $Br(\pi^{-1}(y)) = 0$ by Tsen's theorem. Thus Artin approximation may be used as in [5] to lift a trivialization of an Azumaya algebra on the fibre to a trivialization of the algebra on X.

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CITY COLLEGE
CUNY
NEW YORK, NY 10031