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A UNIFIED THEOREM ON CONTINUOUS SELECTIONS

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A selection theorem is proved which unifies and generalizes some known results.

1. Introduction.

The purpose of this note is to prove the following theorem, which unifies and generalizes previously known results.

THEOREM 1.1. *Let X be paracompact, Y a Banach space, $Z \subset X$ with* $\dim_X Z \leqq 0$, *and* $\phi: X \to \mathscr{F}(Y)$ l.s.c. *with $\phi(x)$ convex for all $x \in X - Z$. Then ϕ admits a selection.*

Recall that a map $\phi: X \to \mathscr{F}(Y)$, where $\mathscr{F}(Y)$ denotes $\{S \subset Y: S \neq \varnothing,\ S \text{ closed in } Y\}$, is *lower semi-continuous*, or l.s.c., if $\{x \in X: \phi(x) \cap V \neq \varnothing\}$ is open in X for every open V in Y. A *selection* for a map $\phi: X \to \mathscr{F}(Y)$ is a continuous $f: X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in X$. Finally, if $Z \subset X$ then $\dim_X Z \leqq 0$ means that $\dim E \leqq 0$ for every set $E \subset Z$ which is closed in X (where $\dim E$ denotes the covering dimension of E)[1].

Theorem 1.1 incorporates several known results: The case $Z = \varnothing$ is [1, Theorem 1], the case $Z = X$ implies [1, Theorem 2], and the case where Z is open in X and $\phi(x)$ is a singleton for all $x \in X - Z$ implies [3, Theorem 1.2][2].

The conclusion of Theorem 1.1 can be strengthened to assert that, if $A \subset X$ is closed, then every selection g for $\phi | A$ extends to a selection f for ϕ: In fact, if we define $\phi_g: X \to \mathscr{F}(Y)$ by $\phi_g(x) = \phi(x)$ for $x \notin A$ and $\phi_g(x) = \{g(x)\}$ for $x \in A$, then ϕ_g is l.s.c. by [2, Example 1.3], so ϕ_g has a selection f by Theorem 1.1, and this f is a selection for ϕ which extends g.

2. Proof of Theorem 1.1.

As in the proofs of the special cases of Theorem 1.1 which were obtained in [1], it will suffice to show that for each $\varepsilon > 0$ there exists a continuous $f: X \to Y$ such that $f(x) \in B_\varepsilon(\phi(x))$[3] for all $x \in X$. Once that is done, one can obtain the required selection for ϕ as the limit of a uniformly Cauchy sequence of continuous functions $f_n: X \to Y$ such that $f_n(x) \in B_{1/n}(\phi(x))$ for all $x \in X$.

[1] Observe that, for normal X, $\dim_X Z \leqq 0$ is valid if either $\dim Z \leqq 0$ or $\dim X \leqq 0$.

[2] In the latter two cases, Theorem 1.1 is valid if Y is any complete metric space, since such a space is always homeomorphic to a closed subset of a Banach space.

[3] $B_\varepsilon(S)$ denotes the open ε-neighborhood of S.

So let $\varepsilon > 0$ be given. For each $y \in Y$, let $U_y = \{x \in X: y \in B_\varepsilon(\phi(x))\}$. Then $\{U_y: y \in Y\}$ is an open cover of X because ϕ is l.s.c., so there exists a locally finite, open cover $\{V_y: y \in Y\}$ of X with $\bar{V}_y \subset U_y$ for all $y \in Y$. For each $x \in X$, let $F_x = \{y \in Y: x \in \bar{V}_y\}$; then F_x is finite, and $F_x \subset B_\varepsilon(\phi(x))$. Let $S = X - Z$, and for each $s \in S$ define

$$G_s = \{x \in X: \operatorname{conv} F_s \subset B_\varepsilon(\phi(x))\} - \bigcup\{\bar{V}_y: y \notin F_s\} .$$

Then $s \in G_s$ because $B_\varepsilon(\phi(s))$ is convex, and G_s is open in X because ϕ is l.s.c. and $\operatorname{conv} F_s$ is compact (see [**3**, Lemma 11.3]). For later use, let us also note that $F_x \subset F_s$ for all $x \in G_s$.

Let $G = \bigcup\{G_s: s \in S\}$, and let $E = X - G$. Then E is closed in X and $E \subset Z$, so $\dim E \leqq 0$. Hence the relatively open cover $\{V_y \cap E: y \in Y\}$ of E has a relatively open, disjoint refinement $\{D_y: y \in Y\}$[4].

Let $W_y = V_y \cap (D_y \cup G)$. The $\{W_y: y \in Y\}$ is a locally finite, open cover of X, and thus has a partition of unity $\{p_y: y \in Y\}$ subordinated to it. Define

$$f(x) = \sum_{y \in Y} (p_y(x))y .$$

Clearly f is continuous, so we need only check that $f(x) \in B_\varepsilon(\phi(x))$ for all $x \in X$.

If $x \in E$, the $f(x) = y \in B_\varepsilon(\phi(x))$ for the unique $y \in Y$ such that $x \in D_y$. So suppose that $x \in G$. Then $x \in G_s$ for some $s \in S$, so

$$f(x) \in \operatorname{conv} F_x \subset \operatorname{conv} F_s \subset B_\varepsilon(\phi(x)) .$$

That completes the proof.

REMARK. The above proof implies that X need only be assumed normal and countably paracompact if Y is separable, and that X need only be normal if $\bigcup_{x \in X} \phi(x)$ is contained in a compact subset of Y.

REFERENCES

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[4] This follows, for instance, from [**1**, Proposition 2].