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EXPONENTIAL DIOPHANTINE EQUATIONS

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CONTENTS

We study equations in which the unknowns are the exponents. (Work in this field originated with C. Størmer and D. H. Lehmer. More recently, Leo J. Alex has extended their results; his work relates to classification of nonabelian simple groups.)

(i) For the equation $k+7^a=3^c+5^d$, $k=3^b, 5^b, 13^b$, or 17^b, and for many similar 4-term equations, we find all integral solutions.

(ii) We find all integral solutions of $3^a+7^b=3^c+5^d+2$.

(iii) We prove that there are infinitely many odd m such that $m^a+7^b=3^c+5^d$ has only the solutions $(a, b, c, d) = (0, 0, 0, 0)$, $(0, 1, 1, 1)$.

1. Introduction. By an *exponential Diophantine equation* (*eDe*) we mean an equation in which the bases are (given or unknown) integers; the exponents are unknown integers. In this article the exponents are nonnegative as well. Examples are the equations

$$1 + 2^a + 7^b = 3^c + 5^d ,$$
$$3^a + 7^b = 3^c + 5^d + 2 .$$

(All solutions of these equations are determined in §6 of this article.)

Isolated examples of *eDe*'s occur very early in the history of the theory of numbers (Mersenne, Fermat). The equation $x^y = y^x$ is another hoary example. To compute accurately the logarithms of primes, Størmer [12] found all solutions of the equation $1 + 2^a3^b5^c = 2^d3^e5^f$; his method depended on the theory of Pell equations. Lehmer [8] refined Størmer's methods. This approach is still useful for computing the logarithm of a prime to high precision.

Another application [2a] of *eDe*'s is to the theory of finite nonabelian simple groups. Suppose G is such a group, $p, q_1, \cdots, q_s$ are the distinct primes dividing $|G|$, $p^2 \nmid |G|$, and $m = [N: C]$ where N, C are respectively the normalizer and centralizer in G of a Sylow p-subgroup, S_p. Then the degree equation for a p-block of highest defect must have the form

$$\sum_{i=1}^{m} \pm x_i^{a(i)} = 0 \, , \tag{1.01}$$

where $q_1, \cdots, q_s$ are the only possible prime factors of x_i, $i = 1, 2, \cdots, m$. (If one of the terms of (1.01) is 1, the p-block involved is the principal p-block.) One step in the solution of (1.01) is to find all solutions when the x_i are themselves primes. In particular the finite nonabelian simple groups of order $2^a3^b7^cp$ and $2^a3^b5^c7^dp$ with $m = 2$ and $C = S_p$ were classified by L. J. Alex [2c, d] using *eDe*'s.

If the conditions in equation (1.01) are changed to read: the prime factors of x_i are all contained in the finite set M_i, and $M_i \cap M_j = \varnothing$ if $i \neq j$, then it is known [11] that there are only finitely many solutions. For certain problems of this type we find all solutions by elementary methods; but we cannot prove that elementary methods always suffice to this end, even when $M_i = \{q_i\}$, q_i a single prime. Here "elementary methods" means reduction of (1.01) to a congruence with respect to single (perhaps a large) modulus.

We note especially that we find all solutions in several cases to which the general results of [11] do not apply at all. But characterization of those equations solvable by elementary methods has eluded us.

A third application is to the study of the class equation for a finite group G. If g is the order of G, then

$$g = 1 + h_1 + \cdots + h_c \, , \tag{1.02}$$

where c is the number of nontrivial classes, and where h_i is the cardinality of the ith class. It is known that h_i divides g, so that for each c, and for a given g, (1.02) is an *eDe*. For fixed g, the equation clearly has only a finite number of solutions. It is interesting to find the solutions (for given g) for each c. If x_i $(x_i | g)$ satisfy

$$g = 1 + x_1 + \cdots + x_c \, , \tag{1.03}$$

it is usually a separate question to decide whether a finite group with c nontrivial classes C_i exists with $|C_i| = x_i$. (If g is too small, there is no solution to (1.02); this puts a bound on the order of a group with a preassigned number of classes.)

An index of *eDe*'s is given in §9 of this article. When confronted with a new *eDe*, one should first use parity checks; if all solutions are not immediately apparent, one may nevertheless find that a parity check will reduce the problem to one of the problems in our index. Again, by choosing special values of certain exponents in the equations in the index, one may be able to subsume the equation in question under an indexed equation. Finally, if neither of these approaches is effective, it may be necessary to apply our methods (if not our results) in some detail. Hence, a fairly complete explanation of the methods is given in the body of the paper.

We have some comments about these methods. With a few classes of exceptions, we have been able to find all solutions of each equation studied in this article by applying elementary methods exclusively. This is not to say that the approach is easy: in some cases our goal was reached only after expending energy comparable to that needed to hike long distances through a heavy snowstorm. Our work has led us to certain conjectures (see §8).

2. Summary. This article is separated into several sections, as indicated in the table of contents (which appears immediately following the abstract). The interconnection of the various sections is as follows. Section 3 is concerned primarily with rather easy examples of exponential Diophantine equations of the form $1 + x^a = y^b + z^c$, where x, y, z are given. All of these equations are completely solved by the use of modular arithmetic. Some generalizations of these equations are considered in §4. Here we adhere to the same methods but the difficulties encountered are much greater. In fact, in some cases, for brevity, we refer to the rather potent results of §6 (which is independent of the other sections). The logical sequence of sections is thus different from the actual progression that appears. For, we have anticipated that some readers might prefer to peruse the material without concerning themselves with the truth of all statements. *We emphasize that when we claim to find all solutions of an eDe, all of the arguments to support our claim are given in full in this article.* (However, exceedingly trivial proofs are omitted entirely.)

In §8 we note that some *eDe*'s cannot be solved using modular arithmetic and a finite number of moduli. The equation $3^a + 7^b = 3^c + 5^d$ *can* be solved completely (i.e., all integer solutions can be found, with proof that there are no others) using a finite number of moduli. On the other hand, it is not clear how to find all solutions of the equation $3^a + 5^b = 3^c + 5^d$; we can however prove that no finite number of moduli is sufficient for this purpose.

In several sections, in particular in §5, we give examples of infinite sets of *eDe*'s that can be solved using only a finite number of moduli. For certain infinite sets of *eDe*'s we show that no nontrivial solutions exist. (A solution is called trivial if all exponents are zero.) On the other hand, there are infinitely many *eDe*'s of the form $1 + (pq)^a = p^b + q^c$ which cannot be solved using a finite number of moduli.

The problems in §7 vary considerably in difficulty. Further, it is easy to invent equations of the form of this section that have no solution.

Additional problems (without solutions) and conjectures are presented in §8. In §9 we attempt to list all *eDe*'s with which we are familiar (in addition to those solved in this article).

3. Equations of the Form $1 + x^a = y^b + z^c$.

Some equations of the form considered in this section (see [5]) provided the motivation for this paper. Those included here are rather easy to solve. Further, each of these equations (with x, y, z fixed) can undoubtedly be generalized in many ways to an equation of the form $X^a + Y^b = Z^c + W^d$.

3.01 THEOREM. *The solutions of*

$$1 + 2^a = 2^b + 3^c \tag{3.02}$$

in nonnegative integers are: $(a, b, c) = (2, 1, 1)$, $(4, 3, 2)$ *or* $(t, t, 0)$, t *arbitrary*.

Proof. Let (a, b, c) be another solution. Clearly $c > 0$. Using mod 3 we find that a is even, b is odd. Thus, using mod 8, $b \geqq 3$ and c is even. Either $b = 3$ or $b \geqq 4$. Suppose $b \geqq 4$. Then using mod 16, $c \equiv 0 \pmod 4$ so that $2^a \equiv 2^b \pmod 5$, a contradiction. Hence, $b = 3$, $2^a = 7 + 3^c$. Clearly $c \geqq 4$. Therefore, using mod 81, $a \equiv 16 \pmod{54}$. Using mod 19, $c \equiv 16 \pmod{18}$. Thus we have a contradiction mod 109 and there is no other solution. □

3.03 THEOREM. *The solutions to*

$$1 + 2^a = 2^b + 5^c \tag{3.04}$$

a	b	c
t	t	0
3	2	1
5	3	2
7	2	3

3.05 Table. The solutions of (3.04).

in nonnegative integers are given in Table 3.05. *Here t is arbitrary.*

Proof. Let (a, b, c) be another solution. Clearly $a, b, c > 0$. Since $5^c - 1 = 2^b(2^{a-b} - 1)$, examining cases we find that $c \geqq 4$ so that $a \geqq 10$. Using mod 16 we conclude that $b > 1$. We consider three cases: $b = 2$, $b = 3$ and $b \geqq 4$.

3.051 *Case.* $b = 2$. Thus $2^a = 3 + 5^c$, $5^c \equiv -3 \pmod{1024}$, $c \equiv 163 \pmod{256}$. Thus we have a contradiction mod 257.

3.052 *Case.* $b = 3$. Thus $2^a = 7 + 5^c$, $5^c \equiv -7 \pmod{64}$, $c \equiv 10 \pmod{16}$. Then, using mod 3, a is odd. This yields a contradiction mod 17.

3.053 *Case.* $b \geqq 4$. Then $5^c \equiv 1 \pmod{16}$ so that $c \equiv 0 \pmod 4$. Hence, using mod 13, $a \equiv b \pmod{12}$. Thus we have a contradiction mod 5. □

3.06 THEOREM. *The only solutions to*

$$1 + 2^a = 2^b + 7^c \tag{3.07}$$

in nonnegative integers are $(a, b, c) = (3, 3, 1)$, $(6, 4, 2)$ *or* $(t, t, 0)$, t *arbitrary.*

Proof. Let (a, b, c) be another solution. Clearly $b, c > 0$. Since $7^c - 1 = 2^b(2^{a-b} - 1)$ we can easily verify that $c \geqq 4$, $a > 11$. Further using mod 7, $(a, b) \equiv (0, 1) \pmod 3$. If $b = 1$ we immediately have a contradiction mod 16. Also, if $b = 2$ or 3 we have a contradiction mod 7. Hence we have two remaining cases: $b = 4$ and $b \geqq 5$.

3.071 *Case.* $b = 4$. Then $2^a = 15 + 7^c$. Hence $7^c \equiv 113 \pmod{128}$ so that $c \equiv 10 \pmod{16}$. Thus $2^a \equiv 15 + 2 \equiv 0 \pmod{17}$, a contradiction.

3.072 *Case.* $b \geqq 5$. Thus $1 \equiv 7^c \pmod{32}$ so that $c \equiv 0 \pmod 4$. Then, using mod 9, $(a, b, c) \equiv (3, 1, 1)$ or $(0, 4, 2)$ $(\text{mod}\,(6, 6, 3))$, so that, in fact $(a, b, c) \equiv (3, 1, 4)$ or $(0, 4, 8)$ $(\text{mod}\,(6, 6, 12))$. Using mod 13 we conclude that $(a, b, c) \equiv (6, 10, 8) \pmod{12}$. Examining cases mod 19 we conclude that $(a, b) \equiv (6, 22) \pmod{36}$. This yields a contradiction mod 37. □

3.08 THEOREM. *The only solutions of*

$$1 + 5^a = 3^b + 3^c \tag{3.09}$$

in nonnegative integers are $(a, b, c) = (0, 0, 0)$ *or* $(1, 1, 1)$.

Proof. Let (a, b, c) be another solution. Clearly $abc \neq 0$. Hence using mod 3, a is odd, so that, using mod 8, b and c are also odd. Either $b = 1$ or $b > 1$. Suppose $b = 1$. Then $c > 1$ so that, using mod 9, $a \equiv 5 \pmod 6$. Thus, using mod 7, $c \equiv 0 \pmod 6$ and we have a contradiction mod 13. Hence $b > 1$ and by symmetry $c > 1$. Thus from mod 9, $a \equiv 3 \pmod 6$. This yields a contradiction mod 7. $\square$

3.10 THEOREM. *The only solutions to*

$$1 + 5^a = 3^b + 23^c \tag{3.11}$$

in nonnegative integers are $(a, b, c) = (0, 0, 0)$ *or* $(2, 1, 1)$.

Proof. Let (a, b, c) be another solution. Clearly $abc \neq 0$. Using mod 8 we conclude that $a \equiv 0$, $b \equiv c \pmod 2$. Hence, using the moduli 5, 16 successively we have $b \equiv c \equiv 1$, $a \equiv 2 \pmod 4$. We consider two cases: $b = 1$ and $b \geqq 2$.

3.111 *Case.* $b = 1$. Then $5^a = 2 + 23^c$. Clearly $a > 2$ so that $23^c \equiv -2 \pmod{125}$. Hence $c \equiv 21 \pmod{100}$, $5^a \equiv 66 \pmod{101}$, a contradiction.

3.112 *Case.* $b \geqq 2$. Then $1 + 5^a \equiv 5^c \pmod 9$ so that $(a, c) \equiv (0, 5)$, $(2, 3)$ or $(4, 1) \pmod 6$. Using mod 13, $b \equiv c \pmod 6$, so that, using mod 7, $(a, b, c) \equiv (0, 5, 5) \pmod 6$. Thus, using mod 19, $(a, b, c) \equiv (3, 5, 2)$ or $(6, 11, 5) \pmod{(9, 18, 9)}$ so that in fact $(a, b, c) \equiv (30, 5, 29)$ or $(6, 29, 5) \pmod{36}$. Thus we have a contradiction mod 37. $\square$

3.12 THEOREM. *The only solutions of*

$$1 + 5^a = 7^b + 19^c \tag{3.13}$$

in nonnegative integers are $(a, b, c) = (0, 0, 0)$ *and* $(2, 1, 1)$.

Proof. Let (a, b, c) be another solution. Clearly $b, c > 0$, $a \geqq 3$. Using mod 3, a is even. Further, using mod 5, b and c are odd. Thus, using mod 7 we conclude $(a, c) \equiv (2, 1)$ or $(4, 5) \pmod 6$, so that, using mod 9, $(a, b, c) \equiv (2, 1, 1)$ or $(4, 5, 5) \pmod 6$. We consider two cases: $b = 1$ and $b \geqq 2$.

3.131 *Case.* $b = 1$. Then $5^a = 6 + 19^c$. Using mod 125 we find that $c \equiv 31 \pmod{50}$ so that using mod 11, $a \equiv 2 \pmod 5$. Thus we have a contradiction mod 101.

3.132 *Case.* $b \geqq 2$. Thus $1 + 5^a \equiv 19^c \pmod{49}$ so that $(a, b, c) \equiv (14, 1, 1)$ or $(28, 5, 5) \pmod{(42, 6, 6)}$. In each case there is a contradiction mod 43. □

The following theorem deals with several very easy examples of the form $1 + p^a = q^b + q^c$, p, q distinct primes.

3.14 THEOREM. *The only solution to each of the following equations in nonnegative integers is* $(a, b, c) = (0, 0, 0)$:

$$1 + 5^a = 7^b + 7^c \tag{3.141}$$

$$1 + 5^a = 11^b + 11^c \tag{3.142}$$

$$1 + 5^a = 17^b + 17^c \tag{3.143}$$

$$1 + 13^a = 17^b + 17^c \tag{3.144}$$

$$1 + 17^a = 7^b + 7^c \tag{3.145}$$

$$1 + 67^a = 17^b + 17^c\,. \tag{3.146}$$

Proof. (3.141) Use mod 7 and mod 3. (3.142) Use mod 11. (3.143) Use the moduli 8, 3 and 5 successively. (3.144) Use mod 17 and mod 16. (3.145) Use mod 7 and mod 9. (3.146) Use mod 17 and mod 8. □

We conclude this section with an obvious general theorem:

3.15 THEOREM. *Let* p, q, r *be distinct primes such that* $q \equiv 1 \pmod{p}$. *Then*

$$1 + p^a = q^b + r^c \tag{3.16}$$

has only the trivial solution.

4. Equations of the Form $x^a + y^b = z^c + w^d$. In this section we find all solutions in nonnegative integers of several equations with four variable exponents. While some of these are quite easy, others are rather difficult.

4.01 THEOREM. *The solutions to*

$$3^a + 7^b = 3^c + 5^d \tag{4.02}$$

in nonnegative integers are given in Table (4.03). *Here* t *is arbitrary.*

a	b	c	d
t	0	t	0
1	1	2	0
0	1	1	1
1	2	3	2
3	0	1	2
3	1	2	2

4.03 Table. The solutions of (4.02).

Proof. Suppose (a, b, c, d) is another solution. Referring to Theorem 6.01 we conclude that $a \neq 1, 2$. Further from [5], $a \neq 0$ so that in fact $a \geqq 3$. Using mod 3, clearly $c > 0$. The theorem is a consequence of the following five lemmas:

4.011 LEMMA. $b > 0$.

Proof. Suppose $b = 0$. Then $3^a + 1 = 3^c + 5^d$. Clearly $d > 0$. We consider two cases: $c = 1$ and $c > 1$.

4.0111 *Case.* $c = 1$. Then $3^a = 2 + 5^c$. Referring to Theorem 6.05 (with $b = c = 0$) we have a contradiction. Hence $c > 1$.

4.0112 *Case.* $c > 1$. Then $1 \equiv 5^d \pmod 9$ so that $d \equiv 0 \pmod 6$. Thus, using mod 7, $a \equiv c \pmod 6$. Hence, using mod 13, $d \equiv 0 \pmod 4$ so that $d \equiv 0 \pmod{12}$. Further, using mod 5 we conclude that $a \equiv 3$, $c \equiv 1 \pmod 4$. Also, using mod 31 we conclude that $a \equiv c \pmod{30}$. Thus $1 \equiv 5^d \pmod{11}$ so that $d \equiv 0 \pmod 5$. In particular $d \equiv 0 \pmod{20}$ so that using mod 41 we conclude $a \equiv c \pmod 8$, a contradiction. Hence $b > 0$. $\square$

4.012 LEMMA. $d > 0$.

Proof. Assume the contrary. Then $3^a + 7^b = 3^c + 1$. Clearly $c > 1$. Hence $7^b \equiv 1 \pmod 9$, $b \equiv 0 \pmod 3$. Thus, using mod 19, $a \equiv c \pmod{18}$. This yields a contradiction mod 7. $\square$

4.013 LEMMA. $c > 1$.

Proof. Assume $c = 1$. Then $3^a + 7^b = 3 + 5^d = 3^0 + 5^d + 2$. Thus from Theorem 6.05 we have a contradiction. $\square$

4.014 LEMMA. $(a, b, d) \equiv (3, 1, 2) \pmod{36}$ *and* $c = 2$.

Proof. Since $7^b \equiv (5^2)^b \equiv 5^d \pmod 9$, we conclude that d is even.

Using the moduli 5, 16 successively we have the following table:

a	b	c	d
1	2	3	2
1	3	0	2
3	0	1	2
3	1	2	2

4.04 Table. $(a, b, c, d) \bmod 4$.

In particular, $d \equiv 2 \pmod 4$, a is odd and $b \not\equiv c \pmod 2$. Thus, considering the moduli 7 and 9 successively we have the following table:

a	b	c	d
1	5	0	4
1	3	2	0
1	4	3	2
3	0	5	0
3	1	2	2
3	5	4	4
5	3	4	0
5	1	0	2
5	2	1	4

4.05 Table. $(a, b, c, d) \bmod 6$.

Combining the above results and using mod 13 we conclude that $(a, b, c, d) \equiv (1, 3, 2, 0)$ or $(3, 1, 2, 2) \pmod{(6, 12, 6, 6)}$. Thus $(a, b, c, d) \equiv (1, 3, 8, 6)$ or $(3, 1, 2, 2) \pmod{12}$. Using mod 19 we conclude $(a, b, c, d) \equiv (13, 0, 14, 3)$, $(3, 1, 2, 2)$ or $(3, 1, 8, 5) \pmod{(18, 3, 18, 9)}$ so that in fact $(a, b, c, d) \equiv (13, 3, 32, 30)$, $(3, 1, 2, 2)$ or $(3, 1, 26, 14) \pmod{(36, 12, 36, 36)}$. Finally, using mod 37 we conclude $(a, b, c, d) \equiv (3, 1, 2, 2) \pmod{36}$. Now suppose $c \geqq 3$. Then, since $b \equiv 1 \pmod 9$ we have $7 \equiv 5^d \pmod{27}$ so that $d \equiv 14 \pmod{18}$, a contradiction. Hence $(a, b, d) \equiv (3, 1, 2) \pmod{36}$ and $c = 2$. $\square$

4.055 Lemma. $c \neq 2$. (*Thus* (a, b, c, d) *does not exist.*)

Proof. Suppose on the contrary that $c = 2$. Then $3^a + 7^b = 9 + 5^d$. We consider two cases: $b = 1$, $b > 1$.

4.0551 *Case.* $b = 1$. Then $3^a = 2 + 5^d$ so that $3^a + 7^0 = 3^0 + 5^d + 2$. Hence referring to Theorem 6.05 we have a contradiction.

4.0552 *Case.* $b > 1$. Then $3^a \equiv 9 + 5^d \pmod{49}$ so that $(a, d) \equiv (3, 14)$, $(9, 2)$, $(15, 32)$, $(21, 20)$, $(27, 8)$, $(33, 38)$ or $(39, 26) \pmod{42}$. In each case we have a contradiction mod 43. □

4.06 THEOREM. *The solutions to*

$$5^a + 7^b = 3^c + 5^d \tag{4.07}$$

in nonnegative integers are given in the following table. Here t is arbitrary.

a	b	c	d
t	0	0	t
0	1	1	1
2	1	3	1
2	3	5	3

4.08 Table. The solutions of (4.07).

Proof. Let (a, b, c, d) be another solution. From [5], $a, d > 0$. Further, from 6.01, $a > 1$. The remainder of the proof consists of the following seven lemmas:

4.081 LEMMA. $b > 0$.

Proof. Suppose $b = 0$. Then $5^a + 1 = 3^c + 5^d$. Clearly $c > 0$. Hence, using mod 3, $a \equiv 0$, $d \equiv 1 \pmod 2$. Thus we have a contradiction mod 8. □

4.082 LEMMA. $c > 0$.

Proof. Suppose $c = 0$. Then $5^a + 7^b = 1 + 5^d$. Thus $7^b \equiv 1 \pmod 5$ so that $b \equiv 0 \pmod 4$. Using mod 3, $a \equiv d \pmod 2$. Examining cases mod 7 we conclude $(a, d) \equiv (3, 1)$ or $(4, 0) \pmod 6$. In either case, using mod 9 we conclude that $b \equiv 1 \pmod 3$ so that in fact $b \equiv 4 \pmod{12}$. Applying these results mod 13 we have a contradiction. □

4.083 LEMMA. $d > 1$.

Proof. Suppose $d = 1$. Then $5^a + 7^b = 3^c + 5$. Clearly $c \geqq 4$. Using mod 3 we conclude that a is even. Hence, using mod 8, b and c are odd. Thus, using the moduli 7 and 9 successively we have $(a, b, d) \equiv (0, 5, 1)$ or $(2, 1, 3) \pmod 6$. Applying the above results mod 13 we conclude that $(a, b, c) \equiv (2, 1, 3) \pmod{(12, 12, 6)}$. If $b = 1$

then $5^a + 2 = 3^c$ and we have a contradiction by Theorem 6.05. Hence $b > 1$. Thus $5^a \equiv 3^c + 5 \pmod{49}$. Hence $(a, c) \equiv (2, 39)$, $(8, 15)$, $(14, 33)$, $(20, 9)$, $(26, 27)$, $(32, 3)$ or $(38, 21) \pmod{42}$. In each case we have a contradiction mod 43. Hence $d > 1$. □

4.084 LEMMA. *$(a, b, c, d) \equiv (2, 1, 3, 1)$ or $(2, 3, 5, 3) \pmod{36}$.*

Proof. If $c = 1$ then $7^b \equiv 3 \pmod{25}$, an absurdity. Hence $c > 1$. Using the moduli 3 and 8 successively we conclude that $a \equiv 0$, $b \equiv c \equiv d \equiv 1 \pmod 2$. Thus, using the moduli 7 and 9 successively we have the following table:

a	b	c	d
0	5	1	1
0	3	5	5
2	1	3	1
2	3	5	3
4	5	1	3
4	1	3	5

4.09 Table. (a, b, c, d) mod 6.

Using the moduli 19 and 27 successively (noting that $c \geqq 3$) we have the following table:

a	b	c	d
6	11	7	7
0	11	13	13
0	9	5	11
2	1	3	1
14	7	9	13
2	3	5	3
8	9	17	3
4	17	7	15
10	7	3	17
4	13	9	11

4.10 Table. (a, b, c, d) mod 18.

Considering each of these cases mod 37 we have the following possibilities: $(a, b, c, d) \equiv (6, 11, 7, 25)$, $(2, 1, 3, 1)$, $(2, 3, 5, 3)$ or $(4, 17, 7, 33)$ (mod (36, 18, 18, 36)). Using mod 13 we conclude $(a, b, c, d) \equiv (2, 1, 3, 1)$ or $(2, 3, 5, 3)$ (mod (36, 36, 18, 36)). Finally using mod 5, $(a, b, c, d) \equiv (2, 1, 3, 1)$ or $(2, 3, 5, 3) \pmod{36}$. □

4.105 LEMMA. *$(a, b, c, d) \equiv (2, 3, 5, 3) \pmod{36}$.*

Proof. Suppose on the contrary that $(a, b, c, d) \equiv (2, 1, 3, 1) \pmod{36}$. Since $d \geqq 2$ (Lemma 4.083) we have $7 \equiv 3^c \pmod{25}$ so that $c \equiv 15 \pmod{20}$ and in fact $c \equiv 15 \pmod{30}$. Thus, using mod 31 we conclude that $b \equiv 8 \pmod{15}$, a contradiction. $\square$

4.106 LEMMA. $a = 2$.

Proof. Suppose on the contrary $a > 2$. Since $b \equiv 3 \pmod 4$ we conclude that $7^3 \equiv 3^c \pmod{25}$, $c \equiv 5 \pmod{20}$ and in fact $c \equiv 5 \pmod{180}$. Further $5^2 + 7^b \equiv 3^5 + 1 \pmod{31}$ so that $7^b \equiv 2 \pmod{31}$, $b \equiv 3 \pmod{15}$ and $b \equiv 3 \pmod{180}$. Examining cases mod 61 we conclude that $a \equiv 2$, $d \equiv 3 \pmod{30}$ so that $(a, b, c, d) \equiv (2, 3, 5, 3) \pmod{180}$. Since $a, d \geqq 3$, $7^b \equiv 3^c \pmod{125}$, so that $93 \equiv 3^c \pmod{125}$, $c \equiv 65 \pmod{100}$. Applying these results mod 101 we conclude that $(a, b, c, d) \equiv (12, 63, 65, 13)$ or $(22, 3, 65, 8) \pmod{(25, 100, 100, 25)}$. In particular, $(a, b, c, d) \equiv (62, 63, 65, 63)$ or $(122, 3, 65, 183) \pmod{300}$. Substituting these results mod 151 we have a contradiction. Hence $a = 2$. $\square$

4.107 LEMMA. $a \neq 2$ (*and hence* (a, b, c, d) *does not exist*).

Proof. Suppose that $a = 2$. Then $25 + 7^b = 3^c + 5^d$. (Recall that $(b, c, d) \equiv (3, 5, 3) \pmod{36}$.) We consider two cases: $d = 3$ and $d > 3$.

4.1071 *Case.* $d = 3$. Then $7^b = 3^c + 100$. Further (from Table 4.08), $c \geqq 6$. Hence $7^b \equiv 100 \pmod{729}$ so that $b \equiv 84 \pmod{243}$ and in fact $b \equiv 327 \pmod{486}$. Since 3 is a primitive root mod the prime 487 and since 7 has order 162 (mod 487) and $327 \equiv 3 \pmod{162}$ we conclude immediately that $c \equiv 5 \pmod{486}$. We consider our equation modulo the prime 1459. Since 3 is a primitive root and the order of 7 is 243, we have the following possibilities: $7^{84} \equiv 3^{5+486k} + 100 \pmod{1459}$, $k = 0, 1, 2$. Since $7^{84} \equiv 1016$, $3^{486} \equiv 339 \pmod{1459}$, in each case we have a contradiction. Hence $d > 3$.

4.1072 *Case.* $d > 3$. Since $7^3 \equiv 3^c \pmod{25}$ we conclude that $c \equiv 5 \pmod{20}$, $c \equiv 5 \pmod{180}$. Using mod 31 we conclude that $b \equiv 3 \pmod{15}$, $b \equiv 3 \pmod{180}$. Thus, using mod 11 it immediately follows that $d \equiv 3 \pmod 5$, $d \equiv 3 \pmod{180}$. Further, since $25 + 7^b \equiv 3^c \pmod{125}$ we conclude that $118 \equiv 3^c \pmod{125}$, $c \equiv 5 \pmod{100}$. Examining possibilities mod 101 we have: $b \equiv 3 \pmod{100}$, $d \equiv 3 \pmod{25}$. Hence $368 \equiv 25 + 7^3 \equiv 3^c \pmod{625}$, $c \equiv 205 \pmod{500}$. We now consider our equation mod 251. Since 5 has order 25 and 3, 7 have order 125 we have: $25 + 7^{3+25k} \equiv 3^{80} + 5^3 \pmod{251}$, $0 \leqq$

$k \leqq 4$. Since $7^{25} \equiv 149$, $3^{80} \equiv 63 \pmod{251}$, in each case we deduce a contradiction. This completes the proof of the lemma and the theorem. □

4.11 THEOREM. *The only solutions to*

$$13^a + 7^b = 3^c + 5^d \tag{4.12}$$

are $(a, b, c, d) = (0, 0, 0, 0)$, $(0, 1, 1, 1)$ *and* $(1, 0, 2, 1)$.

Proof. Let (a, b, c, d) be another solution. From [5], $a > 0$. Using mod 3 we conclude that either $c = 0$ and d is even, or $c > 0$ and d is odd. In the former case $b > 0$ so that using the moduli 8 and 7 successively we have a contradiction. Hence $c > 0$, d odd, so that using mod 8 we conclude $(a, b, c, d) \equiv (1, 0, 0, 1)$ or $(0, 1, 1, 1) \pmod 2$. The theorem is a consequence of the following three lemmas.

4.121 LEMMA. $b > 0$.

Proof. Suppose that $b = 0$. Then $13^a + 1 = 3^c + 5^d$, $(a, c, d) \equiv (1, 0, 1) \pmod 2$. Using mod 5 we conclude that $a \equiv 1$, $c \equiv 2 \pmod 4$, so that, using mod 16, $d \equiv 1 \pmod 4$. Thus, using mod 13, $c \equiv 2 \pmod{12}$. Then, using mod 7 we conclude that $d \equiv 1 \pmod 6$, $d \equiv 1 \pmod{12}$. We consider two cases: $d = 1$, $d > 1$.

4.1211 *Case.* $d = 1$. Then $13^a = 3^c + 4$. Clearly $c \geqq 3$. Thus, using mod 27, $a \equiv 7 \pmod 9$ so that $a \equiv 25 \pmod{36}$. This yields a contradiction mod 37. Hence $d > 1$.

4.1212 *Case.* $d > 1$. Then $13^a + 1 \equiv 3^c \pmod{25}$. Therefore, using mod 25 we have the following possibilities: $(a, c) \equiv (1, 18)$, $(5, 14)$, $(9, 10)$, $(13, 6)$ or $(17, 2) \pmod{20}$. Thus, using mod 11 we conclude that $(a, c, d) \equiv (1, 38, 49) \pmod{60}$. This yields a contradiction mod 61. Hence $b > 0$. □

4.122 LEMMA. $(a, b, c, d) \equiv (0, 1, 1, 1) \pmod 2$.

Proof. Suppose the contrary. Then $(a, b, c, d) \equiv (1, 0, 0, 1) \pmod 2$. Using mod 7 we conclude that $(c, d) \equiv (0, 1) \pmod 6$ (and in particular $c \geqq 6$). Using mod 13 we conclude that $b \equiv 4 \pmod{12}$, $d \equiv 3 \pmod 4$. Hence, using mod 9, $a \equiv 2 \pmod 3$ so that $a \equiv 5 \pmod 6$. Thus, using mod 19, $(a, b, c, d) \equiv (11, 4, 12, 4) \pmod{(18, 12, 18, 9)}$ so that in fact $(a, b, c, d) \equiv (11, 4, 12, 13) \pmod{(18, 12, 18, 18)}$. Therefore, using mod 27 we have $(a, b, c, d) \equiv (11, 16, 12, 13) \pmod{(18, 36, 18, 18)}$ which yields a contradiction mod 37. Hence $(a, b, c, d) \equiv (0, 1, 1, 1) \pmod 2$. □

4.123 LEMMA. $(a, b, c, d) \not\equiv (0, 1, 1, 1) \pmod 2$ *(and hence (a, b, c, d) does not exist).*

Proof. Assume the contrary. Using mod 13 (and the fact that c is odd) we have $(b, c, d) \equiv (7, 3, 1)$, $(9, 1, 1)$ or $(5, 1, 3)$ (mod (12, 6, 4)). Then, using mod 7 we have $(b, c, d) \equiv (9, 1, 1)$ or $(5, 1, 7)$ (mod (12, 6, 12)). Thus, using mod 5, $(a, b, c, d) \equiv (0, 9, 1, 1)$ or $(0, 5, 1, 7)$ (mod (4, 12, 12, 12)). We consider two cases: $c = 1$ and $c > 1$.

4.1231 *Case.* $c = 1$. Then $13^a + 7^b = 3 + 5^d$. Clearly $d > 1$. Using mod 9 we conclude that $(a, b, d) \equiv (8, 9, 1)$ or $(4, 5, 7)$ (mod 12). In either case, using mod 25 we find that $a \equiv 8 \pmod{20}$ so that $(a, b, d) \equiv (8, 9, 1)$ or $(28, 5, 7)$ (mod (60, 12, 12)). Hence, using mod 31 we conclude that $(a, b) \equiv (8, 45) \pmod{60}$. This yields a contradiction mod 11. Hence $c > 1$.

4.1232 *Case.* $c > 1$. (Thus in fact $c \geqq 13$.) Using mod 9 we conclude that $(a, b, c, d) \equiv (4, 9, 1, 1)$ or $(0, 5, 1, 7)$ (mod 12). Then, using the moduli 19 and 27 successively we find $(a, b, c, d) \equiv (28, 3, 7, 1)$, $(0, 2, 13, 31)$ or $(12, 8, 7, 7)$ (mod (36, 9, 18, 36)). In each case we have a contradiction mod 37. Hence no other solution exists. □

4.13 THEOREM. *The only solutions to*

$$17^a + 7^b = 3^c + 5^d \tag{4.14}$$

are $(a, b, c, d) = (0, 0, 0, 0)$ *and* $(0, 1, 1, 1)$.

Proof. Let (a, b, c, d) be another solution. From [5], $a > 0$. We consider two cases: $c = 0$ and $c > 0$.

4.141 *Case.* $c = 0$. Then $17^a + 7^b = 1 + 5^d$. Using mod 3, $a \equiv d \pmod 2$. Hence, using mod 16, $b \equiv 0 \pmod 2$, $d \equiv 0 \pmod 4$ and thus a is even. Hence we have a contradiction mod 5.

a	b	c	d
0	13	1	13
0	9	5	1
0	5	9	5
0	1	13	9
2	3	3	7
2	15	7	11
2	11	11	15
2	7	15	3

4.15 Table. (a, b, c, d) mod (4, 16, 16, 16).

4.142 *Case.* $c > 0$. Then $2^a + 1 \equiv 2^d \pmod 3$ so that $(a, d) \equiv (0, 1) \pmod 2$. Hence, using the moduli 16, 5 successively we conclude: $(a, b, c, d) \equiv (0, 1, 1, 1)$ or $(2, 3, 3, 3) \pmod 4$. Using mod 17 we have the possibilities listed in Table 4.15.

In each of the above cases we have a contradiction mod 64. Hence there is no other solution. □

4.16 THEOREM. *The solutions to*

$$5^a + 5^b = 3^c + 7^d \tag{4.17}$$

are given in the following table:

a	b	c	d
0	0	0	0
1	1	1	1
1	1	2	0
1	3	4	2
1	5	6	4
3	1	4	2
5	1	6	4
2	2	0	2
2	3	5	1

4.18 Table. The solutions of (4.17).

Proof. Let (a, b, c, d) be another solution. By [5], $a, b > 0$. By Theorem 6.05, $a, b > 1$. Immediately we conclude that $c \neq 1$ (using mod 25). The remainder of the proof is divided into five lemmas.

4.181 LEMMA. $c \geqq 2$.

Proof. Assume the contrary. Then $c = 0$ and $5^a + 5^b = 1 + 7^d$. If $a = 2$ then $1 + 2^1 + 7^d = 3^3 + 5^b$ and we have a contradiction by Theorem 6.01. Hence $a > 2$ and by symmetry $b > 2$. Thus $0 \equiv 1 + 7^d \pmod{125}$ so that $d \equiv 10 \pmod{20}$. Further, using mod 3, a, b are even. Therefore, using the moduli 7 and 9 successively we conclude that $a \equiv b \equiv 2 \pmod 6$, $d \equiv 2 \pmod 3$ so that in fact $d \equiv 50 \pmod{60}$. Thus we have a contradiction mod 31. Hence $c \geqq 2$. □

4.182 LEMMA. $d > 0$.

Proof. Suppose $d = 0$. Then $5^a + 5^b = 3^c + 1$, $3^c \equiv -1 \pmod{25}$, $c \equiv 10 \pmod{20}$. Using mod 3, a and b are odd. Thus, using mod 7, $(a, b, c) \equiv (1, 1, 2), (3, 3, 4), (3, 5, 0)$ or $(5, 3, 0) \pmod 6$. Thus $(a, b, c) \equiv$

(1, 1, 50), (3, 3, 10), (3, 5, 30) or (5, 3, 30) (mod (6, 6, 60)). In each case we have a contradiction mod 31. Hence $d > 0$. □

4.183 LEMMA. $(a, b, c, d) \equiv (3, 3, 5, 1) \pmod{60}$.

Proof. Using mod 3, a and b are odd. Also, using mod 8, $c \equiv d \pmod 2$. Since $0 \equiv 3^c + 7^d \pmod{25}$, we have $c \equiv 10 + 15d \pmod{20}$ so that $c \equiv 0 \pmod 5$. Using these results and the moduli 7, 9 successively we have the following table of possibilities:

a	b	c	d
1	1	25	3
1	3	10	2
1	5	0	4
3	1	10	2
3	3	5	1
3	5	20	0
5	1	0	4
5	3	20	0
5	5	15	5

4.19 Table. (a, b, c, d) mod (6, 6, 30, 6).

Hence, using mod 31 we have $(a, b, c, d) \equiv (1, 1, 25, 21)$, $(3, 3, 5, 1)$ or $(5, 5, 15, 11)$ (mod (6, 6, 30, 30)). (In particular c and d are odd.) Thus, using mod 13 we conclude $(a, b, c, d) \equiv (3, 3, 5, 1)$ or $(11, 11, 15, 11)$ (mod (12, 12, 30, 60)). Thus, in either case, using mod 16, $c \equiv 1 \pmod 4$. Hence, using mod 5, $d \equiv 1 \pmod 4$ so that $(a, b, c, d) \equiv (3, 3, 5, 1)$ (mod (12, 12, 60, 60)). Thus, using mod 11 we conclude $(a, b, c, d) \equiv (51, 27, 5, 1)$, $(27, 51, 5, 1)$ or $(3, 3, 5, 1) \pmod{60}$. Therefore, using mod 61, $(a, b, c, d) \equiv (3, 3, 5, 1) \pmod{60}$. □

4.194 LEMMA. $d > 1$.

Proof. Suppose $d = 1$. Then $5^a + 5^b = 3^c + 7$. By the previous lemma, $a, b \geqq 3$ so that $3^c \equiv -7 \pmod{125}$ and hence $c \equiv 5 \pmod{100}$ and in fact $c \equiv 5 \pmod{300}$. Therefore, using mod 101 we conclude that $(a, b, c) \equiv (3, 3, 5) \pmod{300}$. We consider two cases: $a = 3$ and $a > 3$.

4.1941 *Case.* $a = 3$. Then $118 + 5^b = 3^c$. Clearly $b > 3$. Thus, $118 \equiv 3^c \pmod{625}$ so that $c \equiv 305 \pmod{500}$. We have an immediate contradiction mod 251. Hence $a > 3$.

4.1942 *Case.* $a > 3$. By symmetry, $b > 3$ also. Hence $0 \equiv 3^c + 7$

(mod 625) so that $c \equiv 105 \pmod{500}$. Using mod 251 we again have a contradiction. Hence $d > 1$. $\square$

4.195 LEMMA. $d = 1$. (*Hence* (a, b, c, d) *does not exist.*)

Proof. Suppose $d > 1$. Using mod 71 we conclude that $c \equiv 5 \pmod{35}$, $d \equiv 1 \pmod{70}$. In particular $(c, d) \equiv (5, 1) \pmod{42}$. Thus, using mod 49 we conclude that $(a, b) \equiv (3, 39)$, $(9, 33)$, $(15, 27)$, $(21, 21)$, $(27, 13)$, $(33, 9)$, or $(39, 3) \pmod{42}$. Applying these results mod 43 we conclude that $(a, b) \equiv (15, 27)$ or $(27, 15) \pmod{42}$. Thus we have a contradiction mod 29. Hence no other solution exists. $\square$

While the previous theorems have not been easy, the following theorem, which would appear to be quite similar, is extremely easy.

4.20. THEOREM. *The only solutions to*

$$5^a + 7^b = 3^c + 7^d \tag{4.21}$$

in nonnegative integers are $(a, b, c, d) = (0, t, 0, t)$, t *arbitrary.*

Proof. Let (a, b, c, d) be another solution. Using mod 3 we conclude that $c = 0$ and a is even. Thus $a \geqq 2$ so that $7^b \equiv 1 + 7^d \pmod{25}$, a contradiction. $\square$

4.22 THEOREM. *The only nonnegative integral solutions to*

$$15^a + 5^b = 3^c + 7^d \tag{4.23}$$

are $(a, b, c, d) = (0, 0, 0, 0)$, $(1, 0, 2, 1)$ *and* $(2, 2, 5, 1)$.

Proof. Let (a, b, c, d) be another solution. Clearly from [5], $a > 0$. The theorem follows from several lemmas.

4.221 LEMMA. $c > 0$.

Proof. Assume the contrary. Then $15^a + 5^b = 1 + 7^d$. Clearly $b, d > 0$. Using mod 3 we find that b is odd so that mod 8 we have an absurdity. $\square$

4.222 LEMMA. $b > 0$.

Proof. Suppose $b = 0$. Then $15^a + 1 = 3^c + 7^d$. Clearly $a > 1$, $c, d > 0$. Thus, using mod 7, $c \equiv 2 \pmod 6$. (In particular, $c \geqq 2$.) Hence, using mod 9, $d \equiv 0 \pmod 3$. Thus, from mod 15, $d \equiv 1 \pmod 4$

so that $d \equiv 9 \pmod{12}$. Also, $c \equiv 2 \pmod 4$ so that $c \equiv 2 \pmod{12}$. Therefore, from mod 13, $a \equiv 4 \pmod{12}$. We consider two cases: $c = 2$ and $c \geqq 3$.

4.2221 *Case*. $c = 2$. Then $15^a = 8 + 7^d$. Clearly $d > 1$, $15^a \equiv 8 \pmod{49}$, $a \equiv 4 \pmod 7$, $a \equiv 4 \pmod{28}$, $7^d \equiv 12 \pmod{29}$, a contradiction. Hence $c > 2$.

4.2222 *Case*. $c \geqq 3$. Since $a \geqq 4$ we have $1 \equiv 7^d \pmod{27}$, $d \equiv 0 \pmod 9$, $d \equiv 9 \pmod{36}$. Using mod 19 we conclude that $(a, c) \equiv (4, 2)$, $(28, 14)$ or $(16, 26) \pmod{36}$. In each case we consider our equation mod 73 and deduce a contradiction. Hence $b \geqq 1$.

4.223 LEMMA. $d > 0$.

Proof. Suppose $d = 0$. Then $15^a + 5^b = 3^c + 1$ (and $abc > 0$). Using mod 3, b is even. Using mod 5, $c \equiv 2 \pmod 4$. Hence, using mod 8, a is even (and $a \geqq 2$). Thus, using mod 9, $b \equiv 0 \pmod 6$. Hence, using mod 7, $c \equiv 0 \pmod 6$ and in fact $c \equiv 6 \pmod{12}$. Combining the above results and using mod 16 we conclude that $b \equiv 2 \pmod 4$. Thus, from mod 13, $a \equiv 4 \pmod{12}$. Hence $5^b \equiv 1 \pmod{27}$ so that $b \equiv 0 \pmod{18}$. Considering all possibilities mod 19 we deduce a contradiction. Hence $abcd > 0$. □

4.224 LEMMA. $(a, b, c, d) \equiv (2, 2, 5, 1) \pmod{60}$.

Proof. Using mod 3, b is even (and so $b \geqq 2$). Using mod 5, $d \equiv c \pmod 2$. Hence, using mod 8, a is even ($a \geqq 2$). Using the moduli 7 and 9 successively we have the following possibilities: $(b, c, d) \equiv (0, 2, 0)$, $(2, 5, 1)$ or $(4, 1, 5) \pmod 6$. Further, $0 \equiv 3^c + 7^d \equiv 3^c + 3^{15d} \pmod{25}$ so that $c \equiv 10 + 15d \pmod{20}$ and, in particular, $c \equiv 0 \pmod 5$. Hence, using mod 31 we have $(a, b, c, d) \equiv (2, 0, 20, 6)$ or $(2, 2, 5, 1) \pmod{(10, 6, 30, 30)}$. Thus, using mod 11, we have $(a, b, c, d) \equiv (2, 2, 5, 1) \pmod{(10, 30, 30, 30)}$. Using mod 61 we conclude that $a \equiv 2 \pmod{15}$, $d \equiv 1 \pmod{60}$. Hence $a \equiv 2 \pmod{30}$. Therefore, using mod 5, $c \equiv 1 \pmod 4$ so that $c \equiv 5 \pmod{60}$. Thus, using mod 16, $b \equiv 2 \pmod 4$, $b \equiv 2 \pmod{60}$. Hence, using mod 13, $a \equiv 2 \pmod{12}$ and in fact $(a, b, c, d) \equiv (2, 2, 5, 1) \pmod{60}$ as asserted. □

4.225 LEMMA. $d > 1$.

Proof. Assume the contrary. Then $15^a + 5^b = 3^c + 7$. Since $(a, b, c) \equiv (2, 2, 5) \pmod{60}$, clearly $c \geqq 65$. We consider two cases: $a = 2$ and $a > 2$.

4.2251 *Case.* $a = 2$. Then $218 + 5^b = 3^c$, so that $218 + 5^b \equiv 0 \pmod{2187}$. Hence $b \equiv 650 \pmod{1458}$. Using mod 487 we easily conclude that $c \equiv 5 \pmod{486}$. Thus, using mod 1459 we have a contradiction. Hence $a > 2$.

4.2252 *Case.* $a > 2$. Recall that $c \geqq 5$ so that $5^b \equiv 7 \pmod{27}$, $b \equiv 14 \pmod{18}$. Thus we have a contradiction mod 19. $\square$

4.226 LEMMA. $d = 1$ *(so that (a, b, c, d) does not exist).*

Proof. Suppose instead that $d > 1$. Using mod 43 we have the following table:

a	b	c
2	2	5
2	38	29
8	8	11
8	20	23
8	32	17
38	14	17
38	20	35
20	8	41
20	26	11

4.24 Table. (a, b, c) mod 42.

(Recall that $d \equiv 1 \pmod{60}$.) Thus, using mod 49 we conclude that $(a, b, c) \equiv (8, 8, 11)$ or $(20, 8, 41) \pmod{42}$. Hence, using mod 29, $(a, b, c, d) \equiv (22, 22, 25, 5) \pmod{28}$. Thus, combining all of our results we have a contradiction when our equation is considered modulo 71. $\square$

4.25 THEOREM. *The only solutions to*

$$15^a + 3^b = 5^c + 7^d \tag{4.26}$$

in nonnegative integers are $(a, b, c, d) = (0, 0, 0, 0)$ *or* $(2, 5, 3, 3)$.

Proof. By [5], $a > 0$. Thus, using mod 3, $b \neq 0$ and c is odd. Then $3^b \equiv 2^d \pmod 5$ so that $b \equiv d \pmod 2$. Hence, using mod 8 we conclude that a is even and b and d are odd. Thus, using mod 7, $(b, c) \equiv (5, 3) \pmod 6$, $b \geqq 5$. Thus $0 \equiv 5^3 + 7^d \pmod 9$ so that $d \equiv 0 \pmod 3$, $d \equiv 3 \pmod 6$. Using mod 13 we conclude $(a, b, c, d) \equiv (0, 5, 9, 3)$, $(2, 5, 9, 9)$ or $(2, 5, 3, 3)$ (mod (12, 6, 12, 12)). Thus, using mod 16, $(a, b, c, d) \equiv (0, 11, 9, 3)$, $(2, 11, 9, 9)$ or $(2, 5, 3, 3) \pmod{12}$.

Using mod 19 and combining results we have the following table:

a	b	c	d
24	11	9	3
2	23	21	9
14	11	33	9
2	5	3	3
14	11	15	3

4.27 Table. $(a, b, c, d) \bmod (36, 36, 36, 12)$.

The remainder of the proof is divided into two lemmas.

4.271 LEMMA. *$a = 2$ and $(b, c, d) \equiv (5, 3, 3) \pmod{36}$.*

Proof. Assume $a > 2$. Then $0 \equiv 5^c + 7^d \pmod{27}$. Thus, $c \equiv 14d - 9 \pmod{18}$. Combining results, $(c, d) \equiv (9, 27), (21, 33), (33, 21), (3, 15)$ or $(15, 3) \pmod{36}$. In each case we have a contradiction mod 37. Hence $a = 2$. Thus, from Table 4.27, $(b, c, d) \equiv (23, 21, 9)$ or $(5, 3, 3) \pmod{(36, 36, 12)}$. In either case, $9 \equiv 5^3 + 7^d \pmod{27}$ so that $d \equiv 3 \pmod 9$. Thus, in the former case, we have a contradiction mod 37. Hence $(b, c, d) \equiv (5, 3, 3) \pmod{36}$. □

4.272 LEMMA. *$c > 3$.*

Proof. Assume the contrary. Then $c = 3$ and $100 + 3^b = 7^d$, $b \geqq 6$. From Lemma 4.107, case 4.1071, we immediately have a contradiction. □

4.273 LEMMA. *$c \leqq 3$ (and hence (a, b, c, d) does not exist).*

Proof. Assume the contrary. (Recall that $a = 2$, $(b, c, d) \equiv (5, 3, 3) \pmod{36}$.) Since $3^b \equiv 18 \pmod{25}$, $b \equiv 5 \pmod{20}$, $b \equiv 5 \pmod{180}$. Thus, using mod 31, $d \equiv 3 \pmod{15}$, $d \equiv 3 \pmod{180}$. Immediately then, using mod 11, $c \equiv 3 \pmod 5$, $c \equiv 3 \pmod{180}$. Since $3^b \equiv 118 \pmod{125}$, $b \equiv 5 \pmod{100}$. Thus, using mod 101, $c \equiv 3 \pmod{25}$, $d \equiv 3 \pmod{100}$ and in fact $(b, c, d) \equiv (5, 3, 3) \pmod{900}$. Thus $3^b \equiv 118 \pmod{625}$, $b \equiv 305 \pmod{500}$, $b \equiv 1805 \pmod{4500}$. Hence, using mod 251, $d \equiv 53 \pmod{125}$ so that $d \equiv 1803 \pmod{4500}$. Finally, we consider the prime 751. (3, 5, and 7 have orders 750, 375 and 250 respectively.) Thus $225 + 3^{305} \equiv 5^{3+75k} + 7^{53} \pmod{751}$, $0 \leqq k \leqq 4$. Since $3^{305} \equiv 630$, $5^{75} \equiv 569$ and $7^{53} \equiv 27$, in each case we have a contradiction. Hence no other solution exists. □

5. Theorems concerning equations of the general form $p^a + q^b = r^c + s^d$. Many exponential diophantine equations (e.g., $13^a + 5^b = 3^c + 7^d$, $19^a + 5^b = 3^c + 7^d$) can be shown to have only the trivial solution. In such cases proofs are easily generalized to include infinite classes of equations.

5.01 THEOREM. *The equation*

$$n^a + 3^b = 5^c + 7^d \tag{5.02}$$

has no nontrivial solution if the three conditions $n \equiv 1 \pmod 6$, $n \not\equiv 0 \pmod 5$, $n \not\equiv \pm 1 \pmod 8$ *all hold.*

Proof. Suppose (a, b, c, d) is another solution. Using mod 3 we conclude that $b = 0$ and c is even. Hence, using mod 8, a and d are even. Since $b = 0$ we conclude $c \neq 0$. Hence we have a contradiction mod 5. $\square$

We have two immediate corollaries:

5.021 COROLLARY. *Equation* 5.02 *has no nontrivial solutions for the primes* $n = 13, 19, 37, 43$, *and* 61.

5.022 COROLLARY. *Equation* 5.02 *has no nontrivial solutions for* $n \equiv 13 \pmod{120}$ (*and hence for infinitely many prime* n).

5.03 THEOREM. *The equation*

$$n^a + 5^b = 3^c + 7^d \tag{5.04}$$

has no nontrivial solution if $n \equiv 1 \pmod 6$, $n \equiv \pm 1 \pmod 5$ *and* $n \not\equiv \pm 1 \pmod 8$.

Proof. Let (a, b, c, d) be another solution. Using mod 3 we conclude that $c = 0$ and b is even. Hence, using mod 8, a and d are even. Thus, using mod 5 we have a contradiction. $\square$

Therefore we have:

5.041 COROLLARY. *Equation* 5.04 *has no nontrivial solution for the primes* $n = 19, 61, 109, 139$, *and* 181.

5.042 COROLLARY. *Equation* 5.04 *has no nontrivial solution if* $n \equiv 19 \pmod{120}$.

5.05 THEOREM. *Equation* 5.04 *has no nontrivial solution if*

$n \equiv 1 \pmod 6$, $n \equiv \pm 2 \pmod 5$ *and* $n \not\equiv \pm 1 \pmod 8$.

Proof. Let (a, b, c, d) be another solution. Using mod 3 we conclude that $c = 0$ and b is even. Therefore, using mod 8, a and d are even. Thus we have a contradiction mod 5. □

5.051 COROLLARY. *Equation* 5.04 *has no nontrivial solution for the primes* $n = 13, 37, 43, 67, 133$ *and* 157.

5.052 COROLLARY. *Equation* 5.04 *has no nontrivial solution if* $n \equiv 13 \pmod{120}$.

5.06 THEOREM. *Equation* 5.04 *has no nontrivial solution if* $n \equiv 1$ *or* $13 \pmod{42}$ *and* $n \not\equiv \pm 1 \pmod 8$.

Proof. Let (a, b, c, d) be another solution. Using mod 3 we conclude that $c = 0$ and b is even. Thus, using mod 8, a and d are even. Since $c = 0$, $d > 0$ and we have a contradiction mod 7. □

5.061 COROLLARY. *Equation* 5.04 *has only the trivial solution for the primes* $n = 211, 307, 349, 379, 421, 547, 643$ *and* 757.

5.062 COROLLARY. *Equation* 5.04 *has no nontrivial solution for* $n \equiv 13 \pmod{168}$.

5.07 THEOREM. *Equation* 5.04 *has no nontrivial solution if* $n \equiv 17 \pmod{120}$.

Proof. Let (a, b, c, d) be another solution. Using mod 3 we find that either $c = 0$ and a and b are even or $c > 0$ and a and b are odd. Thus, using mod 8, $c = 0$ and is even. Therefore $b > 0$ and we deduce a contradiction using mod 5.

The following two theorems are generalizations of [5].

5.08 THEOREM. *Let* $n = 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot m$ *where* $m \equiv -1 \pmod 8$. *Then the equation*

$$n^a + 7^b = 3^c + 5^d \tag{5.09}$$

has only the solutions $(a, b, c, d) = (0, 0, 0, 0), (0, 1, 1, 1)$.

Proof. Suppose (a, b, c, d) is another solution of (5.09). By [5], $a \neq 0$. Hence, using mod 3, $c > 0$ and d is even. Thus, using mod 8, b and c are even. We consider two cases: $b = 0$ and $b > 0$.

5.091 CASE. $b = 0$. Then $n^a + 1 = 3^c + 5^d$. Clearly $d > 0$ so that in fact $d \geqq 2$. Hence $1 \equiv 3^c \pmod{25}$ so that $c \equiv 0 \pmod{20}$. Thus we have a contradiction mod 11. Hence $b > 0$.

5.092 CASE. $b > 0$. Then $0 \equiv 3^c + 5^d \pmod 7$, a contradiction. □

5.10 THEOREM. *Let* $n \equiv 1 \pmod{2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13}$. *Then the only solutions of* (5.09) *are* $(a, b, c, d) = (0, 0, 0, 0)$ *and* $(0, 1, 1, 1)$.

Proof. Suppose (a, b, c, d) is another solution. Using mod 3 we conclude $d > 0$. Hence $7^b \equiv 3^c + 4 \pmod{20}$ so that $b \equiv c \equiv 1 \pmod 4$. Thus, using mod 16, $d \equiv 1 \pmod 4$. Therefore, using mod 13, $c \equiv 1 \pmod 3$ and $b \equiv 1 \pmod{12}$. Hence $c \equiv 1 \pmod 6$, $5^d \equiv 5 \pmod 7$ and $d \equiv 1 \pmod 6$. Using mod 25 we conclude that $c > 1$. Finally we have a contradiction mod 9.

We present the following easy general theorems without proof.

5.11 THEOREM. *Let p, q, r be distinct primes. Suppose* $p \equiv q \equiv 1 \pmod r$, $p \equiv 1 \pmod q$. *Then the only solutions of*

$$p^a + q^b = p^c + r^d \tag{5.12}$$

are $(a, b, c, d) = (t, 0, t, 0)$, *where t is any integer.*

5.121 COROLLARY. *If*

$$43^a + 7^b = 43^c + 3^d \tag{5.13}$$

then $a = c$, $b = d = 0$.

5.14 THEOREM. *Let p be prime. Then the only solutions of*

$$p^a + p^b = p^c + p^d \tag{5.15}$$

are $(a, b, c, d) = (s, t, s, t)$ *and* $(a, b, c, d) = (s, t, t, s)$ *where s, t are arbitrary integers.*

5.16 THEOREM. *Let p, q be distinct odd primes such that* $q \equiv 1 \pmod p$. *Then the equation*

$$p^a + p^b = q^c + q^d \tag{5.17}$$

has only the trivial solution.

5.18 REMARK. The essential hypotheses in Theorems 5.11, 5.14 and 5.16 are the congruence relations. The primality of the bases plays no rôle in the arguments.

6. The equations $1 + 2^a + 7^b = 3^c + 5^d$, $3^a + 7^b = 3^c + 5^d + 2$. The equations of this section have numerous solutions. The proofs given here are self-contained in the sense that they do not depend on results from any other section. On the other hand, the results established here are used in §§3 and 4.

6.01 THEOREM. *The only solutions of*

$$1 + 2^a + 7^b = 3^c + 5^d \tag{6.02}$$

in nonnegative integers are given in Table 6.03.

a	b	c	d
1	0	1	0
1	1	2	0
1	2	3	2
2	0	0	1
3	0	2	0
5	0	2	2
5	2	4	0

6.03 Table. The solutions of (6.02).

Proof. Suppose (a, b, c, d) is another solution.

6.031 LEMMA. $b > 0$.

Proof. Suppose $b = 0$ so that $2 + 2^a = 3^c + 5^d$. We consider several cases.

6.0311 *Case.* $d = 0$. Then $1 + 2^a = 3^c$. Clearly $c \geqq 3$, so that using mod 27, $a \equiv 9 \pmod{18}$. This yields a contradiction mod 19.

6.0312 *Case.* $d = 1$. Then $2^a = 3^c + 3$. Clearly $c > 0$ so that using mod 3 we have a contradiction.

6.0313 *Case.* $c = 0$, $d > 1$. Then $1 + 2^a = 5^d$. Clearly $a \geqq 3$. Using mod 8 we conclude that d is even. But then we have a contradiction mod 3.

6.0314 *Case.* $c = 1$, $d > 1$. Then $2^a = 1 + 5^d$. Clearly $a \geqq 3$ so that we have a contradiction mod 8.

6.0315 *Case.* $c, d \geqq 2$. Then $2 + 2^a = 3^c + 5^d$. Clearly $a \geqq 6$. Using mod 8 we see that c and d are even. Hence, using mod 3, a

is odd. Therefore, using mod 5, $a \equiv 1$, $c \equiv 2 \pmod 4$, so that, using mod 16, $d \equiv 2 \pmod 4$. Hence (using mod 13), $a \equiv 5 \pmod{12}$, $c \equiv 2 \pmod 3$ so that in fact $c \equiv 2 \pmod{12}$. Using mod 9 we conclude that $d \equiv 2 \pmod{12}$. Using mod 19 we conclude that $(a, c, d) \equiv (5, 2, 2)$, $(5, 8, 14)$ or $(17, 8, 2) \pmod{(36, 18, 36)}$. Thus, using mod 37 we have $(a, c, d) \equiv (5, 2, 2) \pmod{36}$. Therefore, using mod 73, $d \equiv 2 \pmod{72}$ so that using mod 32, $c \equiv 2 \pmod 8$. Hence, using mod 64 we find that $(c, d) \equiv (2, 10)$ or $(10, 2) \pmod{16}$. Finally, we have a contradiction mod 17. □

6.032 LEMMA. $c > 0$.

Proof. Suppose $c = 0$. Then $2^a + 7^b = 5^d$. Using mod 3 we find that d is odd and a is even. Hence, using mod 8, $a = 2$. Thus we have a contradiction mod 7. □

6.033 LEMMA. $a > 0$.

Proof. Suppose $a = 0$. Then $2 + 7^b = 3^c + 5^d$. This reduces to an absurdity modulo 3. □

6.034 LEMMA. $d > 0$.

Proof. Suppose $d = 0$. Then $2^a + 7^b = 3^c$. Clearly $c \geqq 3$. Also, using mod 3, a is odd. We consider several cases:

6.0341 *Case*. $a = 1$. Then $2 + 7^b = 3^c$, $c \geqq 3$. Using mod 7, we find $c \equiv 2 \pmod 6$. Further, using mod 27, $b \equiv 4 \pmod 9$. Thus we have a contradiction mod 37.

6.0342 *Case*. $a = 3$. Then $8 + 7^b = 3^c$. Using mod 7, $c \equiv 0 \pmod 6$. Also, using mod 9, $b \equiv 0 \pmod 3$. This yields a contradiction mod 19.

6.0343 *Case*. $a = 5$. Then $32 + 7^b = 3^c$. Using mod 8 we conclude that b, c are even. Also, clearly $b \geqq 4$. Thus $c \geqq 5$, $7^b \equiv -32 \equiv (-2)^5 \pmod{243}$. From this it follows easily that $b \equiv 29 \pmod{81}$. Hence in fact $b \equiv 110 \pmod{162}$. Thus using mod 163 we have $c \equiv 95 \pmod{162}$, so that c is odd, a contradiction.

6.0344 *Case*. $a \geqq 7$. Then $2^a + 7^b = 3^c$, $a \geqq 7$. Using mod 16 we find that b is even and $c \equiv 0 \pmod 4$. Hence using mod 5, $a \equiv 1$, $b \equiv 2 \pmod 4$. Thus, using mod 64 we deduce that $(b, c) \equiv (2, 12)$ or $(6, 4) \pmod{(8, 16)}$. Applying these results mod 17 we have

$$(a, b, c) \equiv (1, 10, 12) \pmod{(8, 16, 16)}\,. \tag{6.0345}$$

Further, using mod 13, $(a, b, c) \equiv (1, 6, 0)$, $(5, 2, 1)$ or $(9, 10, 2)$ (mod $(12, 12, 3)$). Thus, using mod 7, $(a, b, c) \equiv (5, 2, 1)$ (mod $(12, 12, 3)$), so that in fact, from (6.0345), $(a, b, c) \equiv (17, 26, 28)$ (mod $(24, 48, 48)$). Finally, we have a contradiction mod 97. $\square$

6.0346 LEMMA. $a > 1$.

Proof. Suppose $a = 1$. Then $3 + 7^b = 3^c + 5^d$. Thus $c \geqq 2$. Using mod 3 we conclude that d is even. Using mod 8, $b \not\equiv c \pmod 2$. Using mod 7 and mod 9 we conclude $(b, c, d) \equiv (3, 0, 4)$, $(1, 2, 0)$ or $(2, 3, 2) \pmod 6$. Using mod 105 we find $(b, c, d) \equiv (3, 0, 4)$, $(7, 8, 0)$ or $(2, 3, 2)$ (mod $(12, 12, 6)$). Applying these results mod 13 we conclude $(b, c, d) \equiv (2, 3, 2) \pmod{12}$, so that, using mod 37, $(b, c, d) \equiv (2, 3, 2) \pmod{36}$. Further, using mod 109 we find that $(b, c, d) \equiv (2, 3, 2) \pmod{27}$. If $c > 3$ we have a contradiction mod 81. Hence $c = 3$, $d > 2$, $7^b \equiv 24 \pmod{125}$, $b \equiv 6 \pmod{20}$. This yields a contradiction mod 11. $\square$

6.0347 LEMMA. $a \equiv 1$, $b \equiv c \equiv d \equiv 0 \pmod 2$.

Proof. Using mod 3, $a \equiv 1$, $d \equiv 0 \pmod 2$. Thus using mod 8, $b \equiv c \equiv 0 \pmod 2$. $\square$

6.0348 LEMMA. $a < 3$. (*This eliminates all remaining cases and completes the proof of the theorem.*)

Proof. Suppose $a \geqq 3$. If $a = 3$ we immediately have a contradiction mod 5. Hence $a \geqq 5$. Using mod 80 we conclude $(a, b, c, d) \equiv (1, 0, 2, 2) \pmod 4$. Using mod 13 therefore, $(a, b, c, d) \equiv (5, 0, 2, 2)$ or $(9, 4, 10, 2)$ (mod $(12, 12, 12, 4)$). Thus we have a contradiction mod 7. $\square$

The following theorem concerns a five term equation with infinitely many solutions.

6.05 THEOREM. *The only solutions of*

$$3^a + 7^b = 3^c + 5^d + 2 \tag{6.06}$$

in nonnegative integers are given in Table 6.07. *In this table, t denotes an arbitrary nonnegative integer.*

Proof. Let (a, b, c, d) be another solution.

a	b	c	d
t	1	t	1
1	0	0	0
2	0	1	1
3	0	0	2
3	3	5	3
4	2	1	3
6	4	1	5

6.07 Table. The solutions of (6.06).

6.071 LEMMA. $a, d > 0$.

Proof. Either $a = 0$ or $a > 0$. Suppose $a = 0$. Using mod 3 we conclude $c = 0$, d odd. Hence in particular, $7^b \equiv 2 \pmod{25}$, a contradiction. Hence $a > 0$. Next suppose $a \neq 0$, $d = 0$. Using mod 3 we conclude that $c = 0$, a contradiction. □

6.072 LEMMA. $d > 1$.

Proof. Suppose $d = 1$. Then $3^a + 7^b = 3^c + 7$. If b were 0 we would have a contradiction mod 9. Hence $b \geqq 2$. Using mod 7 we conclude $a \equiv c \pmod 6$ so that $7^b \equiv 7 \pmod{13}$, $b \equiv 1 \pmod{12}$. Hence $b \equiv 1 \pmod 6$. Now $3^a \equiv 3^c + 7 \pmod{49}$ so that $a \not\equiv c$ (mod 42). This contradicts the fact that $3^a \equiv 3^c \pmod{43}$. (For, 3 is a primitive root for each of the moduli 49, 43. Also, $b \equiv 1$ (mod 6) and $7^6 \equiv 1 \pmod{43}$.) □

6.073 LEMMA. $b > 0$.

Proof. Suppose $b = 0$, $3^a = 3^c + 5^d + 1$. We consider two cases: $c = 0$, $c > 0$.

6.0731 *Case.* $c = 0$. Then $3^a = 5^d + 2$. Using mod 3, d is even. Hence $3^a > 27$. Using the moduli 9, 7 successively we have: $d \equiv 2$, $a \equiv 3 \pmod 6$. Therefore, using mod 19 we conclude $a \equiv 3 \pmod{18}$, $d \equiv 2 \pmod 9$. Since $a \geqq 4$ we have $5^d \equiv 79 \pmod{81}$, $d \equiv 20 \pmod{54}$. Thus we have a contradiction mod 109.

6.0732 *Case.* $c > 0$. Using mod 3, d is odd. Also, by Lemma 2, $d \geqq 3$. Applying mod 8 we find that a is even and c is odd. By inspection, $a \geqq 4$. Furthermore, using mod 5, $a \equiv 2$, $c \equiv 1 \pmod 4$, so that, using mod 16, $d \equiv 1 \pmod 4$. Suppose $c = 1$, $3^a = 5^d + 4$. Using mod 27 we conclude that $d \equiv 13 \pmod{18}$. Hence using mod 7, $a \equiv 2 \pmod 6$, and thus we have a contradiction mod 19. Therefore,

$c > 1$ and in fact $c \geqq 5$. Using mod 9 we have $d \equiv 3 \pmod 6$, which yields a contradiction mod 7. □

6.074 LEMMA. $c > 0$.

Proof. Suppose $c = 0$, $3^a + 7^b = 5^d + 3$. Using mod 3 we conclude that d is even. Thus, using mod 8, a is odd and b is even. Hence, using mod 5, $a \equiv 3 \pmod 4$. Therefore, using mod 7, $a \equiv 5$, $d \equiv 4 \pmod 6$, so that using mod 9, $b \equiv 4 \pmod 6$ (since b is even). Hence we have a contradiction mod 13. □

6.075 LEMMA. $a > 1$.

Proof. Suppose the contrary. Then $a = 1$, $1 + 7^b = 3^c + 5^d$. Using mod 20, $b \equiv c \equiv 1 \pmod 4$. Hence, using mod 16, $d \equiv 1 \pmod 4$. Examining possibilities mod 13 we have $b \equiv 1 \pmod{12}$, $c \equiv 1 \pmod 3$. Therefore $c \equiv 1 \pmod 6$, $5 \equiv 5^d \pmod 7$ and hence $d \equiv 1 \pmod 6$. Thus, if $c > 1$ we have a contradiction mod 9. Further, if $c = 1$, $d > 1$ we have a contradiction mod 25. □

6.076 LEMMA. $a > 2$.

Proof. Suppose $a = 2$. Then $7 + 7^b = 3^c + 5^d$. Using mod 3, d is odd. Hence using the moduli 7, 8 successively we conclude that c is even, b is odd. Thus, using mod 5, $b \equiv 1$, $c \equiv 2 \pmod 4$. Using mod 13, $(b, c, d) \equiv (1, 2, 1) \pmod{(12, 3, 4)}$, and, in fact $c \equiv 2 \pmod{12}$. Hence, using mod 7, $d \equiv 1 \pmod 6$. Therefore, using mod 49 we conclude $(c, d) \equiv (2, 19)$, $(8, 25)$, $(14, 31)$, $(20, 37)$, $(26, 1)$, $(32, 7)$ or $(38, 13) \pmod{42}$. Consideration of our equation mod 43 in each of these cases yields a contradiction. □

6.077 LEMMA. $c > 1$.

Proof. Suppose $c = 1$, $3^a + 7^b = 5^d + 5$. Using mod 3, d is odd. Using mod 8, $a \equiv b \pmod 2$. Using mod 7, $(a, d) \equiv (1, 1)$, $(4, 3)$ or $(0, 5) \pmod 6$. Thus, using mod 9, $(b, d) \equiv (0, 1)$, $(2, 3)$ or $(1, 5) \pmod{(3, 6)}$. Combining these results we conclude that $(a, b, d) \equiv (1, 3, 1)$, $(4, 2, 3)$ or $(0, 4, 5) \pmod 6$. Applying mod 13 we have: $(a, b, d) \equiv (1, 2, 3)$ or $(0, 4, 1) \pmod{(3, 12, 4)}$ so that in fact $(a, b, d) \equiv (4, 2, 3)$ or $(0, 4, 5) \pmod{(6, 12, 12)}$. Using mod 5, $(a, b, d) \equiv (4, 2, 3)$ or $(6, 4, 5) \pmod{12}$. Applying these results mod 37 we conclude that $(a, b, d) \equiv (4, 2, 3)$ or $(6, 4, 5) \pmod{36}$. Hence, using mod 109,

$$(a, b, d) \equiv (4, 2, 3) \quad \text{or} \quad (6, 4, 5) \pmod{108}\,. \tag{6.078}$$

Consideration of our equation mod 163 then yields $(a, b, d) \equiv (4, 2, 3)$ or $(6, 4, 5) \pmod{(162, 162, 54)}$. We consider three cases: $a > 6$, $a = 4$, $a = 6$.

6.0781 *Case.* $a > 6$. Using mod 243 we have $(a, b, d) \equiv (4, 2, 111)$ or $(6, 4, 5) \pmod{162}$. Applying mod 487 we find $(a, b, d) \equiv (4, 2, 111)$ or $(6, 4, 5) \pmod{(486, 162, 162)}$. Using mod 729 we conclude $(a, b, d) \equiv (4, 2, 435), (4, 326, 273), (4, 164, 111), (6, 4, 5), (6, 328, 329)$ or $(6, 166, 167) \pmod{486}$. Consideration of our equation in each of these cases mod 1459 yields $(a, b, d) \equiv (6, 4, 5) \pmod{(1458, 486, 486)}$. Therefore, using mod 2187 we conclude $(a, b, d) \equiv (6, 4, 977)$, $(6, 247, 491)$ or $(6, 490, 5) \pmod{(1458, 729, 1458)}$. Since $(a, b, d) \equiv (6, 4, 5) \pmod{36}$ (from (6.08) and the fact that $a \equiv 0 \pmod 6$) we conclude $(a, b, d) \equiv (6, 4, 977)$, $(6, 976, 491)$, or $(6, 1948, 5) \pmod{2916}$. In each case we have a contradiction mod 2917. Hence $a \leqq 6$ so that $a = 4$ or $a = 6$.

6.0782 *Case.* $a = 4$. Then $76 + 7^b = 5^d$. Hence $d \geqq 4$, $7^b \equiv 549 \pmod{625}$ so that $b \equiv 22 \pmod{100}$. Thus, using mod 101 we have a contradiction.

6.0783 *Case.* $a = 6$. $724 + 7^b = 5^d$. Hence $d > 5$, $7^b \equiv 14901 \pmod{15625}$. Thus $b \equiv 2004 \pmod{2500}$ so that $b \equiv 27004 \pmod{67500}$. (For, from (6.08), $b \equiv 4 \pmod{108}$.) Using mod 101, $d \equiv 5 \pmod{25}$. Hence $d \equiv 5 \pmod{2700}$. Therefore, using mod 11251 we have a contradiction. Thus $c > 1$. □

6.079 Lemma. $b > 1$.

Proof. Suppose $b = 1$, $3^a + 5 = 3^c + 5^d$. Using mod 9, we conclude $d \equiv 1 \pmod 6$. Hence $3^a + 5 \equiv 3^c \pmod{25}$. Thus (since 3 is a primitive root mod 25), $a \not\equiv c \pmod{20}$. Also $a \not\equiv c \pmod{10}$ (or we would have $2 \cdot 3^c \equiv 5 \pmod{25}$). Since $5^d \equiv 5 \pmod{31}$ and 3 is a primitive root mod 31 we have a contradiction. □

6.0795 Lemma. $c > 2$.

Proof. Suppose on the contrary that $c = 2$, $3^a + 7^b = 5^d + 11$. Using mod 3, d is odd. Thus, using mod 8, a is even, b is odd. Hence applying mod 7, $a \equiv 2$, $d \equiv 1 \pmod 6$. Consideration of our equation mod 5 yields $a \equiv 2$, $b \equiv 1 \pmod 4$ so that, from mod 16, $d \equiv 1 \pmod 4$. Now, using mod 27 we have $(b, d) \equiv (1, 13), (4, 7)$ or $(7, 1) \pmod{(9, 18)}$. In particular $b \equiv d \equiv 1 \pmod 3$. Using mod 19 we conclude that $(a, b, d) \equiv (2, 25, 1) \pmod{36}$. Thus we have a contradiction mod 37. □

6.0796 LEMMA. *$(a, b, c, d) \equiv (t, 1, t, 1)$ or $(3, 3, 5, 3) \pmod 6$, t arbitrary.*

Proof. Using mod 3, d is odd. Further, using mod 8 we have the following table of possibilities mod 2:

a	b	c	d
0	0	1	1
0	1	0	1
1	1	1	1

6.08 Table. (a, b, c, d) mod 2.

Using mod 7 and mod 9 successively and referring to Table 6.09 we have the following table of solutions mod 6, where $0 \leqq t \leqq 5$:

a	b	c	d
t	1	t	1
0	2	1	5
1	5	5	5
2	3	0	3
2	5	4	5
3	3	5	3
4	0	1	3
4	2	3	5

6.09 Table. (a, b, c, d) mod 6.

Using mod 13, the lemma follows. □

6.095 LEMMA. $a > 3$.

Proof. Suppose on the contrary that $a = 3$, $25 + 7^b = 3^c + 5^d$. Since d is odd and $d > 1$ we have two cases: $d = 3$, $d \geqq 4$.

6.0951 *Case.* $d = 3$. $7^b = 3^c + 100$. By Lemma 10, $c \geqq 6$ so that $7^b \equiv 100 \pmod{729}$, $b \equiv 84 \pmod{243}$. Thus, using mod 1459 we have $c \equiv 248 \pmod{1458}$, contradicting the fact that $c \equiv 5 \pmod 6$. Hence $d > 3$.

6.0952 *Case.* $d \geqq 4$. From Lemma 6.0796, $(b, c, d) \equiv (1, 3, 1)$ or $(3, 5, 3) \pmod 6$. Using mod 13 we conclude $(b, c, d) \equiv (1, 3, 1)$ or $(3, 5, 3) \pmod{(12, 6, 12)}$. Further, using mod 25, $(b, c) \equiv (1, 15)$ or $(3, 5) \pmod{(4, 20)}$ so that $(b, c, d) \equiv (1, 15, 1)$ or $(3, 5, 3) \pmod{(12, 60, 12)}$. Thus, using mod 31 we have $(b, c, d) \equiv (3, 5, 3) \pmod{(60, 60, 12)}$. Therefore, using mod 11, $(b, c, d) \equiv (3, 5, 3) \pmod{60}$. Hence $25 +$

$7^b \equiv 3^c$, $3^c \equiv 118 \pmod{125}$ so that $c \equiv 5 \pmod{100}$. Hence, using mod 101 and combining results, $(b, c, d) \equiv (3, 5, 3) \pmod{300}$. Further $25 + 7^b \equiv 3^c$, $368 \equiv 3^c \pmod{625}$ so that $c \equiv 205 \pmod{500}$. Hence, using mod 251 we have a contradiction. Thus $a \geqq 4$. □

6.0953 Lemma. $(a, b, c, d) \not\equiv (3, 3, 5, 3) \pmod 6$.

Proof. Suppose the contrary. Using mod 13 we conclude $b \equiv 3 \pmod{12}$, $d \equiv 3 \pmod 4$. Hence, using mod 5, $c \equiv 1$, $a \equiv 3 \pmod 4$. Thus $(a, b, c, d) \equiv (3, 3, 5, 3) \pmod{12}$. Using mod 27 we conclude that $(b, d) \equiv (0, 9), (3, 3)$ or $(6, 15) \pmod{(9, 18)}$. Hence, $(b, d) \equiv (0, 27)$, $(3, 3)$ or $(6, 15) \pmod{(9, 36)}$. Consideration of our equation mod 37 yields $(a, b, c, d) \equiv (3, 3, 5, 3) \pmod{36}$. Therefore, using mod 81, $(b, d) \equiv (3, 39), (12, 21)$ or $(21, 3) \pmod{(27, 54)}$. Applying these results mod 109 we find $(a, b, c, d) \equiv (12, 12, 5, 21) \pmod{27}$ so that in fact $(a, b, c, d) \equiv (39, 39, 5, 21) \pmod{54}$. Consideration of cases mod 163 yields a contradiction. Hence $(a, b, c, d) \not\equiv (3, 3, 5, 3) \pmod 6$. □

6.0954 Lemma. *For any positive integer t, $(a, b, c, d) \not\equiv (t, 1, t, 1) \pmod 6$. (Thus the proof of the theorem is completed.) Note that the equality $(a, b, c, d) = (t, 1, t, 1)$ is excluded in this assertion.*

Proof. Suppose the contrary. Using mod 13, $b \equiv 1 \pmod{12}$, $d \equiv 1 \pmod 4$. Hence, using mod 5, $a \equiv c \pmod 4$. Thus $(a, b, c, d) \equiv (t, 1, t, 1) \pmod{12}$. Using mod 25 we have the following table of pairs $(a, c) \pmod{20}$:

a	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
c	4	9	18	15	8	13	2	19	12	17	6	3	16	1	10	7	0	5	14	11

6.10 Table. (a, c) mod 20.

Applying the above results and using mod 31 we arrive at the following table of possibilities mod 60:

a	b	c
25	25	13
31	37	43
57	49	45
58	13	34
51	13	3
56	25	20
38	37	14
28	49	22

6.11 Table. (a, b, c) mod 60.

Thus, using mod 11 we have the following table of possibilities (mod 60):

a	b	c	d
25	25	13	13
57	49	45	37
58	13	34	25
51	13	3	49
38	37	14	1

6.12 Table. (a, b, c, d) mod 60.

In each of the above cases we have a contradiction if we consider our equation mod 61. □

(We note that (6.06) also has solutions in negative integers. It is easily seen (using mod 25) that these are the quadruples $(a, b, c, d) = (t, 1, t, 1)$, t negative.)

7. Equations of the Form $p^a + q^b + r^c = s^d$. An equation of the form $p^a + q^b + r^c = s^d$ does not admit the trivial solution; but nonetheless it can be difficult to find all of its solutions. However, in many cases the equation is quite tractable.

7.01 THEOREM. *The only solutions to*

$$5^a + 5^b + 3^c = 3^d \tag{7.02}$$

in nonnegative integers are $(a, b, c, d) = (0, 1, 1, 2)$, $(0, 2, 0, 3)$ *and* $(0, 0, 0, 1)$.

Proof. Let (a, b, c, d) be another solution. Either $a = 0$ or $a > 0$.

Suppose that $a = 0$. Then $1 + 5^b = 3^d - 3^c$ and clearly $b > 2$. Further $d > c$. There are two cases: $c = 0$ and $c > 0$.

7.021 *Case.* $c = 0$. Then $2 + 5^b = 3^d$. Clearly $d \geqq 4$, $2 + 5^b \equiv 0 \pmod{81}$. $b \equiv 20 \pmod{54}$. This yields a contradiction mod 109.

7.022 *Case.* $c > 0$. Using mod 3 we conclude that b is odd. Hence, using mod 8, d must be even and c must be odd. Using mod 5, $d \equiv 2$, $c \equiv 1 \pmod 4$. Thus using mod 25 we have the following table:

c	9	1	13	5	17
d	2	6	10	14	18

7.03 Table. (c, d) mod 20.

Thus, using mod 11, $(b, c, d) \equiv (3, 9, 2) \pmod{(10, 20, 20)}$. This gives a contradiction mod 61.

If, on the other hand $a > 0$ (and by symmetry $b > 0$), it is seen (using mod 5) that $c \equiv d \pmod 4$. This yields a contradiction mod 8. $\square$

7.04 THEOREM. *The only solutions to*

$$3^a + 5^b + 7^c = 11^d \tag{7.05}$$

in nonnegative integers are $(a, b, c, d) = (1, 0, 1, 1)$ *and* $(2, 0, 0, 1)$.

Proof. Let (a, b, c, d) be another solution. Using the moduli 3 and 8 successively we conclude that $a \neq 0$. Further we have the following table:

a	b	c	d
0	0	0	1
1	0	1	1

7.06 Table. (a, b, c, d) mod 2.

The remainder of the proof requires the following three lemmas.

LEMMA 7.041. $b = 0$.

Proof. Assume the contrary. Using mod 5 we conclude that $a \equiv 1$, $c \equiv 3 \pmod 4$. Thus, using mod 7 we have the following table:

a	b	d
1	0	1
3	4	3
5	2	5

7.07 Table. (a, b, d) mod 6.

Using the moduli 9, 13 successively in each case we have a contradiction. Hence $b = 0$. $\square$

LEMMA 7.042. $c > 0$.

Proof. Suppose that $c = 0$. Then $3^a + 2 = 11^d$. Clearly $a \geqq 4$ so that $2 \equiv 11^d \pmod{81}$, $d \equiv 25 \pmod{54}$. Using mod 19, $a \equiv 2 \pmod{18}$. Hence using mod 109 we have a contradiction. Thus $c > 0$. $\square$

LEMMA 7.043. $a > 1$.

Proof. Suppose $a = 1$. Then $4 + 7^c = 11^d$. Clearly $c > 1$ so that $4 \equiv 11^d \pmod{49}$, $d \equiv 16 \pmod{21}$, and, in fact $d \equiv 37 \pmod{42}$. Hence, using mod 9, $c \equiv 1 \pmod 3$ so that (from Table 7.06) $c \equiv 1 \pmod 6$. We immediately have a contradiction mod 43. Hence $a > 1$. □

To complete the proof of the theorem, observe that, using mod 7, $(a, d) \equiv (0, 5)$ or $(1, 1) \pmod 6$. Thus, using mod 9, $(a, c, d) \equiv (0, 2, 5)$ or $(1, 3, 1) \pmod 6$. In each case we have a contradiction mod 13. Thus (a, b, c, d) does not exist. □

7.08 THEOREM. *The only integral solution of*

$$5^a + 5^b + 7^c = 17^d \tag{7.09}$$

is $(a, b, c, d) = (1, 1, 1, 1)$.

Proof. Let (a, b, c, d) be another solution. Clearly $a, b, c, d \geqq 0$. Using mod 8 we conclude that $(a, b, c) \equiv (0, 0, 1)$ or $(1, 1, 1) \pmod 2$. Hence using mod 3, $a \equiv b \equiv c \equiv d \equiv 1 \pmod 2$. Thus, using the moduli 7, 9 successively we have $(a, b, c, d) \equiv (1, 1, 1, 1)$, $(3, 3, 3, 5)$ or $(5, 5, 5, 3) \pmod 6$. Using mod 16 we find that $a \equiv b \pmod 4$. Also, using mod 5, $c \equiv d \pmod 4$. Thus, using mod 13 we have $(a, b, c, d) \equiv (1, 1, 1, 1)$ or $(5, 5, 11, 3) \pmod{12}$. Then, using mod 31 we find $(a, b, d) \equiv (1, 1, 1)$, $(1, 1, 19)$ or $(5, 5, 27)$ (mod (12, 12, 30)) so that in fact $(a, b, c, d) \equiv (1, 1, 1, 1)$, $(1, 1, 1, 9)$ or $(5, 5, 11, 7)$ (mod (12, 12, 12, 20)). Hence using mod 25, $a = b = 1$. (Therefore, $c \equiv 1 \pmod{12}$.) Thus $10 + 7^c = 17^d$, $c \geqq 2$, so that $10 \equiv 17^d \pmod{49}$, $d \equiv 19 \pmod{42}$. Finally we have a contradiction mod 43.

7.10 THEOREM. *The equation*

$$2^a + 2^b + 7^c = 29^d \tag{7.11}$$

has no integral solution.

Proof. Let (a, b, c, d) be a solution. Clearly $a, b, c, d \geqq 0$. By consideration of cases using mod 7 and mod 3 we easily conclude that $a, b \geqq 4$. Hence, using mod 16, $c \equiv 0 \pmod 2$, $d \equiv 0 \pmod 4$. Thus using mod 3, $a \not\equiv b \pmod 2$. Therefore, using mod 5 we conclude (without restricting generality) that $a \equiv 3$, $b \equiv 2$, $c \equiv 2 \pmod 4$. In particular, $c > 0$ so that consideration of our equation mod 7 yields $a \equiv b \equiv 2 \pmod 3$. Hence $a \equiv 11$, $b \equiv 2 \pmod{12}$. Thus $7^c \equiv 2^d$

(mod 9) so that $(c, d) \equiv (0, 0)$, $(1, 4)$ or $(2, 2)$ (mod (3, 6)). Hence $(c, d) \equiv (6, 0)$, $(10, 4)$, or $(2, 2)$ (mod (12, 6)) so that we have a contradiction mod 13. □

7.12 THEOREM. *The equation*

$$2^a + 3^b + 5^c = n^d \tag{7.13}$$

where $n \equiv 7$ (mod 150) *and* a, b, c, d *are nonnegative integers has* (i) *only the solution* $(a, b, c, d) = (0, 0, 1, 1)$ *if* $n = 7$; (ii) *no solutions if* $n > 7$.

Proof. Suppose $n \equiv 7$ (mod 150) and (a, b, c, d) is a nonnegative solution of (7.13). Clearly $a = 0$. Further, if $b = 0$ and $n > 7$ then $2 \equiv 7^d$ (mod 25), an impossibility. In the remaining case, $b > 0$, we have a contradiction mod 3. □

7.14 THEOREM. *The equation*

$$3^a + 5^b + 7^c = n^d \tag{7.15}$$

has no nonnegative integral solution if $n \equiv 19$ (mod 24).

Proof. Let (a, b, c, d) be a solution. Using mod 3 we conclude that $a = 0$, b odd. Thus, using mod 8 we have a contradiction. □

7.16 COROLLARY. *Equation* 7.15 *has no solution for the primes* $n = 19$, 43, 67, 139 *and* 163.

The following general theorem is obvious.

7.17 THEOREM. *The equation*

$$p^a + q^b + r^c = s^d \tag{7.18}$$

has no solution in nonnegative integers if $p, q, r, s \equiv 1$ (mod m) *for some integer* $m > 2$.

8. Problems and conjectures. In this section we list some conjectures that are suggested by the results of this article.

8.01 PROBLEM. Let k be a given integer, $\varepsilon_i = \pm 1$. Suppose that distinct primes p_i are also given. Show that all nonnegative integral solutions of the equation

$$\sum \varepsilon_i p_i^{a(i)} = k \tag{8.02}$$

can be found by reducing (8.02) to a finite number of congruences with respect to certain integral moduli m_j, $j = 1, 2, \cdots, r$.

8.03 PROBLEM. If $p_1 = p_2$ but the hypotheses of 8.01 are otherwise unchanged, what is the situation?

8.031 COMMENT. It is not difficult to show that for any finite set of moduli $\{m_i\}$ the equation

$$2^a + 3^b = 2^c + 3^d \tag{8.032}$$

which has the obvious solutions $a = c$, $b = d$ and the exotic solutions $(a, b, c, d) = (2, 0, 1, 1)$, $(4, 0, 3, 2)$, $(5, 1, 3, 3)$ or $(8, 1, 4, 5)$ (with $a > c$) cannot be fully investigated by reducing it to the congruences $2^a + 3^b \equiv 2^c + 3^d \pmod{m_i}$. However it seems likely that (8.032) has no more solutions.

8.033 COMMENT. Similar remarks apply to the equation

$$p^a + q^b = p^c + q^d \tag{8.034}$$

where p and q are any given distinct primes.

8.035 COMMENT. The equation

$$3^a = 1 + 2^b + 2^c \tag{8.036}$$

was brought to our attention by R. L. Graham. The equation is a special case of one solved by Pillai [10]. Moreover, using a few small moduli, L. J. Alex [2b] established that the only solutions are $(a, b, c) = (1, 0, 0)$, $(2, 2, 2)$, $(4, 6, 4)$ or $(4, 4, 6)$.

8.037 COMMENT. The class of equations

$$1 + (pq)^a = p^b + q^c \tag{8.038}$$

where p, q are given distinct primes, does not seem to be amenable to our methods.

8.039 COMMENT. As is demonstrated in this article, there are many equations of the form (8.02) with $p_1 = p_2$ which are solvable by our methods.

8.04 PROBLEM. If the bases p_i in 8.01 are replaced by arbitrary pre-assigned pairwise relatively prime integers, can (8.02) be solved using a finite set of moduli?

9. Index of *eDe*'s. This index includes all of the *eDe*'s that have been brought to our attention. There are many others that can be solved by transparent arguments. (See §1.)

Two term equations

$x^y = y^x$ [4], [7]
$x^{y^z} = z^{x^y}$ (We know no reference.)

Three term equations

$y = 1 + x$ (xy divisible by no primes except 2, 3, 5) [1], [8], [9]
$z = x + y$ (xyz divisible by no primes except 2, 3, 5, 7) [1]

Four term equations—three terms on one side

$3^a = 1 + 2^b + 2^c$ (8.035)
$3^d = 3^c + 5^a + 5^b$ (7.01)
$11^d = 3^a + 5^b + 7^c$ (7.04)
$17^d = 5^a + 5^b + 7^c$ (7.09)
$29^d = 2^a + 2^b + 7^c$ (7.10)
$n^d = 2^a + 3^b + 5^c$ ($n \equiv 7 \pmod{150}$) (7.13)
$n^d = 3^a + 5^b + 7^c$($n \equiv 19 \pmod{24}$, e.g., $n = 19, 43, 67, 139, 163$) (7.14)
$s^d = p^a + q^b + r^c$ ($p \equiv q \equiv r \equiv s \equiv 1 \pmod{m}$, $m > 2$) (7.17)

Four term equations—two terms on each side

$1 + 2^a = 2^b + 3^c$ (3.01)
$1 + 2^a = 2^b + 5^c$ (3.03)
$1 + 2^a = 2^b + 7^c$ (3.06)
$1 + 2^a = 4 \cdot 3^b + 5^c$ [3]
$1 + 3^a = 5^b + 7^c$ [5]
$1 + 5^a = 3^b + 7^c$ [5]
$1 + 5^a = 3^b + 3^c$ (3.09)
$1 + 5^a = 3^b + 23^c$ (3.10)
$1 + 5^a = 7^b + 7^c$ (3.141)
$1 + 5^a = 7^b + 19^c$ (3.12)
$1 + 5^a = 11^b + 11^c$ (3.142)
$1 + 5^a = 17^b + 17^c$ (3.143)
$1 + 5^a = 2 \cdot 3^b + 3 \cdot 2^c$ [3]
$1 + 7^a = 3^b + 5^c$ [5]
$1 + 13^a = 17^b + 17^c$ (3.144)
$1 + 17^a = 7^b + 7^c$ (3.145)
$1 + 67^a = 17^b + 17^c$ (3.146)
$1 + p^a = q^b + r^c$ ($q \equiv 1 \pmod{p}$) (3.15)
$1 + (pq)^a = p^b + q^c$ (8.038)
$9 + 5^a = 3^b + 7^c$ [6]
$2^a + 3^b = 2^c + 3^d$ (8.031)

$3^c + 5^d = 5^a + 7^b$ (4.06)
$3^a + 5^b = 11^c + 7^d$ [2]
$3^c + 5^d = 13^a + 7^b$ (4.11)
$3^c + 5^d = 17^a + 7^b$ (4.13)
$3^c + 5^d = m^a + 7^b$ $(m = 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot r,\ r \equiv -1 \pmod 8)$ (5.08)
$3^c + 5^d = m^a + 7^b$ $(m \equiv 1 \pmod{2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13})$ (5.10)
$3^c + 7^d = 5^a + 7^b$ (4.16)
$3^c + 7^d = 5^a + 7^b$ (4.20)
$3^a + 7^b = 5^c + 11^d$ [2]
$3^c + 7^d = 5^a + 15^b$ (4.22)
$3^c + 7^d = n^a + 5^b$ ($n \equiv 1 \pmod 6$, $n \equiv \pm 1 \pmod 5$, $n \not\equiv \pm 1 \pmod 8$) simultaneously; e.g., $n = 19, 61, 109, 139, 181$ or $n \equiv 19 \pmod{120}$) (5.03)
$3^c + 7^d = n^a + 5^b$ ($n \equiv 1 \pmod 6$, $n \equiv \pm 2 \pmod 5$, $n \not\equiv \pm 1 \pmod 8$) simultaneously; e.g., $n = 13, 37, 43, 67, 133, 157$ or $n \equiv 13 \pmod{120}$) (5.05)
$3^c + 7^d = n^a + 5^b$ ($n \equiv 1$ or $13 \pmod{42}$, $n \not\equiv \pm 1 \pmod 8$) simultaneously; e.g., $n = 211, 307, 349, 379, 421, 547, 643, 757$ or $n \equiv 13 \pmod{168}$) (5.06)
$3^c + 7^d = n^a + 5^b$ $(n \equiv 17 \pmod{120})$ (5.07)
$3^a + 11^b = 5^c + 7^d$ [2]
$3^d + 43^c = 7^b + 43^a$ (5.121)
$3^b + n^a = 5^c + 7^d$ ($n \equiv 1 \pmod 6$, $n \not\equiv 0 \pmod 5$, $n \not\equiv \pm 1 \pmod 8$) simultaneously; e.g., $n = 13, 19, 37, 43, 61$ or $n \equiv 13 \pmod{120}$) (5.01)
$p^a + q^b = p^c + r^d$ ($p \equiv q \equiv 1 \pmod r$, $p \equiv 1 \pmod q$ simultaneously) (5.11)
$p^a + p^b = p^c + p^d$ (5.14)
$p^a + p^b = q^c + q^d$ $(q \equiv 1 \pmod p)$ (5.16)

Equations with more than four terms

$3^c + 5^d = 1 + 2^a + 7^b$ (6.01)
$3^a + 7^b = 2 + 3^c + 5^d$ (6.05)
$\sum \varepsilon_i p_i^{a(i)} = k$ (8.01)

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