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BUNDLES OVER CONFIGURATION SPACES

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Let $F(R^n, k)$ be the configuration space of ordered sets of k distinct points in R^n. $F(R^n, k)$ is acted upon freely by the symmetric group on k letters, Σ_k. In this paper we calculate the order of the vector bundles

$$\xi_{n,k}\colon F(R^n, k) \times_{\Sigma_k} R^k \to F(R^n, k)/\Sigma_k.$$

Applications to the study of iterated loop spaces of spheres are also discussed.

1. The study of the stable homotopy type of the spaces $\Omega^n S^{n+r}$ has received much attention in recent years [**2, 8, 13**]. The starting point for this study was Snaith's stable descomposition [**18**]:

$$\Omega^n S^{n+r} \simeq_s \bigvee_{k \geq 0} F(\mathbf{R}^n, k)^+ \wedge_{\Sigma_k} S^{r^{(k)}},$$

where $F(\mathbf{R}^n, k)^+$ is the configuration space of k ordered distinct points in $\mathbf{R}^n$ together with a disjoint basepoint, $S^{r^{(k)}}$ is the k-fold smash product of S^r with itself, Σ_k is the symmetric group of k letters, and where "$\simeq_s$" denotes stable homotopy equivalence.

The space $F(\mathbf{R}^n, k)^+ \wedge_{\Sigma_k} S^{r^{(k)}}$ is clearly the Thom complex of the r-fold Whitney sum of the vector bundle

$$\xi_{n,k}\colon F(\mathbf{R}^n, k) \times_{\Sigma_k} \mathbf{R}^k \to F(\mathbf{R}^n, k)/\Sigma_k.$$

If $M(\xi_{n,k})$ is the associated Thom spectrum, then Snaith's theorem gives an equivalence of spectra

$$\Sigma^\infty \Omega^n S^{n+r} \simeq \bigvee_{k \geq 0} \Sigma^{rk} M(r\xi_{n,k}),$$

where Σ^∞ is the stabilization functor which assigns to a space its associated suspension spectrum.

If $\phi_{n,k}$ is the stable order of $\xi_{n,k}$ (i.e., $\phi_{n,k}$ is the smallest integer such that $\phi_{n,k}\xi_{n,k}$ is stably trivial) then we have the obvious periodicity

$$M((r + \phi_{n,k})\xi_{n,k}) \simeq \Sigma^{k\phi_{n,k}} M(r\xi_{n,k}).$$

This, together with Snaith's theorem gives clear interrelationships amongst the stable homotopy types of the spaces $\Omega^n S^{n+r}$ as r varies.

The case $n = 2$ is well understood by the work of F. Cohen, M. Mahowald, and R. J. Milgram [**5**], who proved that $\phi_{2,k} = 2$ for all k. The resulting periodicity in the homotopy type of the associated Thom spectra was used by M. Mahowald [13] and R. Cohen [8] to construct new infinite families in the stable homotopy ring π_*^s.

It is the purpose of this paper to compute the orders $\phi_{n,k}$ for general n and k. Our main result can be stated as follows. Let

$$a_{n,k} = 2^{\rho(n-1)} \prod_{3 \leq p \leq k} p^{[(n-1)/2]}$$

where p denotes an odd prime, and where $\rho(m)$ is Adam's vector field number: $\rho(m)$ = the number of positive integers $\leq m$ that are congruent to 0, 1, 2, or 4 mod 8.

THEOREM 1.1. *If* $n \not\equiv 0 \bmod 4$, *then* $\phi_{n,k} = a_{n,k}$. *Furthermore, if* $n \equiv 0 \bmod 4$, *then* $a_{n,k} \mid \phi_{n,k}$ *and* $\phi_{n,k} \mid 2a_{n,k}$.

REMARKS. 1. The bundle $\xi_{n,2}$ is easily seen to be stably isomorphic to the canonical line bundle over $\mathbf{R}P^{n-1}$, so the fact that $\phi_{n,2} = 2^{\rho(n-1)}$ is the classical result of Adams [1].

2. Using the Atiyah-Hirzebruch spectral sequence converging to the KO-theory of $F(\mathbf{R}^n, p)/\Sigma_p$, S. W. Yang computed the order of $\xi_{n,p}$, and proved that $a_{n,k} \mid \phi_{n,k}$ [20].

3. The conjecture that $\phi_{n,k} = a_{n,k}$ was made by Yang, Mahowald, and F. Cohen.

The essential idea in the proof of 1.1 is to notice that the classifying map

$$f_{n,k}\colon F(\mathbf{R}^n, k)/\Sigma_k \to BO$$

of $\xi_{n,k}$ factors as a composition of maps, one of which is the natural inclusion

$$i_n\colon \Omega_0^n S^n \hookrightarrow Q_0 S^0,$$

where $QX = \lim_{m\to\infty} \Omega^m \Sigma^m X$, and where $\Omega_k^n S^n$ denotes the component of $\Omega^n S^n$ containing maps of degree k. We then study the order of i_n localized at a prime p, using the results of F. Cohen, J. Moore, and J. Neisendorfer [6, 7, 15] and of Toda [19].

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2. Proof of Theorem 1.1. Our first object is to identify the classifying maps of the bundles $\xi_{n,k}$. This is done easily by recalling first that $F(\mathbf{R}^\infty, k) = \lim_{n\to\infty} F(\mathbf{R}^n, k)$ is a contractible space, acted upon freely by Σ_k, and therefore $F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k$. For a proof of this, see for instance [**14**].

Thus the bundle

$$\xi_{\infty,k}\colon F(\mathbf{R}^\infty, k) \times_{\Sigma_k} \mathbf{R}^k \to F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k$$

is classified by the map

$$f_k\colon B\Sigma_k \to BO(k)$$

induced by the regular representation of Σ_k in $O(k)$. Moreover, since the bundle $\xi_{n,k}$ is the pull-back of $\xi_{\infty,k}$ under the inclusion $F(\mathbf{R}^n, k)/\Sigma_k \subset F(\mathbf{R}^\infty, k)/\Sigma_k$, $\xi_{n,k}$ is classified by the map

$$f_{n,k}\colon F(\mathbf{R}^n, k)/\Sigma_k \subset F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k \underset{f_k}{\to} BO(k).$$

The stable order $\phi_{n,k}$ of $\xi_{n,k}$ is the order of the class determined by $f_{n,k}$ in the abelian group $[F(\mathbf{R}^n, k)/\Sigma_k, BO]$. In order to determine $\phi_{n,k}$ we first recall some of May's iterated loop space machinery [**14**].

Recall first the "approximations"

$$\alpha_n\colon C_n X \to \Omega^n \Sigma^n X$$

of [**14**]. $C_n X$ is a filtered space which approximates $\Omega^n\Sigma^n X$ in the sense that α_n is a weak homotopy equivalence if X is connected. For $X = S^0$,

$$C_n(S^0) \simeq \coprod_k F(\mathbf{R}^n, k)/\Sigma_k$$

and the map $\alpha_n\colon \coprod_k F(\mathbf{R}^n, k)/\Sigma_k \to \Omega^n S^n$ takes $F(\mathbf{R}^n, k)/\Sigma_k$ to $\Omega^n_k S^n$.

Now consider the map

$$\beta\colon \coprod_k BO(k) \to BO \times \mathbf{Z}$$

which includes $BO(k)$ into $BO \times \{k\}$ in the obvious manner. Let $\eta\colon QS^0 \to BO \times \mathbf{Z}$ be the infinite loop map induced by the map $S^0 \to BO \times \mathbf{Z}$ which sends 0 to the basepoint in $BO \times \{0\}$ and 1 to the basepoint in $BO \times \{1\}$. We then have

LEMMA 2.1. *The following diagram homotopy commutes for all positive integers n and k.*

$$\begin{array}{ccccc}
F(\mathbf{R}^n, k)/\Sigma_k \subset \coprod_j F(\mathbf{R}^n, j)/\Sigma_j & \longrightarrow & \coprod_j F(\mathbf{R}^\infty, j)/\Sigma_j & \xrightarrow{\coprod f_j} & \coprod_j BO(j) \\
\downarrow \alpha_n & & \downarrow \alpha_\infty & & \downarrow \beta \\
\Omega^n S^n & \xrightarrow[i_n]{} & QS^0 & \xrightarrow[\eta]{} & BO \times \mathbf{Z} \\
\downarrow *[-k] & & \downarrow *[-k] & & \downarrow *[-k] \\
\Omega^n S^n & \xrightarrow[i_n]{} & QS^0 & \xrightarrow[\eta]{} & BO \times \mathbf{Z}
\end{array}$$

where $*[-k]$ *translates components by* $-k$.

Proof. This follows directly from May's iterated loop space machinery, and an explicit proof is found in [**4**].

Note that the classifying map $f_{n,k}$: $F(\mathbf{R}^n, k)/\Sigma_k \to BO = BO \times \{0\} \subset BO \times \mathbf{Z}$ of $\xi_{n,k}$ is the composition obtained by going along the top and then down the right-hand side of the diagram in Lemma 2.1. Now since η is a map of infinite loop spaces, and therefore like i_n is an H-map, Lemma 2.1 implies that the power of p in the prime factorization of $\phi_{n,k}$ is bounded by the order of the localization at p of $i_n \in [\Omega_0^n S^n, Q_0 S^0]$.

PROPOSITION 2.2. *For a prime* p, *let* $i_{n,p}$: $\Omega_0^n S_{(p)}^n \to Q_0 S_{(p)}^0$ *be the localization of* i_n. *Then in* $[\Omega_0^n S_{(p)}^n, Q_0 S_{(p)}^0]$ *the order of* $i_{n,p}$ *divides* p^q, *where*

$$q = \begin{cases} \left[\dfrac{n-1}{2}\right] & \text{if } p \text{ is odd} \\ \rho(n-1) & \text{if } p = 2 \text{ and } n \not\equiv 0 \bmod 4 \\ \rho(n-1) + 1 & \text{if } p = 2 \text{ and } n \equiv 0 \bmod 4. \end{cases}$$

Notice that Theorem 1.1 is a corollary of Proposition 2.2 in view of Yang's results [**20**] (see the second remark following the statement of Theorem 1.1), and the fact that if $k < p$, $F(\mathbf{R}^\infty, k)/\Sigma_k = B\Sigma_k$ is homology p-equivalent to a point.

Proof of 2.2. We prove Proposition 2.2 in several cases.

Case 1. p odd and n odd (say $n = 2m + 1$).

Recent results of Selick [**17**], Cohen, Moore and Neisendorfer [**6**, **7**], and Neisendorfer [**15**] imply that the identity element

$$1 \in \left[\Omega_0^{2m+1} S_{(p)}^{2m+1}, \Omega_0^{2m+1} S_{(p)}^{2m+1}\right]$$

has order p^m. Since i_n is an H-map, the result follows in this case.

Case 2. $p = 2$, n odd.

To verify this case we shall use the Kahn-Priddy theorem [**10**]:

THEOREM 2.3. *There exist maps* s: $Q\mathbf{R}P^\infty \to Q_0S^0$ *and* j: $Q_0S^0 \to Q\mathbf{R}P^\infty$ *such that when localized at the prime* 2, $s \circ j$ *is a homotopy equivalence.*

In [**16**], Segal gave a proof of this theorem in which he showed that when restricted to $\Omega_0^n S^n \subset Q_0S^0$, j factors through a map j_n: $\Omega_0^n S^n \to Q\mathbf{R}P^{n-1}$. In [**3**], Caruso, Cohen, May, and Taylor also gave a proof of the Kahn-Priddy theorem, obtaining Segal's factorization, and in which explicit formulae for the maps j_n, j, and s are given.

In any case, using the proof and the formulae in [**3**] of this theorem, N. Kuhn verified that the maps j_n and j are one-fold loop maps [**12**]. The fact that j is an H-map actually follows immediately from Kahn's work in [**11**]. Using these results, we shall consider the following homotopy commutative diagram of spaces localized at 2.

$$\begin{tikzcd}
\Omega_0^n S^n \arrow[rr, "i_n"] \arrow[d, "j_n"'] & & Q_0S^0 \arrow[dll, "j"'] \\
Q\mathbf{R}P^{n-1} \subset Q\mathbf{R}P^\infty \arrow[rr, "s"'] & & Q_0S^0 \arrow[u, "(s \circ j)^{-1}"']
\end{tikzcd}$$

where $(s \circ j)^{-1}$ is a homotopy inverse to $s \circ j$. Since s is an infinite loop map, and j deloops once, $s \circ j$ and therefore $(s \circ j)^{-1}$ are maps of loop spaces. Thus the order of i_n (localized at 2) divides the order of the identity of $Q\mathbf{R}P^{n-1}$, which Toda showed to be $2^{\rho(n-1)}$ when n is odd [**19**]. This proves the proposition in this case.

Case 3. $n = 2m$.

Consider the following fibration of James [**9**].

$$S^{2m-1} \overset{e}{\hookrightarrow} \Omega S^{2m} \overset{h}{\to} \Omega S^{4m-1}$$

This fibration yields the classical EHP sequence in homotopy groups. Apply Ω^{2m-1} to this fibration and consider the following diagram.

$$
\begin{tikzcd}
\Omega^{2m-1}S^{2m-1} \arrow[r, "T"] \arrow[d, "e"'] & \Omega^{2m-1}S^{2m-1} \arrow[d, "e"] & \\
\Omega_0^{2m}S^{2m} \arrow[r, "T"] \arrow[d, "h"'] & \Omega_0^{2m}S^{2m} \arrow[r, "i_{2m}"] \arrow[d, "h"] & Q_0S^0 \\
\Omega^{2m}S^{4m-1} \arrow[r, "T"'] \arrow[ur, "{[i,i]'}"] & \Omega^{2m}S^{4m-1} &
\end{tikzcd}
$$

where T is twice the identity map, and $[i, i]' = \Omega^{2m}[i, i]$, where $[i, i]$: $S^{4m-1} \to S^{2m}$ is the Whitehead product of the identity with itself.

LEMMA 2.4. *In the above diagram we have*
(a) *both squares commute,*
(b) *the lower triangle commutes, and*
(c) $i_{2m} \circ [i, i]'$ *is null homotopic.*

Proof. The commutativity of the two squares is obvious, and the commutativity of the lower triangle follows from the standard fact that the Hopf invariant of $[i, i]$ is 2. Similarly, the fact that $i_{2m} \circ [i, i]' = 0$ follows from the standard fact that the Whitehead product $[i, i]$ stabilizes to zero.

COROLLARY 2.5. *There exists a map* g: $\Omega_0^{2m}S^{2m} \to \Omega_0^{2m-1}S^{2m-1}$ *so that* $T \simeq [i, i]' \circ h + e \circ g$.

Proof. By the lemma, $h \circ (T - [i, i]' \circ h)$ is null homotopic, and therefore $T - [i, i]' \circ h$ lifts to a map g: $\Omega_0^{2m}S^{2m} \to S^{2m-1}$ satisfying the required property.

We are now ready to prove the proposition in this final case. Localizing at 2, we have that

$$
\begin{aligned}
2^{\rho(2m-2)+1}i_{2m} &= 2^{\rho(2m-2)}(i_{2m} \circ T) \\
&= 2^{\rho(2m-2)}\big(i_{2m} \circ [i, i]' \circ h + i_{2m} \circ e \circ g\big)
\end{aligned}
$$

by 2.5, and which equals $2^{\rho(2m-2)}(i_{2m-1} \circ g)$ by 2.4 part c and the fact that $i_{2m-1} = i_{2m} \circ e$. But $2^{\rho(2m-2)}i_{2m-1}$ is null homotopic by the result in case 2. We may therefore conclude that

$$2^{\rho(2m-2)+1}i_{2m} = 0.$$

Similarly, localized at p odd and using the result of case 1, we obtain that $2p^{[(n-1)/2]}i_{2m}$ is null homotopic, and therefore so is $p^{[(n-1)/2]}i_{2m}$.

Thus we have proved the proposition when p is odd, and summarizing the results in $p = 2$, we have:

$$2^{\rho(n-1)}i_n = 0 \qquad \text{if } n \text{ is odd,}$$
$$2^{\rho(n-2)+1}i_n = 0 \qquad \text{if } n \text{ is even,}$$
$$\text{and} \quad 2^{\rho(n)}i_n = 0 \qquad \text{if } n \text{ is even.}$$

The last equation follows from the first since i_{2m} factors through i_{2m+1}.

Notice that if $n \equiv 2 \bmod 8$, $\rho(n-1) = \rho(n-2) + 1$ and therefore $2^{\rho(n-1)}i_n = 0$. If $n \equiv 6 \bmod 8$, $\rho(n-1) = \rho(n)$ so $2^{\rho(n-1)}i_n = 0$. Thus if $n \not\equiv 0 \bmod 4$, $2^{\rho(n-1)}i_n$ is null homotopic. If $n \equiv 0 \bmod 4$, $\rho(n-1) = \rho(n-2)$ so $2^{\rho(n-1)+1}i_n = 0$.

This completes the proof of Proposition 2.2, and therefore of Theorem 1.1.

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