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NOTES ON THE FEYNMAN INTEGRAL, III: THE SCHROEDINGER EQUATION

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In the setting of Cameron and Storvick's recent theory we show that the solution of an integral equation formally equivalent to the Schroedinger equation is expressible as the analytic Feynman integral of a function on ν-dimensional Wiener space of the form $F(\vec{X}) = \exp\{\int_0^t \theta(t-s, \vec{X}(s)+\vec{\xi})\,ds\}\psi(\vec{X}(t)+\vec{\xi})$. Here $\vec{X}$ is an $\mathbf{R}^\nu$-valued continuous function on $[0, t]$ such that $\vec{X}(0)=\vec{0}$, $\vec{\xi}\in\mathbf{R}^\nu$, and ψ and $\theta(s,\cdot)$ are Fourier-Stieltjes transforms.

1. Introduction. Let $L_2^\nu[0, t_0] = L_2^\nu$ denote the space of $\mathbf{R}^\nu$-valued, Lebesgue measurable, square integrable functions on $[0, t_0]$. Let $C^\nu[0, t_0]$ denote Wiener space, that is the space of $\mathbf{R}^\nu$-valued, continuous functions $\vec{X}$ on $[0, t_0]$ such that $\vec{X}(0) = \vec{0}$. In a recent paper [4], Cameron and Storvick introduced a Banach algebra S of (equivalence classes of) functions on Wiener space which are a kind of stochastic Fourier transform of Borel measures on L_2^ν. (Precise definitions will be given in §2.) For such functions they showed that the analytic Feynman integral, defined by analytic continuation of the Wiener integral, exists. Further they showed that functions of the form

$$F(\vec{X}) = \exp\left\{\int_0^{t_0}\theta(s, \vec{X}(s))\,ds\right\} \tag{1.1}$$

are in S where they assumed that the "potential" θ: $[0, t_0]\times\mathbf{R}^\nu\to\mathbf{C}$ satisfies: (i) For each s in $[0, t_0]$, $\theta(s,\cdot)$ is the Fourier-Stieltjes transform of an element σ_s of $M(\mathbf{R}^\nu)$, the space of $\mathbf{C}$-valued, countably additive (and hence bounded) Borel measures on $\mathbf{R}^\nu$; that is ($\langle\ ,\ \rangle$ denotes inner product in $\mathbf{R}^\nu$)

$$\theta(s, \vec{U}) = \int_{\mathbf{R}^\nu}\exp\{i\langle\vec{U},\vec{V}\rangle\}\,d\sigma_s(\vec{V}). \tag{1.2}$$

(ii) For each Borel subset E of $[0, t_0]\times\mathbf{R}^\nu$, $\sigma_s(E^{(s)})$ is a Borel measurable function of s on $[0, t_0]$. Here $E^{(s)}$ denotes the s-section of E. (iii) The total variation $\|\sigma_s\|$ of σ_s is bounded as a function of s.

For the case $\nu = 1$ and under strengthened measurability assumptions, Cameron and Storvick showed in [5] that the analytic Feynman integral of functions F which are essentially of the form (1.1) gives a solution to an integral equation formally equivalent to Schroedinger's equation. In [5] Cameron and Storvick make use of the fact that F is in S. This was established by them in [4] using several intermediate spaces ($\mathfrak{M}'$, S', $\mathfrak{M}''$, S'', $\mathfrak{M}''_n$, S''_n, $\hat{S_n}$) and some rather elaborate machinery. In our paper [**12**] we substantially simplified the proof that F is in S and, in particular, avoided the use of the intermediate spaces.

The main purpose of the present paper is to extend the results of [**5**] *to arbitrary dimension* ν and to do this in such a way as to avoid dependence on the use of the machinery from [**4**]. This seems worthwhile: The physical motivation for this theory is found in quantum mechanics, and, even in keeping track of the probability amplitude for the position of a single quantum particle in space, requires $\nu = 3$ dimensions. Multiple particle systems require additional dimensions. The arguments of Cameron and Storvick as given do not extend to more dimensions. Specifically, after making a certain estimate, they obtain the function $(t_0 - s)^{-1/2}$. This function is in $L_1[0, t_0]$, and they use this fact to finish their argument. If one attempts to extend their argument to general ν, one encounters the function $(t_0 - s)^{-\nu/2}$ which fails to be in $L_1[0, t_0]$ for all $\nu \geq 2$. We use a different summation procedure for some conditionally convergent integrals that enter into the discussion, and this enables us to replace certain estimates with actual calculations. By doing this and proceeding very carefully in certain places, we are able to obtain the result for arbitrary ν. (A sketch of the relationship between the summation procedures used here and in [**5**] will be given in §6 below.)

We obtain our results under somewhat less stringent conditions on θ than are employed in [**5**]. (i) above is unchanged. (ii) is replaced by the equivalent but formally weaker assumption that for each Borel subset B of $\mathbf{R}^\nu$, $\sigma_s(B)$ is Borel measurable as a function of s on $[0, t_0]$. We show in Corollary 3.1 below that $\|\sigma_s\|$ is measurable as a function of s and then (iii) is replaced by the weaker assumption that $\|\sigma_s\|$ is in $L_1[0, t_0]$. Now in their Schroedinger equation paper [**5**] (but not in their earlier paper [**4**]), Cameron and Storvick have $\nu = 1$, and, rather than casting their hypotheses in terms of a family of measures $\{\sigma_s\colon 0 \leq s \leq t\}$, they work with a complex-valued function of two variables $h(s, u)$ which, for each s in $[0, t_0]$, is a function of bounded variation on $\mathbf{R}$. They require h to be Borel measurable as a function of two variables. If one restricts attention to the case $\nu = 1$ and recasts our assumptions above in terms of a function of two variables h, we are requiring that $h(s_0, \cdot)$ be of bounded variation for every s_0 in $[0, t_0]$ and that $h(s, u_0)$ be a Borel measurable function of s for every u_0 in $\mathbf{R}$. This of course does not imply that h is Borel measurable as a function of 2 variables (See example 21, pp. 142–144 of [**7**].) The key to this weaker measurability assumption is Corollary 3.2 below which shows that $\theta(s, u)$ is Borel measurable as a function of two variables even when h is not.

It has recently been shown [**9**] that the study of the Banach algebra S and of the Banach algebra $\mathfrak{F}(H)$ of Fresnel integrable functions [**1, 2, 17**] are "equivalent". When the results of [**9**], especially Theorem 3 and Corollary 1, are combined with the results of the present paper, one sees immediately that the solution to our integral equation can also be expressed as a Fresnel integral. When $\theta(s, \vec{U}) = \theta(\vec{U})$ is independent of s (that is, θ is time independent), related results are well known [**2**; Theorem 3.2], but for θ dependent on s, this result seems to be new.

2. Definitions and preliminaries. Let ν be a positive integer and let m^ν denote ν-dimensional Wiener measure. A subset A of $C^\nu[0, t_0]$ is said to be scale-invariant measurable provided ρA is Wiener measurable for every $\rho > 0$. It is easy to see that the class $\mathfrak{S}$ of scale-invariant measurable sets forms a σ-algebra. A set N in $\mathfrak{S}$ is said to be scale-invariant null provided $m^\nu(\rho N) = 0$ for every $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). For a rather detailed discussion of scale-invariant measurability and its relation with other topics see [**11**].

Let F be a complex-valued function on $C^\nu[0, t_0]$ which is s-a.e. defined and scale-invariant measurable and such that the Wiener integral

$$J(\lambda) = \int_{C^\nu[0,\, t_0]} F(\lambda^{-1/2}\vec{X})\, dm^\nu(\vec{X})$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}^+ \equiv \{\lambda \in \mathbf{C}\colon\ \operatorname{Re}\lambda > 0\}$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the *analytic Wiener integral* of F over $C^\nu[0, t_0]$ with parameter λ, and, for λ in $\mathbf{C}^+$, we write

$$\int_{C^\nu[0,\, t_0]}^{anw_\lambda} F(\vec{X})\, dm^\nu(\vec{X}) \equiv J^*(\lambda).$$

Let q be a real parameter ($q \neq 0$) and let F be a function whose analytic Wiener integral exists for λ in $\mathbf{C}^+$. If the following limit exists, we call it the *analytic Feynman integral* of F over $C^\nu[0, t_0]$ with parameter q and we write

$$\int_{C^\nu[0,\, t_0]}^{anf_q} F(\vec{X})\, dm^\nu(\vec{X}) \equiv \lim_{\lambda \to -iq} \int_{C^\nu[0,\, t_0]}^{anw_\lambda} F(\vec{X})\, dm^\nu(\vec{X})$$

where λ approaches $-iq$ through $\mathbf{C}^+$.

The Banach algebra S consists of functions F on $C^\nu[0, t_0]$ expressible in the form

$$F(\vec{X}) = \int_{L_2^\nu} \exp\left\{ i \sum_{j=1}^{\nu} \int_0^{t_0} v_j(s)\, \tilde{d}x_j(s) \right\} d\sigma(\vec{V})$$

for s-a.e. $\vec{X} = (x_1, \ldots, x_\nu)$ in $C^\nu[0, t_0]$ where σ is an element of $M(L_2^\nu)$, the space of $\mathbf{C}$-valued, countably additive Borel measures on L_2^ν, and the integrals $\int_0^{t_0} v_j(s)\,\tilde{d}x_j(s)$ are Paley-Wiener-Zygmund integrals [**16**, or **12**, or **13**]. Letting $\|F\| \equiv \|\sigma\|$, the total variation norm of σ, Cameron and Storvick show that S is a Banach algebra and that the analytic Feynman integral exists for every F in S [**4**; Theorem 5.1]. (Actually the elements of S are equivalence classes $[F]$ of functions which are s-a.e. equal to an F as above. In certain arguments [**9**] it is important to distinguish between the functions and the equivalence classes. However this distinction is not especially important in this paper, and so we follow the usual convention and blur the distinction between functions and equivalence classes.)

3. **Some measurability results.** The first 3 general measurability results below, especially Theorem 3.1, are perhaps of some independent interest. They were discovered in conjunction with this paper, and, indeed, they will be used several times in what follows. We omit the rather straightforward proofs of these results as it is anticipated that they will be included in a semi-expository paper on this subject which is in preparation [**10**]. There are some related results in the literature [**3**, **6**, **14**, **15**], and it would not be too surprising if these results themselves appear somewhere.

LEMMA 3.1. *Let $(Y, \mathfrak{Y}, \gamma)$ be a σ-finite measure space and let $(Z, \mathfrak{Z})$ be a measurable space. For γ-a.e. y, let σ_y be a* **C**-*valued, countably additive measure on $(Z, \mathfrak{Z})$ of finite total variation. Suppose that for every $B \in \mathfrak{Z}$, $\sigma_y(B)$ is a $\mathfrak{Y}$-measurable function of y. Then for every E in the product σ-algebra $\mathfrak{Y} \times \mathfrak{Z}$, $\sigma_y(E^{(y)})$ is a $\mathfrak{Y}$-measurable function of y.*

LEMMA 3.2. *Let the assumptions of Lemma 3.1 be satisfied. Then for any bounded,* **C**-*valued, $\mathfrak{Y} \times \mathfrak{Z}$ measurable function $\phi(y, z)$ on $Y \times Z$,*

$$\int_Z \phi(y, z)\, d\sigma_y(z)$$

is a $\mathfrak{Y}$-measurable function of y.

THEOREM 3.1. *Let the assumptions of Lemma 3.1 be satisfied and suppose, in addition, that $\|\sigma_y\| \le h(y) \in L_1(Y, \mathfrak{Y}, \gamma)$. Then if μ is defined on $\mathfrak{Y} \times \mathfrak{Z}$ by*

$$\mu(E) \equiv \int_Y \sigma_y(E^{(y)})\, d\gamma(y), \tag{3.1}$$

μ is a **C**-*valued, countably additive measure on $\mathfrak{Y} \times \mathfrak{Z}$ with $\|\mu\| \le \|h\|_1$.*

Furthermore, if $\phi(y, z)$ is bounded and $\mathfrak{Y} \times \mathfrak{Z}$ measurable, then $\int_Z \phi(y, z) d\sigma_y(z)$ is in $L_1(Y, \mathfrak{Y}, \gamma)$, and we have

$$\int_Y \left[\int_Z \phi(y, z) d\sigma_y(z)\right] d\gamma(y) = \int_{Y \times Z} \phi(y, z) d\mu(y, z). \tag{3.2}$$

We will find the following corollary useful. It is clear that related more general results can be proven.

COROLLARY 3.1. *Let* $\{\sigma_s\colon 0 \le s \le t_0\}$ *be a family from* $M(\mathbf{R}^\nu)$ *such that* $\sigma_s(B)$ *is a Borel measurable function of s for every B in* $\mathfrak{B}(\mathbf{R}^\nu)$, *the Borel class of* $\mathbf{R}^\nu$. *Then* $\|\sigma_s\|$ *is a Borel measurable function of s.*

Proof. Let $C_0(\mathbf{R}^\nu, \mathbf{C})$ be the space of $\mathbf{C}$-valued continuous functions on $\mathbf{R}^\nu$ which vanish at ∞. Let f be in $C_0(\mathbf{R}^\nu, \mathbf{C})$. The function $\int_{\mathbf{R}^\nu} f(\vec{V})\, d\sigma_s(\vec{V})$ is a Borel measurable function of s by Lemma 3.2; to see this, let $(Y, \mathcal{Y}, \gamma) = ([0, t_0], \mathfrak{B}([0, t_0]), \text{Lebesgue measure})$, $(Z, \mathcal{Z}) = (\mathbf{R}^\nu, \mathfrak{B}(\mathbf{R}^\nu))$, and $\phi(s, \vec{V}) = f(\vec{V})$.

Let D be a countable dense subset of the unit ball of $(C_0(\mathbf{R}^\nu, \mathbf{C}), \|\cdot\|_\infty)$. Recall that the dual of the Banach space $(C_0(\mathbf{R}^\nu), \mathbf{C})$ is isometrically isomorphic to $M(\mathbf{R}^\nu)$. Hence

$$\|\sigma_s\| = \sup\left\{\left|\int_{\mathbf{R}^\nu} f(\vec{V})\, d\sigma_s(\vec{V})\right| : f \in D\right\}.$$

Thus $\|\sigma_s\|$ is the supremum of a countable number of Borel measurable functions of s and so is itself Borel measurable.

DEFINITION 3.1. Let $\mathcal{G}$ be the set of all $\mathbf{C}$-valued functions on $[0, t_0] \times \mathbf{R}^\nu$ of the form

$$\theta(s, \vec{U}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{U}, \vec{V}\rangle)\, d\sigma_s(\vec{V}) \tag{3.3}$$

where $\{\sigma_s\colon 0 \le s \le t_0\}$ is a family from $M(\mathbf{R}^\nu)$ satisfying the following 2 conditions:

(3.4a) For every $B \in \mathfrak{B}(\mathbf{R}^\nu)$, $\sigma_s(B)$ is Borel measurable in s.

$$\|\sigma_s\| \in L_1[0, t_0]. \tag{3.4b}$$

Let $\mathcal{G}_0$ denote the subset of $\mathcal{G}$ obtained by replacing condition (3.4b) by the condition:

(3.4c) There exists $M \ge 0$ such that $\|\sigma_s\| \le M$ for all s in $[0, t_0]$.

REMARK 3.1. In [12], θ was a $\mathbf{C}$-valued function on $[0, t_0] \times \mathbf{R}$ given by

$$\theta(s, u) = \int_{\mathbf{R}} \exp(iuv)\, d\sigma_s(v)$$

where $\{\sigma_s\colon 0 \le s \le t_0\}$ was a family from $M(\mathbf{R})$ satisfying

$$\text{(3.4a)}' \quad \begin{array}{l}\text{for every } E \text{ in } \mathfrak{B}([0,t_0]\times\mathbf{R}) = \mathfrak{B}([0,t_0])\times\mathfrak{B}(\mathbf{R}),\\ \sigma_s(E^{(s)}) \text{ is Borel measurable in } s, \text{ and}\end{array}$$

$$\text{(3.4b)}' \qquad \|\sigma_s\| \le h(s) \in L_1[0,t_0],$$

while in [4] the corresponding conditions were (3.4a)′ and (3.4c) with $\nu = 1$. Note that Lemma 3.1 and Corollary 3.1 now allow us to replace conditions (3.4a)′ and (3.4b)′ by the simpler conditions (3.4a) and (3.4b). In particular we see that if θ is in $\mathcal{G}$ and is given by (3.3) then $\|\sigma_s\|$ is Borel measurable in s (Corollary 3.1). The next corollary will show that θ is Borel measurable as a function of 2 variables.

COROLLARY 3.2. *Let* θ: $[0,t_0]\times\mathbf{R}^\nu \to \mathbf{C}$ *be given by* (3.3) *where* $\{\sigma_s\colon 0\le s\le t_0\}$ *is a family from* $M(\mathbf{R}^\nu)$ *satisfying* (3.4a). *Then* θ *is Borel measurable.*

Proof.. We will use Lemma 3.2 with $(Y,\mathfrak{Y},\gamma) = ([0,t_0]\times\mathbf{R}^\nu, \mathfrak{B}([0,t_0]\times\mathbf{R}^\nu)$, Lebesgue measure), $(Z,\mathfrak{Z}) = (\mathbf{R}^\nu, \mathfrak{B}(\mathbf{R}^\nu))$ and $\phi(s,\vec{U},\vec{V}) = \exp(i\langle\vec{U},\vec{V}\rangle)$. Given $(s,\vec{U})$ in $[0,t_0]\times\mathbf{R}^\nu$, let $\sigma_{(s,\vec{U})} = \sigma_s$. Certainly $\sigma_{(s,\vec{U})}(B) = \sigma_s(B)$ is a Borel measurable function of $(s,\vec{U})$ for every B in $\mathfrak{B}(\mathbf{R}^\nu)$. Also ϕ is bounded and $\mathfrak{B}([0,t_0]\times\mathbf{R}^\nu)\times\mathfrak{B}(\mathbf{R}^\nu)$ measurable. The desired measurability now follows immediately from Lemma 3.2.

REMARK 3.2. Corollary 3.2 and Lemma 3.2 will yield the measurability properties that we will need as we continue. For example, let θ be in $\mathcal{G}$ and for $\lambda > 0$ and $\vec{\xi}$ in $\mathbf{R}^\nu$ let G: $[0,t_0]\times C^\nu[0,t_0]\to\mathbf{C}$ be defined by

$$G(s,\vec{X}) = \theta(s, \lambda^{-1/2}\vec{X}(s) + \vec{\xi}).$$

Then it follows readily from Corollary 3.2 that G is Borel measurable.

PROPOSITION 3.1. *Let* ψ *be in* $\hat{M}(\mathbf{R}^\nu)$; *that is*

$$\psi(\vec{U}) = \int_{\mathbf{R}^\nu} \exp(i\langle\vec{U},\vec{V}\rangle)\, d\phi(\vec{V}) \tag{3.5}$$

where ϕ *is in* $M(\mathbf{R}^\nu)$. *Let* $\vec{\xi}\in\mathbf{R}^\nu$ *be given. Then the function*

$$g(\vec{X}) = g_{\vec{\xi}}(\vec{X}) \equiv \psi(\vec{X}(t_0) + \vec{\xi}) \tag{3.6}$$

is in S.

Proof. We seek a measure $\sigma = \sigma_{\vec{\xi}}$ in $M(L_2^\nu[0, t_0])$ such that for s-a.e. $\vec{X}$ in $C^\nu[0, t_0]$

$$\psi(\vec{X}(t_0) + \vec{\xi}) = \int_{L_2^\nu[0,t_0]} \exp\left\{i \sum_{j=1}^{\nu} \int_0^{t_0} v_j(s)\, \tilde{d}x_j(s)\right\} d\sigma(\vec{V}).$$

Let $\phi_{\vec{\xi}}$ in $M(\mathbf{R}^\nu)$ be defined by

$$\phi_{\vec{\xi}}(B) = \int_B \exp(i\langle \vec{\xi}, \vec{V}\rangle)\, d\phi(\vec{V})$$

for $B \in \mathfrak{B}(\mathbf{R}^\nu)$. Let $\Phi\colon \mathbf{R}^\nu \to L_2^\nu([0, t_0])$ be defined by

$$\Phi(\vec{U})(s) \equiv \vec{U} = (u_1, \ldots, u_\nu),$$

i.e., $u_j(s) \equiv u_j$ for $0 \le s \le t_0$. We claim that $\sigma = \phi_{\vec{\xi}} \circ \Phi^{-1}$ is the desired measure. For let $\rho > 0$ be given. We need to show that for m^ν-a.e. $\vec{X}$ in $C^\nu[0, t_0]$

$$\psi(\rho\vec{X}(t_0) + \vec{\xi}) = \int_{L_2^\nu[0,t_0]} \exp\left\{i \sum_{j=1}^{\nu} \int_0^{t_0} v_j(s)\, \tilde{d}\rho x_j(s)\right\} d\sigma(\vec{V}).$$

But

$$\begin{aligned}
\psi(\rho\vec{X}(t_0) + \vec{\xi}) &= \int_{\mathbf{R}^\nu} \exp\left\{i \sum_{j=1}^{\nu} \left(\rho x_j(t_0) + \xi_j\right) u_j\right\} d\phi(\vec{U}) \\
&= \int_{\mathbf{R}^\nu} \exp\left\{i\rho \sum_{j=1}^{\nu} x_j(t_0) u_j + i\langle \vec{\xi}, \vec{U}\rangle\right\} d\phi(\vec{U}) \\
&= \int_{\mathbf{R}^\nu} \exp\left\{i\rho \sum_{j=1}^{\nu} \int_0^{t_0} u_j \tilde{d}x_j(s) + i\langle \vec{\xi}, \vec{U}\rangle\right\} d\phi(\vec{U}) \\
&= \int_{\mathbf{R}^\nu} \exp\left\{i\rho \sum_{j=1}^{\nu} \int_0^{t_0} u_j \tilde{d}x_j(s)\right\} d\phi_{\vec{\xi}}(\vec{U}).
\end{aligned}$$

Now using the Change of Variable Theorem [**8**, p. 163] this last expression equals

$$\int_{L_2^\nu[0,t_0]} \exp\left\{i\rho \sum_{j=1}^{\nu} \int_0^{t_0} v_j(s)\, \tilde{d}x_j(s)\right\} d\sigma(\vec{V})$$

as desired.

REMARK 3.3. Let $\theta(s, \vec{U})$ be in $\mathcal{G}$. Then, by the ν-dimensional version of Theorem 1 of [**12**], $f(\vec{X}) \equiv \int_0^{t_0} \theta(s, \vec{X}(s))\, ds$ and $\exp(f(\vec{X}))$ are in $S \equiv S(L_2^\nu[0, t_0])$.

PROPOSITION 3.2. *Let θ be in $\mathcal{G}$ and be given by* (3.3). *Let g: $[0, t_0] \to [0, t_0]$ be Borel measurable and be such that $h \circ g \in L_1[0, t_0]$ where $h(s) \equiv \|\sigma_s\|$. Then $\theta_1(s, \vec{U}) \equiv \theta(g(s), \vec{U})$ belongs to $\mathcal{G}$.*

Proof. Let $\tau_s \equiv \sigma_{g(s)}$. Let $B \in \mathfrak{B}(\mathbf{R}^\nu)$. $\tau_s(B)$ is a Borel measurable function of s since $\sigma_{g(s)}(B)$ is the composition of two Borel measurable maps. Also $\|\tau_s\| = \|\sigma_{g(s)}\| = h(g(s))$ is in $L_1[0, t_0]$ by assumption. Hence

$$\theta_1(s, \vec{U}) = \theta(g(s), \vec{U}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{U}, \vec{V}\rangle)\, d\sigma_{g(s)}(\vec{V})$$

$$= \int_{\mathbf{R}^\nu} \exp(i\langle \vec{U}, \vec{V}\rangle)\, d\tau_s(\vec{V})$$

is in $\mathcal{G}$.

The following corollary follows immediately from Remark 3.3 and Proposition 3.2.

COROLLARY 3.3. *If $\theta(s, \vec{U})$ is in $\mathcal{G}$ then $\theta_1(s, \vec{U}) \equiv \theta(t_0 - s, \vec{U})$ is in $\mathcal{G}$ and so $f_1(\vec{X}) \equiv \int_0^{t_0} \theta(t_0 - s, \vec{X}(s))\, ds$ and $\exp(f_1(\vec{X}))$ are in S.*

PROPOSITION 3.3. *Let $\{\sigma_s: 0 \le s \le t_0\}$ be a family from $M(\mathbf{R}^\nu)$ satisfying* (3.4a) *and* (3.4b). *Let $f(s, \vec{U})$ be any $\mathbf{C}$-valued, bounded, Borel measurable function on $[0, t_0] \times \mathbf{R}^\nu$. For B in $\mathfrak{B}(\mathbf{R}^\nu)$ let*

$$\tau_s(B) \equiv \int_B f(s, \vec{U})\, d\sigma_s(\vec{U}).$$

Then $\{\tau_s$: $0 \le s \le t_0\}$ is a family from $M(\mathbf{R}^\nu)$ satisfying (3.4a) *and* (3.4b).

Proof. Clearly τ_s is in $M(\mathbf{R}^\nu)$ for $0 \le s \le t_0$. Given B in $\mathfrak{B}(\mathbf{R}^\nu)$,

$$\tau_s(B) = \int_{\mathbf{R}^\nu} \chi_B(\vec{U}) f(s, \vec{U})\, d\sigma_s(\vec{U})$$

is Borel measurable in s by Lemma 3.2. Also $\|\tau_s\| \le \|f\|_\infty \|\sigma_s\| \in L_1[0, t_0]$.

REMARK 3.4. Let θ in $\mathcal{G}$ be given by (3.3) and let f and τ_s be as in Proposition 3.3. Then

$$\theta_f(s, \vec{U}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{U}, \vec{V}\rangle)\, d\tau_s(\vec{V})$$

is in $\mathcal{G}$ by Proposition 3.3. Also one clearly has

$$\theta_f(s, \vec{U}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{U}, \vec{V}\rangle) f(s, \vec{V})\, d\sigma_s(\vec{V}). \tag{3.7}$$

Taking a fixed $\vec{\xi}$ in $\mathbf{R}^\nu$, letting $f(s, \vec{V}) = \exp(i\langle \vec{V}, \vec{\xi}\rangle)$, and applying Proposition 3.3 and Remark 3.4, we obtain the following corollary.

COROLLARY 3.4. *Let θ be in $\mathcal{G}$. Then for fixed $\vec{\xi}$ in $\mathbf{R}^\nu$ we see that*

$$\theta_{\vec{\xi}}(s, \vec{U}) \equiv \theta(s, \vec{U} + \vec{\xi}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{U} + \vec{\xi}, \vec{V}\rangle)\, d\sigma_s(\vec{V})$$

is in $\mathcal{G}$.

COROLLARY 3.5. *Let θ be in $\mathcal{G}$ and let $\vec{\xi}$ be in $\mathbf{R}^\nu$. Then $\theta_1(s, \vec{U}) \equiv \theta(t_0 - s, \vec{U} + \vec{\xi})$ is in $\mathcal{G}$ and so $g(\vec{X}) \equiv \int_0^{t_0} \theta(t_0 - s, \vec{X}(s) + \vec{\xi})\, ds$ and $\exp(g(\vec{X}))$ are in S.*

4. The expansion of the analytic Feynman integral of F. In this section we will obtain a useful series expansion (in terms of integrals over finite-dimensional spaces) for the analytic Feynman integral of the function

$$\begin{aligned} F(\vec{X}) &\equiv F(t_0, \vec{\xi}; \vec{X}) \\ &\equiv \exp\left\{\int_0^{t_0} \theta(t_0 - s, \vec{X}(s) + \vec{\xi})\, ds\right\} \psi(\vec{X}(t_0) + \vec{\xi}) \end{aligned} \tag{4.1}$$

where $\psi \in \hat{M}(\mathbf{R}^\nu)$ is given by (3.5), $\theta \in \mathcal{G}$ is given by (3.3) and $\vec{\xi} \in \mathbf{R}^\nu$.

First we write $F(\vec{X}) = \sum_{n=0}^{\infty} F_n(\vec{X})$ where

$$F_n(\vec{X}) \equiv F_n(t_0, \vec{\xi}; \vec{X}) \equiv \frac{1}{n!}\left[\int_0^{t_0} \theta(t_0 - s, \vec{X}(s) + \vec{\xi})\, ds\right]^n \psi(\vec{X}(t_0) + \vec{\xi}).$$

Now we know from Proposition 3.1 and Corollary 3.5 that the functions

$$\vec{X} \to \psi(\vec{X}(t_0) + \vec{\xi})$$

and

$$\vec{X} \to \int_0^{t_0} \theta(t_0 - s, \vec{X}(s) + \vec{\xi})\, ds$$

are in S. Let $\tau_{t_0, \vec{\xi}} \equiv \tau$ and $\sigma_{t_0, \vec{\xi}} \equiv \sigma$ be the associated measures in $M(L_2^\nu)$. Since S is a Banach algebra, $F_n(\vec{X})$ is also in S with associated measure $(1/n!)(\sigma * \cdots * \sigma) * \tau$. Now $\|F_n\| \leq (1/n!)\|\sigma\|^n \|\tau\|$ and so $\sum_0^\infty \|F_n\| < \infty$. Hence it follows [4, Theorem 5.4] that F is in S and that

$$\int_{C^\nu[0, t_0]}^{\text{anf}_q} F(\vec{X})\, dm^\nu(\vec{X}) = \sum_0^\infty \int_{C^\nu[0, t_0]}^{\text{anf}_q} F_n(\vec{X})\, dm^\nu(\vec{X}).$$

Thus to get our desired series expansion we will work with

$$\int_{C^\nu[0,t_0]}^{anf_q} F_n(\vec{X})\,dm^\nu(\vec{X}).$$

Let $\lambda > 0$ be given for now. We seek an expression for the Wiener integral $\int_{C^\nu[0,t_0]} F_n(\lambda^{-1/2}\vec{X})\,dm^\nu(\vec{X})$. The measurability questions that arise in the course of this discussion are easily resolved by using Lemma 3.2 and Corollary 3.2. Now for $n \geq 1$

$$\begin{aligned}
(4.2)\quad & \int_{C^\nu[0,t_0]} F_n(\lambda^{-1/2}\vec{X})\,dm^\nu(\vec{X}) \\
&= \frac{1}{n!}\int_{C^\nu[0,t_0]}\left[\int_0^{t_0}\theta\left(t_0 - s, \lambda^{-1/2}\vec{X}(s) + \vec{\xi}\right)ds\right]^n \\
&\qquad\qquad \times\psi\left(\lambda^{-1/2}\vec{X}(t_0) + \vec{\xi}\right)dm^\nu(\vec{X}) \\
&= \frac{1}{n!}\int_{C^\nu[0,t_0]}\left[\int_0^{t_0}\theta\left(s, \lambda^{-1/2}\vec{X}(t_0 - s) + \vec{\xi}\right)ds\right]^n \\
&\qquad\qquad \times\psi\left(\lambda^{-1/2}\vec{X}(t_0) + \vec{\xi}\right)dm^\nu(\vec{X}) \\
&= \frac{1}{n!}\int_{C^\nu[0,t_0]}\left[\prod_{j=1}^n \int_0^{t_0}\theta\left(s_j, \lambda^{-1/2}\vec{X}(t_0 - s_j) + \vec{\xi}\right)ds_j\right] \\
&\qquad\qquad \times\psi\left(\lambda^{-1/2}\vec{X}(t_0) + \vec{\xi}\right)dm^\nu(\vec{X}) \\
&= \frac{1}{n!}\int_{C^\nu[0,t_0]}\int_{[0,t_0]^n}\left[\prod_{j=1}^n \theta\left(s_j, \lambda^{-1/2}\vec{X}(t_0 - s_j) + \vec{\xi}\right)\right] \\
&\qquad\qquad \times\psi\left(\lambda^{-1/2}\vec{X}(t_0) + \vec{\xi}\right)d\vec{s}\,dm^\nu(\vec{X}) \\
&= \int_{C^\nu[0,t_0]}\int_{\Delta_n(t_0)}\left[\prod_{j=1}^n \theta\left(s_j, \lambda^{-1/2}\vec{X}(t_0 - s_j) + \vec{\xi}\right)\right] \\
&\qquad\qquad \times\psi\left(\lambda^{-1/2}\vec{X}(t_0) + \vec{\xi}\right)d\vec{s}\,dm^\nu(\vec{X}) \\
&= \int_{\Delta_n(t_0)}\int_{C^\nu[0,t_0]}\left[\prod_{j=1}^n \theta\left(s_j, \lambda^{-1/2}\vec{X}(t_0 - s_j) + \vec{\xi}\right)\right] \\
&\qquad\qquad \times\psi\left(\lambda^{-1}\vec{X}(t_0) + \vec{\xi}\right)dm^\nu(\vec{X})\,d\vec{s}
\end{aligned}$$

where

$$\begin{aligned}
(4.3)\quad \Delta_n &\equiv \Delta_n(t_0) \\
&\equiv \left\{\vec{s} = (s_1,\ldots,s_n) \in [0,t_0]^n : 0 < s_1 < s_2 < \cdots < s_n < t_0\right\},
\end{aligned}$$

and the use of the Fubini Theorem is justified since

$$\int_{\Delta_n}\int_{C^\nu[0,t_0]}\left[\prod_{j=1}^{n}\left|\theta\left(s_j,\lambda^{-1/2}\vec{X}(t_0-s_j)+\vec{\xi}\right)\right|\right]$$

$$\times\left|\psi\left(\lambda^{-1/2}\vec{X}(t_0)+\vec{\xi}\right)\right|dm^\nu(\vec{X})\,d\vec{s}$$

$$\leq\int_{\Delta_n}\int_{C^\nu[0,t_0]}\left[\prod_{j=1}^{n}\|\sigma_{s_j}\|\right]\|\phi\|dm^\nu(\vec{X})\,d\vec{s}$$

$$=\|\phi\|\int_{\Delta_n}\left[\prod_{j=1}^{n}\|\sigma_{s_j}\|\right]d\vec{s}=\frac{\|\phi\|}{n!}\int_{[0,t_0]^n}\left[\prod_{j=1}^{n}\|\sigma_{s_j}\|\right]d\vec{s}$$

$$=\frac{\|\phi\|}{n!}\left[\int_0^{t_0}\|\sigma_s\|ds\right]^n<\infty.$$

Now applying a basic Wiener integration formula we see that the right side of equation (4.2) equals

$$\int_{\Delta_n}\left[\prod_{j=1}^{n+1}2\pi(s_j-s_{j-1})\right]^{-\nu/2}$$

$$\times\int_{\mathbf{R}^{(n+1)\nu}}\left[\prod_{j=1}^{n}\theta\left(s_j,\lambda^{-1/2}\vec{U}_j'+\vec{\xi}\right)\right]\psi\left(\lambda^{-1/2}\vec{U}_0'+\vec{\xi}\right)$$

$$\times\exp\left\{-\sum_{j=1}^{n+1}\frac{\|\vec{U}_j'-\vec{U}_{j-1}'\|^2}{2(s_j-s_{j-1})}\right\}d\vec{U}_0'\,d\vec{U}_1'\cdots d\vec{U}_n'\,d\vec{s}$$

where $\vec{U}_{n+1}'\equiv\vec{0}$, $s_{n+1}=t_0$, $s_0=0$ and of course $\vec{U}_j'=(u_{j,1}',u_{j,2}',\ldots,u_{j,\nu}')$. Now making the substitution $\vec{U}_j=\lambda^{-1/2}\vec{U}_j'+\vec{\xi}$ for $j=0,1,\ldots,n+1$, and then making use of (3.3) and (3.5) we finally obtain

$$\text{(4.4)}\quad\int_{C^\nu[0,t_0]}F_n(\lambda^{-1/2}\vec{X})\,dm^\nu(\vec{X})$$

$$=\int_{\Delta_n}\left[\prod_{j=1}^{n+1}2\pi(s_j-s_{j-1})/\lambda\right]^{-\nu/2}\int_{\mathbf{R}^{(n+1)\nu}}\left[\prod_{j=1}^{n}\theta\left(s_j,\vec{U}_j\right)\right]\psi(\vec{U}_0)$$

$$\times\exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n+1}\frac{\|\vec{U}_j-\vec{U}_{j-1}\|^2}{s_j-s_{j-1}}\right\}d\vec{U}_0\,d\vec{U}_1\cdots d\vec{U}_n\,d\vec{s}$$

$$= \int_{\Delta_n} \left[\prod_{j=1}^{n+1} 2\pi(s_j - s_{j-1})/\lambda \right]^{-\nu/2}$$

$$\times \int_{\mathbf{R}^{(n+1)\nu}} \left[\prod_{j=1}^{n} \int_{\mathbf{R}^\nu} \exp\left(i\langle \vec{U}_j, \vec{V}_j \rangle\right) d\sigma_{s_j}\left(\vec{V}_j\right) \right]$$

$$\times \int_{\mathbf{R}^\nu} \exp\left(i\langle \vec{U}_0, \vec{V}_0 \rangle\right) d\phi\left(\vec{V}_0\right)$$

$$\times \exp\left\{ -\frac{\lambda}{2} \sum_{j=1}^{n+1} \frac{\|\vec{U}_j - \vec{U}_{j-1}\|^2}{s_j - s_{j-1}} \right\} d\vec{U}_0 \cdots d\vec{U}_n \, d\vec{s}$$

where of course, $s_0 = 0$, $s_{n+1} = t_0$, and $\vec{U}_{n+1} = \vec{\xi}$. One can easily justify integrating first with respect to $\vec{U}_0$, $\vec{U}_1$, etc. in the above expression. We do this next; ignoring, for the time being, the integrations with respect to $\sigma_{s_1}, \ldots, \sigma_{s_n}$, ϕ and $\vec{s}$. That is we consider the expression

$$(4.5) \quad \left[\prod_{j=1}^{n+1} 2\pi(s_j - s_{j-1})/\lambda \right]^{-\nu/2}$$

$$\times \int_{\mathbf{R}^\nu} \overset{(n+1)}{\cdots} \int_{\mathbf{R}^\nu} \exp\{ i\langle \vec{U}_0, \vec{V}_0 \rangle + i\langle \vec{U}_1, \vec{V}_1 \rangle + \cdots + i\langle \vec{U}_n, \vec{V}_n \rangle \}$$

$$\times \exp\left\{ \frac{-\lambda \|\vec{U}_1 - \vec{U}_0\|^2}{2(s_1 - s_0)} \right\} \exp\left\{ \frac{-\lambda \|\vec{U}_2 - \vec{U}_1\|^2}{2(s_2 - s_1)} \right\}$$

$$\cdots \exp\left\{ \frac{-\lambda \|\vec{U}_n - \vec{U}_{n-1}\|^2}{2(s_n - s_{n-1})} \right\} \exp\left\{ \frac{-\lambda \|\vec{\xi} - \vec{U}_n\|^2}{2(t_0 - s_n)} \right\} d\vec{U}_0 \, d\vec{U}_1 \cdots d\vec{U}_n$$

and integrate, first with respect to $\vec{U}_0$, then with respect to $\vec{U}_1, \ldots,$ and finally with respect to $\vec{U}_n$.

To carry out the integrations with respect to the $\vec{U}_j$'s we need the following formula whose 1-dimensional version was stated in [**12**, Lemma 4]: For $\lambda > 0$,

$$(4.6) \quad \left[2\pi(s_j - s_{j-1})/\lambda\right]^{-\nu/2} \int_{\mathbf{R}^\nu} \exp\left\{ \frac{-\lambda \|\vec{U}_j - \vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})} + i\langle \vec{U}_j, \vec{W} \rangle \right\} d\vec{U}_j$$

$$= \exp\left\{ i\langle \vec{U}_{j-1}, \vec{W} \rangle - \frac{(s_j - s_{j-1})\|\vec{W}\|^2}{2\lambda} \right\}.$$

First we use this formula to carry out the integration with respect to $\vec{U}_0$. We obtain

$$[2\pi(s_1 - s_0)/\lambda]^{-\nu/2}\int_{\mathbf{R}^\nu} \exp\left\{\frac{-\lambda\|\vec{U}_1 - \vec{U}_0\|^2}{2(s_1 - s_0)} + i\langle\vec{U}_0, \vec{V}_0\rangle\right\} d\vec{U}_0$$

$$= \exp\left\{i\langle\vec{U}_1, \vec{V}_0\rangle - \frac{(s_1 - s_0)\|\vec{V}_0\|^2}{2\lambda}\right\}.$$

To integrate with respect to $\vec{U}_1$, we first take the expression $\exp\{-(s_1 - s_0)\|\vec{V}_0\|^2/2\lambda\}$ outside the integral and apply formula (4.6) again to calculate

$$[2\pi(s_2 - s_1)/\lambda]^{-\nu/2}\int_{\mathbf{R}^\nu} \exp\left\{\frac{-\lambda\|\vec{U}_2 - \vec{U}_1\|^2}{2(s_2 - s_1)} + i\langle\vec{U}_1, \vec{V}_1 + \vec{V}_0\rangle\right\} d\vec{U}_1$$

$$= \exp\left\{i\langle\vec{U}_2, \vec{V}_1 + \vec{V}_0\rangle - \frac{(s_2 - s_1)\|\vec{V}_1 + \vec{V}_0\|^2}{2\lambda}\right\}.$$

We continue this for a total of $(n + 1)$ integrations, using formula (4.6) each time, and expression (4.5) becomes

$$\exp\Big\{i\langle\vec{\xi}, \vec{V}_n + \vec{V}_{n-1} + \cdots + \vec{V}_0\rangle$$
$$- \frac{1}{2\lambda}\big[(s_1 - s_0)\|\vec{V}_0\|^2 + (s_2 - s_1)\|\vec{V}_1 + \vec{V}_0\|^2$$
$$+ (s_3 - s_2)\|\vec{V}_2 + \vec{V}_1 + \vec{V}_0\|^2 + \cdots$$
$$+ (t_0 - s_n)\|\vec{V}_n + \vec{V}_{n-1} + \cdots + \vec{V}_1 + \vec{V}_0\|^2\big]\Big\}$$

$$= \exp\Big\{i\langle\vec{\xi}, \vec{V}_n + \vec{V}_{n-1} + \cdots + \vec{V}_0\rangle$$
$$- \frac{1}{2\lambda}\left[\sum_{j=0}^{n} \|\vec{V}_j\|^2(t_0 - s_j) + 2\sum_{j=1}^{n} (t_0 - s_j)\left\langle\vec{V}_j, \sum_{k=0}^{j-1}\vec{V}_k\right\rangle\right]\Big\}$$

$$= \exp\Big\{-\frac{1}{2\lambda}\left[\sum_{l=0}^{n}\sum_{j=0}^{l} (2 - \delta_{j,l})(t_0 - s_l)\langle\vec{V}_j, \vec{V}_l\rangle\right]$$
$$+ i\langle\vec{\xi}, \vec{V}_n + \vec{V}_{n-1} + \cdots + \vec{V}_1 + \vec{V}_0\rangle\Big\}.$$

Using this result and (4.3) we finally have that for $\lambda > 0$,

$$\int_{C^\nu[0,t_0]} F_n(\lambda^{-1/2}\vec{X})\, dm^\nu(\vec{X}) = \Gamma^{(n)}(t_0, \vec{\xi}; \lambda), \tag{4.7}$$

where

$$\Gamma^{(n)}(t_0, \vec{\xi}; \lambda)$$

$$\equiv \int_{\Delta_n} \int_{\mathbf{R}^{(n+1)\nu}} \exp\left\{ -\frac{1}{2\lambda}\left[\sum_{l=0}^{n} \sum_{j=0}^{l} (2 - \delta_{j,l})(t_0 - s_l)\langle \vec{V}_j, \vec{V}_l\rangle \right] + i\langle \vec{\xi}, \vec{V}_n + \cdots + \vec{V}_1 + \vec{V}_0\rangle \right\}$$

$$d\sigma_{s_1}(\vec{V}_1) \cdots d\sigma_{s_n}(\vec{V}_n)\, d\phi(\vec{V}_0)\, d\vec{s}.$$

One can show without much difficulty that $\Gamma^{(n)}(t_0, \vec{\xi}; \lambda)$ continues to exist for $\operatorname{Re}\lambda \geq 0$ ($\lambda \neq 0$), that $\Gamma^{(n)}(t_0, \vec{\xi}; \lambda)$ is analytic in $\mathbf{C}^+ = \{\lambda \in \mathbf{C}: \operatorname{Re}\lambda > 0\}$ and that $\lim_{\lambda \to -iq} \Gamma^{(n)}(t_0, \vec{\xi}; \lambda) = \Gamma^{(n)}(t_0, \vec{\xi}; -iq)$. In making the limiting arguments necessary to verify these assertions, it is helpful to keep in mind that the integral in the definition of $\Gamma^{(n)}$ is, at this point, being thought of as an iterated integral. It is also useful to note that for all λ such that $\operatorname{Re}\lambda \geq 0$ ($\lambda \neq 0$), and $\vec{\xi}$ in $\mathbf{R}^\nu$ we have

$$\left|\Gamma^{(n)}(t_0, \vec{\xi}; \lambda)\right| \leq \int_{\Delta_n(t_0)} \prod_{j=1}^{n} \|\sigma_{s_j}\|\, \|\phi\| d\vec{s}$$

$$= \frac{\|\phi\|}{n!} \int_{[0,t_0]^n} \prod_{j=1}^{n} \|\sigma_{s_j}\| d\vec{s} = \frac{\|\phi\|}{n!}\left[\int_0^{t_0} \|\sigma_s\| ds\right]^n < \infty.$$

Thus for $n \geq 1$ we have established that

$$\int_{C^\nu[0,t_0]}^{anf_q} F_n(\vec{X})\, dm^\nu(\vec{X}) = \Gamma^{(n)}(t_0, \vec{\xi}; -iq). \tag{4.8}$$

It is straightforward to show that

$$\int_{C^\nu[0,t_0]}^{anf_q} F_0(\vec{X})\, dm^\nu(\vec{X}) = \int_{\mathbf{R}^\nu} \exp\left\{ i\langle \vec{\xi}, \vec{V}_0\rangle + \frac{t_0\|\vec{V}_0\|^2}{2qi} \right\} d\phi(\vec{V}_0) \tag{4.9}$$

$$\equiv \Gamma^{(0)}(t_0, \vec{\xi}; -iq).$$

Hence, for F given by (4.1), we have the following series expansion for the analytic Feynman integral of F,

(4.10)

$$\int_{C^\nu[0,t_0]}^{\mathrm{anf}_q} F(t_0,\vec{\xi};\vec{X})\,dm^\nu(\vec{X}) = \sum_{n=0}^{\infty} \Gamma^{(n)}(t_0,\vec{\xi};-iq)$$

$$= \sum_{n=0}^{\infty} \int_{\Delta_n(t_0)} \int_{\mathbf{R}^{(n+1)\nu}} \exp\left\{\frac{1}{2qi}\left[\sum_{l=0}^{n}\sum_{j=0}^{l}(2-\delta_{j,l})(t_0-s_l)\langle\vec{V}_j,\vec{V}_l\rangle\right]\right.$$

$$\left. + i\langle\vec{\xi},\vec{V}_n+\cdots+\vec{V}_1+\vec{V}_0\rangle\right\}$$

$$d\sigma_{s_1}(\vec{V}_1)\cdots d\sigma_{s_n}(\vec{V}_n)\,d\phi(\vec{V}_0)\,d\vec{s}.$$

5. An alternate expansion of the analytic Feynman integral. In working with Schroedinger's equation there is another form for the nth term of the series (4.10) which is easier to work with; we obtain this alternate form next.

As in [**4, 12**], let μ be defined for E in $\mathcal{B}([0,t_0]\times\mathbf{R}^\nu)$ by

$$\mu(E) \equiv \int_0^{t_0} \sigma_s(E^{(s)})\,ds. \tag{5.1}$$

We want to use Theorem 3.1 to rewrite the expression in (4.10) for $\Gamma^{(n)}(t_0,\vec{\xi};-iq)$ in terms of integrals with respect to μ. For the purpose of applying Theorem 3.1, let $(Y,\mathcal{Y},\gamma) = ([0,t_0]^n, \mathcal{B}([0,t_0]^n)$, Lebesgue measure), and $(Z,\mathcal{Z}) = (\mathbf{R}^{n\nu},\mathcal{B}(\mathbf{R}^{n\nu}))$. Given $\vec{s}=(s_1,\ldots,s_n)$ in Y, let $\sigma_{(s_1,\ldots,s_n)} = \sigma_{s_1}\times\cdots\times\sigma_{s_n}$. Note that

$$\|\sigma_{(s_1,\ldots,s_n)}\| = \prod_{j=1}^{n}\|\sigma_{s_j}\|$$

which is in $L_1([0,t_0]^n)$ since $\|\sigma_s\|$ is in $L_1[0,t_0]$. To show that Theorem 3.1 is applicable, it remains to show that for every B in $\mathcal{B}(\mathbf{R}^{n\nu})$, $(\sigma_{s_1}\times\cdots\times\sigma_{s_n})(B)$ is a measurable function of $(s_1,\ldots,s_n)$.

Let $\mathcal{C} \equiv \{B\in\mathcal{B}(\mathbf{R}^{n\nu})\colon (\sigma_{s_1}\times\cdots\times\sigma_{s_n})(B)$ is measurable in $(s_1,\ldots,s_n)\}$. $\mathcal{C}$ contains the measurable rectangles because $\sigma_{s_i}(B_i)$ is measurable in s_i for every B_i in $\mathcal{B}(\mathbf{R}^\nu)$, $i=1,2,\ldots,n$. It is easy to show that $\mathcal{C}$ is closed under finite, disjoint unions and so contains the algebra $\mathcal{A}$ of finite, disjoint unions of measurable rectangles. Further one sees easily that $\mathcal{C}$ is a monotone class. Hence, by the Monotone Class Theorem [**8**, p. 27], $\mathcal{C}\supseteq\sigma(\mathcal{A})$. Therefore $\mathcal{C} = \mathcal{B}(\mathbf{R}^\nu)\times\cdots\times\mathcal{B}(\mathbf{R}^\nu) = \mathcal{B}(\mathbf{R}^{n\nu})$.

Theorem 3.1 now tells us that if we let

$$\mu_0(E) \equiv \int_{[0,t_0]^n} \sigma_{(s_1,\ldots,s_n)}\left(E^{(s_1,\ldots,s_n)}\right) d\vec{s}$$

for E in $\mathfrak{B}([0,t_0]^n \times \mathbf{R}^{n\nu})$, then μ_0 is a $\mathbf{C}$-valued, countably additive measure of finite total variation; and, for every bounded Borel function $\Phi(\vec{s},\vec{\vec{V}})$,

$$\text{(5.2)} \qquad \int_{[0,t_0]^n}\left[\int_{\mathbf{R}^{n\nu}} \Phi(\vec{s},\vec{\vec{V}})\, d\sigma_{(s_1,\ldots,s_n)}(\vec{\vec{V}})\right] d\vec{s}$$

$$= \int_{[0,t_0]^n \times \mathbf{R}^{n\nu}} \Phi(\vec{s},\vec{\vec{V}})\, d\mu_0(\vec{s},\vec{\vec{V}})$$

where $\vec{\vec{V}} = (\vec{V}_1,\ldots,\vec{V}_n)$ and $\vec{V}_j = (v_{j,1}, v_{j,2},\ldots,v_{j,\nu})$. In particular, formula (5.2) holds for

$$\Phi(\vec{s},\vec{\vec{V}}) = \int_{\mathbf{R}^\nu} \chi_{\Delta_n(t_0)}(\vec{s})$$

$$\times \exp\left\{\frac{1}{2qi}\left[\sum_{l=0}^{n}\sum_{j=0}^{l}(2-\delta_{j,l})(t_0-s_l)\langle\vec{V}_j,\vec{V}_l\rangle\right]\right.$$

$$\left. + i\langle\vec{\xi},\vec{V}_n+\vec{V}_{n-1}+\cdots+\vec{V}_1+\vec{V}_0\rangle\right\} d\phi(\vec{V}_0).$$

Thus we can write

$$\text{(5.3)} \quad \Gamma^{(n)}(t_0,\vec{\xi};-iq)$$

$$= \int_{[0,t_0]^n}\left[\int_{\mathbf{R}^{n\nu}}\left(\int_{\mathbf{R}^\nu}\chi_{\Delta_n(t_0)}(\vec{s})\right.\right.$$

$$\times \exp\left\{\frac{1}{2qi}\sum_{l=0}^{n}\sum_{j=0}^{l}(2-\delta_{j,l})(t_0-s_l)\langle\vec{V}_j,\vec{V}_l\rangle\right.$$

$$\left.\left.\left. + i\langle\vec{\xi},\vec{V}_n+\cdots+\vec{V}_1+\vec{V}_0\rangle\right\} d\phi(\vec{V}_0)\right) d(\sigma_{s_1}\times\cdots\times\sigma_{s_n})(\vec{\vec{V}})\right] d\vec{s}$$

$$= \int_{[0,t_0]^n \times \mathbf{R}^{n\nu}} \left(\int_{\mathbf{R}^\nu} \chi_{\Delta_n(t_0)}(\vec{s}) \right.$$

$$\times \exp\left\{ \frac{1}{2qi} \sum_{l=0}^{n} \sum_{j=0}^{l} (2 - \delta_{j,l})(t_0 - s_l)\langle \vec{V}_j, \vec{V}_l \rangle \right.$$

$$\left. \left. + i\langle \vec{\xi} \vec{V}_n + \cdots + \vec{V}_1 + \vec{V}_0 \rangle \right\} d\phi(\vec{V}_0) \right) d\mu_0(\vec{s}, \vec{\vec{V}}).$$

Finally, to write $\Gamma^{(n)}$ in terms of μ, we will show that $\mu_0 = \mu \times \cdots \times \mu$. Since μ_0 and $\mu \times \cdots \times \mu$ are both measures on $\mathfrak{B}([0, t_0]^n \times \mathbf{R}^{n\nu})$, it suffices to show that they agree on sets of the form $E_1 \times \cdots \times E_n$ where each E_j is in $\mathfrak{B}([0, t_0] \times \mathbf{R}^\nu)$. But

$$\mu_0(E_1 \times \cdots \times E_n) = \int_{[0,t_0]^n} (\sigma_{s_1} \times \cdots \times \sigma_{s_n})\left((E_1 \times \cdots \times E_n)^{(s_1, \ldots, s_n)}\right) d\vec{s}$$

$$= \int_{[0,t_0]^n} (\sigma_{s_1} \times \cdots \times \sigma_{s_n})\left(E_1^{(s_1)} \times \cdots \times E_n^{(s_n)}\right) d\vec{s}$$

$$= \int_{[0,t_0]^n} \sigma_{s_1}\left(E_1^{(s_1)}\right) \cdots \sigma_{s_n}\left(E_n^{(s_n)}\right) d\vec{s}$$

$$= \prod_{j=1}^{n} \int_0^{t_0} \sigma_{s_j}\left(E_j^{(s_j)}\right) ds_j = \prod_{j=1}^{n} \mu(E_j)$$

as desired. Hence $\Gamma^{(n)}$ may alternately be written as

$$(5.4) \quad \Gamma^{(n)}(t_0, \vec{\xi}; -iq)$$

$$= \int_{[0,t_0]^n \times \mathbf{R}^{n\nu}} \Phi(\vec{s}, \vec{V}) d(\mu \times \cdots \times \mu)(s_1, \vec{V}_1; \cdots; s_n, \vec{V}_n)$$

$$= \int_{\mathbf{R}^\nu} \int_{\Delta_n(t_0) \times \mathbf{R}^{n\nu}} \exp\left\{ \frac{1}{2qi} \left[\sum_{l=0}^{n} \sum_{j=0}^{l} (2 - \delta_{j,l})(t_0 - s_l)\langle \vec{V}_j, \vec{V}_l \rangle \right] \right.$$

$$\left. + i\langle \vec{\xi}, \vec{V}_n + \cdots + \vec{V}_0 \rangle \right\}$$

$$d(\mu \times \cdots \times \mu)(s_1, \vec{V}_1; \cdots; s_n, \vec{V}_n)\, d\phi(\vec{V}_0).$$

We can also of course now rewrite (4.10) as

$$(5.5)\quad \int_{C^\nu[0,t_0]}^{\mathrm{anf}_q} F\big(t_0, \vec{\xi}; \vec{X}\big)\, dm^\nu(\vec{X}) = \sum_{n=0}^{\infty} \Gamma^{(n)}(t, \vec{\xi}; -iq)$$

$$= \sum_{n=0}^{\infty} \int_{\mathbf{R}^\nu} \int_{\Delta_n(t_0)\times \mathbf{R}^{n\nu}} \exp\left\{ \frac{1}{2qi} \left[\sum_{l=0}^{n} \sum_{j=0}^{l} (2 - \delta_{j,l})(t_0 - s_l)\langle \vec{V}_j, \vec{V}_l\rangle \right] \right.$$

$$\left. + i\langle \vec{\xi}, \vec{V}_n + \cdots + \vec{V}_0\rangle \right\}$$

$$d(\mu \times \cdots \times \mu)\big(s_1, \vec{V}_1; \cdots ; s_n, \vec{V}_n\big)\, d\phi\big(\vec{V}_0\big).$$

Since the series expansions (4.10) and (5.5) play a key role in the next section it will be helpful to summarize the main facts in one place in the notation that we will use as we continue.

The series expansion has so far been written in terms of t_0 and $\vec{\xi}$; but what has been done for fixed t_0 and $\vec{\xi}$ can equally well be done for any $(t, \vec{U})$ in $[0, t_0] \times \mathbf{R}^\nu$. From this point on, we will regard q as an arbitrary, but fixed, nonzero real number, and so, we will eliminate q from our notation.

THEOREM 5.1. *Let $\psi \in \hat{M}(\mathbf{R}^\nu)$ be given by* (3.5). *Suppose that θ is given by* (3.3) *and satisfies* (3.4a) *and* (3.4b). *For $(t, \vec{U})$ in $[0, t_0] \times \mathbf{R}^\nu$, let*

$$(5.6)\qquad F(\vec{X}) \equiv F(t, \vec{U}; \vec{X})$$

$$\equiv \exp\left\{ \int_0^t \theta\big(t - s, \vec{X}(s) + \vec{U}\big)\, ds \right\} \psi\big(\vec{X}(t) + \vec{U}\big),$$

and let

$$(5.7)\qquad \Gamma(t, \vec{U}) = \int_{C^\nu[0,t]}^{\mathrm{anf}_q} F(t, \vec{U}; \vec{X})\, dm^\nu(\vec{X}).$$

Then

$$(5.8)\qquad \Gamma(t, \vec{U}) = \sum_{n=0}^{\infty} \Gamma^{(n)}(t, \vec{U})$$

where

(5.9)

$$\Gamma^{(n)}(t,\vec{U}) = \int_{\Delta_n(t)} \int_{\mathbf{R}^{(n+1)\nu}} \exp\left\{ \frac{1}{2qi} \left[\sum_{l=0}^{n} \sum_{j=0}^{l} (2-\delta_{j,l})(t-s_l)\langle \vec{V}_j, \vec{V}_l \rangle \right] + i\langle \vec{U}, \vec{V}_n + \cdots + \vec{V}_0 \rangle \right\} d\sigma_{s_1}(\vec{V}_1)\cdots d\sigma_{s_n}(\vec{V}_n)\, d\phi(\vec{V}_0)\, d\vec{s}$$

$$= \int_{\mathbf{R}^\nu} \int_{\Delta_n(t)\times \mathbf{R}^{n\nu}} \exp\left\{ \frac{1}{2qi} \left[\sum_{l=0}^{n} \sum_{j=0}^{l} (2-\delta_{j,l})(t-s_l)\langle \vec{V}_j, \vec{V}_l \rangle \right] + i\langle \vec{U}, \vec{V}_n + \cdots + \vec{V}_0 \rangle \right\} d(\mu\times\cdots\times\mu)(s_1,\vec{V}_1;\cdots;s_n,\vec{V}_n)\, d\phi(\vec{V}_0)$$

and where $\Delta_n(t) = \{\vec{s}\in[0,t_0]^n\colon\ 0 = s_0 < s_1 < \cdots < s_n < t \le t_0\}$. Furthermore, we have the inequality

$$|\Gamma^{(n)}(t,\vec{U})| \le \frac{\|\phi\|}{n!}\left(\int_0^{t_0} \|\sigma_s\| ds\right)^n \tag{5.10}$$

and so the series (5.8) converges absolutely and uniformly on $[0, t_0]\times \mathbf{R}^\nu$.

REMARK 5.1. It is not really necessary to change the Wiener space in (5.7) every time t is changed. Actually

$$\Gamma(t,\vec{U}) = \int_{C^\nu[0,t_0]}^{anf_q} F(t,\vec{U};\vec{X})\, dm^\nu(\vec{X})$$

since for $\lambda > 0$

$$\int_{C^\nu[0,t_0]} \exp\left\{\int_0^t \theta(t-s,\lambda^{-1/2}\vec{X}(s)+\vec{U})\,ds\right\}\psi(\lambda^{-1/2}\vec{X}(t)+\vec{U})\,dm^\nu(\vec{X})$$

$$= \int_{C^\nu[0,t]} \exp\left\{\int_0^t \theta(t-s,\lambda^{-1/2}\vec{X}(s)+\vec{U})\,ds\right\} \times\psi(\lambda^{-1/2}\vec{X}(t)+\vec{U})\,dm_t^\nu(\vec{X})$$

where m_t^ν denotes Wiener measure on $C^\nu[0,t]$.

6. A summation procedure. In §4, formula (4.6) with $\lambda > 0$, played a key role. It is easy to show, via an analytic continuation argument, that (4.6) holds for all λ in $\mathbf{C}^+$. But we will need a version of (4.6) for $\lambda = -iq$. Since the integrand has constant absolute value one in this case, it is clear that we will need a summation procedure. Let

$$\int_{\mathbf{R}^\nu}^{-} f(\vec{U})d\vec{U} \equiv \lim_{A\to+\infty} \int_{\mathbf{R}^\nu} f(\vec{U})\exp\left\{\frac{-\langle\vec{U},\vec{U}\rangle}{2A}\right\} d\vec{U} \tag{6.1}$$

whenever the expression on the right exists. Of course if f is in $L_1(\mathbf{R}^\nu)$, it is clear using the Dominated Convergence Theorem that

$$\int_{\mathbf{R}^\nu}^{-} f(\vec{U})\,d\vec{U} = \int_{\mathbf{R}^\nu} f(\vec{U})\,d\vec{U}.$$

Note that this is a different summation procedure than that used by Cameron and Storvick in [5]. Recall that in the case $\nu = 1$ they used the summation procedure.

$$\int_{\to-\infty}^{\to\infty} f(u)\,du \equiv \lim_{B,\,B'\to\infty} \int_{-B'}^{B} f(u)\,du \tag{6.2}$$

whenever the expression on the right exists. One major advantage of using the summation procedure (6.1) is its notational simplicity in higher dimensions; i.e., for $\nu > 1$. Another advantage is that for many functions of interest to us one can actually evaluate the integrals that occur on the right side of (6.1) and thus replace certain estimates with actual calculations. It turns out that, for the specific functions under consideration in this paper, the two summation procedures actually agree. However they are not equivalent in general; we will discuss this question briefly in the case $\nu = 1$. The results extend readily to higher dimensions.

First we note that the function $f(u) = e^{iu}$ serves as an example where (6.1) exists but (6.2) does not. On the other hand if we put a very mild growth condition on $f(u)$ (to insure that $\int_{\mathbf{R}}|f(u)|\exp(-u^2/2A)\,du < \infty$ for each $A > 0$), then the existence of (6.2) yields the existence of (6.1). This fact will follow easily from the following lemma and its proof.

LEMMA 6.1. *Suppose that f is integrable over $[-B', B]$ for every B', $B > 0$ and that $\lim_{B,B'\to\infty}\int_{-B'}^{B} f(u)\,du$ exists and equals some finite number L. Then $\lim_{A\to\infty}\int_{\to-\infty}^{\to\infty} f(u)e^{-u^2/2A}\,du = L$.*

Proof. Without loss of generality assume that $f(u) \equiv 0$ on $(-\infty,0)$. First we observe that for each $A > 0$, $\int_0^{\to\infty} f(u)\exp(-u^2/2A)\,du$ exists since for each $B > 0$ we have (integration by parts) that

$$\int_0^B f(u)e^{-u^2/2A}\,du = e^{-B^2/2A}\int_0^B f(t)\,dt + \int_0^B \left[\int_0^u f(t)\,dt\right]\frac{u}{A}e^{-u^2/2A}\,du.$$

Because $\lim_{B\to\infty}\int_0^B f(t)\,dt = L$, the right-hand side clearly has a finite limit as $B \to \infty$. Hence so does the left-hand side.

Next, proceeding formally we see that

$$\lim_{A\to\infty}\int_0^{\to\infty} f(u)e^{-u^2/2A}\,du = \lim_{A\to\infty}\lim_{B\to\infty}\int_0^B f(u)e^{-u^2/2A}\,du$$

$$= \lim_{B\to\infty}\lim_{A\to\infty}\int_0^B f(u)e^{-u^2/2A}\,du = \lim_{B\to\infty}\int_0^B f(u)\,du = L.$$

The third equality above follows from the Dominated Convergence Theorem. The interchange of limits is justified since the expression

$$\int_B^{\to\infty} f(u)e^{-u^2/2A}\,du$$

converges uniformly (as a function of A) to zero as $B \to \infty$. To see this note that for any $0 < B < K$ we (integration by parts) have that

$$\int_B^K f(u)e^{-u^2/2A}\,du = e^{-K^2/2A}\int_B^K f(t)\,dt + \int_B^K\left[\int_B^u f(t)\,dt\right]\frac{u}{A}e^{-u^2/2A}\,du.$$

Thus

$$\begin{aligned}\left|\int_B^K f(u)e^{-u^2/2A}\,du\right| & \\ &\le e^{-K^2/2A}\left|\int_B^K f(t)\,dt\right| + \sup_{B\le u\le K}\left|\int_B^u f(t)\,dt\right|\int_B^K \frac{u}{A}e^{-u^2/2A}\,du \\ &\le \sup_{B\le u\le K}\left|\int_B^u f(t)\,dt\right|\end{aligned}$$

which is independent of A and goes to zero as $B \to \infty$ by the Cauchy criterion.

PROPOSITION 6.1. *Assume that the hypotheses of Lemma* 6.1 *are satisfied and that there exists* $\alpha \in [0,2)$ *and positive constants* M, N *and* R *such that* $|f(u)| \le N\exp(M|u|^\alpha)$ *for all* $|u| \ge R$. *Then the left member below exists and equality holds*

$$\lim_{A\to\infty}\int_{\mathbf{R}} f(u)e^{-u^2/2A}\,du \equiv \int_{\mathbf{R}}^{-} f(u)\,du = L.$$

The desired version of (4.6), namely Corollary 6.1 below, will follow immediately from the next Lemma.

LEMMA 6.2. *For* $A > 0$

$$(6.3)\quad \left[-iq/2\pi(s_j - s_{j-1})\right]^{\nu/2} \times \int_{\mathbf{R}^\nu} \exp\left\{\frac{iq\|\vec{U}_j - \vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})} + i\langle \vec{U}_j, \vec{W}\rangle - \frac{\|\vec{U}_j\|^2}{2A}\right\} d\vec{U}_j$$

$$= \left(-iqA/\left[(s_j - s_{j-1}) - iqA\right]\right)^{\nu/2} \times \exp\left\{\left[2qA\langle \vec{U}_{j-1}, \vec{W}\rangle - A(s_j - s_{j-1})\|\vec{W}\|^2 + iq\left\|\vec{U}_{j-1}\right\|^2\right]/\left[2(s_j - s_{j-1}) - 2iqA\right]\right\}.$$

COROLLARY 6.1.

$$(6.4)\quad \left[-iq/2\pi(s_j - s_{j-1})\right]^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \exp\left\{\frac{iq\|\vec{U}_j - \vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})} + i\langle \vec{U}_j, \vec{W}\rangle\right\} d\vec{U}_j$$

$$= \exp\left\{i\langle \vec{U}_{j-1}, \vec{W}\rangle + \frac{(s_j - s_{j-1})\|\vec{W}\|^2}{2qi}\right\}.$$

In order to prove Lemma 6.2, we will need the formula

$$(6.5)\quad \int_{\mathbf{R}^\nu} \exp\left(-\lambda \sum_{l=1}^{\nu} (u_l - z_l)^2\right) d\vec{U} = \int_{\mathbf{R}^\nu} \exp\left[-\lambda \|\vec{U}\|^2\right] d\vec{U} = (\pi/\lambda)^{\nu/2}$$

for $\lambda \in \mathbf{C}^+$ and $\vec{Z} = (z_1, \ldots, z_\nu) \in \mathbf{C}^\nu$.

This formula is familiar for $\lambda > 0$ and $\vec{Z}$ in $\mathbf{R}^\nu$. The argument is perhaps easiest to think about if one first extends to $\lambda \in \mathbf{C}^+$, $\vec{Z} \in \mathbf{R}^\nu$, and then, finally, to $\lambda \in \mathbf{C}^+$, $\vec{Z} \in \mathbf{C}^\nu$.

Proof of Lemma 6.2. For $A > 0$, and using equation (6.5) in the 4th equality below we have

$$\left[-iq/2\pi(s_j - s_{j-1})\right]^{\nu/2} \times \int_{\mathbf{R}^\nu} \exp\left\{\frac{iq\|\vec{U}_j - \vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})} + i\langle \vec{U}_j, \vec{W}\rangle - \frac{\|\vec{U}_j\|^2}{2A}\right\} d\vec{U}_j$$

$$= \exp\left\{\frac{iq\|\vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})}\right\}\left[-iq/2\pi(s_j - s_{j-1})\right]^{\nu/2} \times \int_{\mathbf{R}^\nu} \exp\left\{\frac{\|\vec{U}_j\|^2}{2}\left[\frac{iq}{s_j - s_{j-1}} - \frac{1}{A}\right] + i\left\langle \vec{W} - \frac{q\vec{U}_{j-1}}{s_j - s_{j-1}}, \vec{U}_j\right\rangle\right\} d\vec{U}_j$$

$$= \exp\left\{\frac{iq\|\vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})}\right\}\left[-iq/2\pi(s_j - s_{j-1})\right]^{\nu/2}$$

$$\times \int_{\mathbf{R}^\nu} \exp\left\{\left[\frac{iq}{2(s_j - s_{j-1})} - \frac{1}{2A}\right]\right.$$

$$\left.\times\left[\|\vec{U}_j\|^2 + \frac{2Ai}{iqA - (s_j - s_{j-1})}\left\langle (s_j - s_{j-1})\vec{W} - q\vec{U}_{j-1}, \vec{U}_j\right\rangle\right]\right\} d\vec{U}_j$$

$$= \exp\left\{\frac{iq\|\vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})} + \left(\frac{iq}{2(s_j - s_{j-1})} - \frac{1}{2A}\right)\right.$$

$$\left.\times\left(A^2 \sum_{l=1}^{\nu}\left[\frac{(s_j - s_{j-1})w_l - qu_{j-1,l}}{iqA - (s_j - s_{j-1})}\right]^2\right)\right\}$$

$$\times\left[-iq/2\pi(s_j - s_{j-1})\right]^{\nu/2}$$

$$\times \int_{\mathbf{R}^\nu} \exp\left\{-\frac{1}{2}\left(\frac{1}{A} - \frac{iq}{s_j - s_{j-1}}\right)\right.$$

$$\left.\times\left(\sum_{l=1}^{\nu}\left[u_{j,l} + \frac{Ai\left((s_j - s_{j-1})w_l - qu_{j-1,l}\right)}{iqA - (s_j - s_{j-1})}\right]^2\right)\right\} d\vec{U}_j$$

$$= \left[-iq/2\pi(s_j - s_{j-1})\right]^{\nu/2}\left[\frac{2\pi A(s_j - s_{j-1})}{(s_j - s_{j-1}) - iqA}\right]^{\nu/2}$$

$$\times \exp\left\{\frac{iq\|\vec{U}_{j-1}\|^2}{2(s_j - s_{j-1})} + \left(\frac{iqA^2}{2(s_j - s_{j-1})} - \frac{A}{2}\right)\right.$$

$$\left.\times\left(\sum_{l=1}^{\nu}\left[\frac{(s_j - s_{j-1})w_l - qu_{j-1,l}}{iqA - (s_j - s_{j-1})}\right]^2\right)\right\}.$$

Formula (6.3) now follows after some algebraic calculations.

LEMMA 6.3. *For $A > 0$, $t \in [0, t_0]$, $q \in \mathbf{R}$ ($q \neq 0$), and $\vec{\xi}$ and $\vec{V}$ in $\mathbf{R}^\nu$ we have the inequality*

(6.6)

$$\left|\exp\left\{\left[2qA\langle\vec{\xi},\vec{V}\rangle - A(t_0 - t)\|\vec{V}\|^2 + iq\|\vec{\xi}\|^2\right]/[2(t_0 - t) - 2iqA]\right\}\right| \leq 1.$$

Proof. The left side of (6.6) equals

$$(6.7)\quad \exp\left\{\operatorname{Re}\left(\frac{2qA\langle\vec{\xi},\vec{V}\rangle - A(t_0 - t)\|\vec{V}\|^2 + iq\|\vec{\xi}\|^2}{2(t_0 - t) - 2iqA}\right)\right\}$$

$$= \exp\left\{\frac{2Aq(t_0 - t)\langle\vec{\xi},\vec{V}\rangle - A(t_0 - t)^2\|\vec{V}\|^2 - q^2A\|\vec{\xi}\|^2}{2(t_0 - t)^2 + 2q^2A^2}\right\}.$$

Now think of A, t, q and $\vec{\xi} = (\xi_1, \ldots, \xi_\nu)$ as arbitrary but fixed and regard the argument of the exponential as a function of $\vec{V} = (v_1, v_2, \ldots, v_\nu)$. A routine calculation shows that its maximum occurs for $\vec{V} = (q/(t_0 - t))\vec{\xi}$. Substituting this value for $\vec{V}$ into (6.7) one obtains $\exp\{0\}$ as desired.

7. The Schroedinger equation. We want to show that $\Gamma(t, \vec{U})$ given by (5.7) satisfies the following integral equation which is formally equivalent to Schroedinger's equation:

$$(7.1)\quad \Gamma(t,\vec{\xi}) = (q/2\pi it)^{\nu/2}\int^{-}_{\mathbf{R}^\nu} \psi(\vec{U})\exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2t}\right\} d\vec{U}$$

$$+ \int_0^t [q/2\pi i(t-s)]^{\nu/2}\int^{-}_{\mathbf{R}^\nu} \theta(s,\vec{U})\Gamma(s,\vec{U})$$

$$\times\exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)}\right\} d\vec{U}\,ds.$$

We begin by finding an alternate expression for the first term on the right side of (7.1); actually we will show it equals $\Gamma^{(0)}(t, \vec{\xi}; -iq)$ as given by (4.9).

LEMMA 7.1. *Let ψ in $\hat{M}(\mathbf{R}^\nu)$ be given by* (3.5). *Then for* $(t, \vec{\xi}) \in [0, t_0] \times \mathbf{R}^\nu$,

$$(7.2)\qquad (q/2\pi i t)^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \psi(\vec{U}) \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2t} \right\} d\vec{U}$$

$$= \int_{\mathbf{R}^\nu} \exp\left\{ \frac{t\|\vec{V}\|^2}{2qi} + i\langle \vec{\xi}, \vec{V} \rangle \right\} d\phi(\vec{V}).$$

Proof. First note that when the limit exists we have

$$(q/2\pi i t)^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \psi(\vec{U}) \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2t} \right\} d\vec{U}$$

$$= \lim_{A \to +\infty} (q/2\pi i t)^{\nu/2} \int_{\mathbf{R}^\nu} \left[\int_{\mathbf{R}^\nu} \exp(i\langle \vec{U}, \vec{V} \rangle)\, d\phi(\vec{V}) \right]$$

$$\times \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2t} - \frac{\|\vec{U}\|^2}{2A} \right\} d\vec{U}$$

$$= \lim_{A \to +\infty} (q/2\pi i t)^{\nu/2} \int_{\mathbf{R}^\nu} \int_{\mathbf{R}^\nu} \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2t} \right.$$

$$\left. + i\langle \vec{U}, \vec{V} \rangle - \frac{\|\vec{U}\|^2}{2A} \right\} d\vec{U}\, d\phi(\vec{V}).$$

We now apply Lemma 6.2 to the last expression above to obtain

$$\lim_{A \to +\infty} \left[\frac{-iqA}{t - iqA} \right]^{\nu/2} \int_{\mathbf{R}^\nu} \exp\left\{ \frac{Aiq\langle \vec{\xi}, \vec{V} \rangle}{it + Aq} - \frac{At\|\vec{V}\|^2}{2t - 2qiA} \right.$$

$$\left. + \frac{iq\|\vec{\xi}\|^2}{2t - 2iqA} \right\} d\phi(\vec{V})$$

$$= \int_{\mathbf{R}^\nu} \exp\left\{ i\langle \vec{\xi}, \vec{V} \rangle + \frac{t\|\vec{V}\|^2}{2qi} \right\} d\phi(\vec{V}).$$

where this last equality follows from Lemma 6.3 and the Dominated Convergence Theorem, which establishes the necessary limits.

THEOREM 7.1. *Let θ, ψ and Γ satisfy the hypotheses of Theorem* 5.1. *Then for* $(t, \vec{\xi}) \in [0, t_0] \times \mathbf{R}^\nu$, $\Gamma(t, \vec{\xi})$ *satisfies the Schroedinger integral equation* (7.1).

Proof. We will prove this result by substituting the series expansion (5.8) for $\Gamma(s, \vec{U})$ into the second term on the right side of (7.1) and show that the resulting expression, which we will denote by $H(t, \vec{\xi})$, equals

$$\Gamma(t, \vec{\xi}) - (q/2\pi i t)^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \psi(\vec{U}) \exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2t}\right\} d\vec{U}.$$

First we see that

$$(7.3)\quad H(t, \vec{\xi}) \equiv \int_0^t \int_{\mathbf{R}^\nu}^{-} [q/2\pi i(t-s)]^{\nu/2} \theta(s, \vec{U}) \Gamma(s, \vec{U}) \times \exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)}\right\} d\vec{U}\, ds$$

$$= \int_0^t \lim_{A\to\infty} \int_{\mathbf{R}^\nu} [q/2\pi i(t-s)]^{\nu/2} \theta(s, \vec{U}) \left[\lim_{N\to\infty} \sum_{n=0}^{N} \Gamma^{(n)}(s, \vec{U})\right] \times \exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)} - \frac{\|\vec{U}\|^2}{2A}\right\} d\vec{U}\, ds.$$

Here the existence of either member of the above equation implies that of the other, and the same is true of the equations (7.4), (7.5), (7.8), (7.9) and (7.10) to follow. Moreover we shall show that the second member of (7.10) does exist, and hence all members of all these equations exist.

Now inequality (5.10) implies that for $0 \le s \le t$,

$$\sum_{n=0}^{N} |\Gamma^{(n)}(s, \vec{U})| \le \|\phi\| \exp\left(\int_0^t \|\sigma_\tau\| d\tau\right)$$

and so, applying the Dominated Convergence Theorem to (7.3) we can write

(7.4)

$$H(t, \vec{\xi}) = \int_0^t \lim_{A\to\infty} \lim_{N\to\infty} \sum_{n=0}^{N} \int_{\mathbf{R}^\nu} [q/2\pi i(t-s)]^{\nu/2} \theta(s, \vec{U}) \Gamma^{(n)}(s, \vec{U}) \times \exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)} - \frac{\|\vec{U}\|^2}{2A}\right\} d\vec{U}\, ds$$

$$= \int_0^t \lim_{A\to\infty} \lim_{N\to\infty} \sum_{n=0}^{N} \int_{\mathbf{R}^\nu} [q/2\pi i(t-s)]^{\nu/2}$$

$$\times \int_{\mathbf{R}^\nu} \exp\big(i\langle \vec{U}, \vec{V}_{n+1}\rangle\big)\, d\sigma_s\big(\vec{V}_{n+1}\big)$$

$$\times \Bigg[\int_{\mathbf{R}^\nu} \int_{\Delta_n(s)\times\mathbf{R}^{n\nu}} \exp\Bigg\{ i\langle \vec{U}, \vec{V}_n + \cdots + \vec{V}_0\rangle + \frac{1}{2qi}$$

$$\times \sum_{l=0}^{n} \sum_{j=0}^{l} (2-\delta_{j,l})\langle \vec{V}_j, \vec{V}_l\rangle (s-s_l) \Bigg\}$$

$$d(\mu\times\cdots\times\mu)\big(s_1,\vec{V}_1;\cdots;s_n,\vec{V}_n\big)\, d\phi\big(\vec{V}_0\big) \Bigg]$$

$$\times \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)} - \frac{\|\vec{U}\|^2}{2A} \right\} d\vec{U}\, ds$$

where the last equality was obtained by substituting for θ and for $\Gamma^{(n)}$ using (5.9). Now applying the Fubini Theorem and then Lemma 6.2 to (7.4) we obtain

$$(7.5)\quad H(t,\vec{\xi}) = \int_0^t \lim_{A\to\infty} \lim_{N\to\infty} \sum_{n=0}^{N} \int_{\mathbf{R}^\nu} \int_{\mathbf{R}^\nu} \int_{\Delta_n(s)\times\mathbf{R}^{n\nu}}$$

$$\times \exp\left\{ \frac{1}{2qi} \sum_{l=0}^{n} \sum_{j=0}^{l} (2-\delta_{j,l})(s-s_l)\langle \vec{V}_j, \vec{V}_l\rangle \right\}$$

$$\times \int_{\mathbf{R}^\nu} [q/2\pi i(t-s)]^{\nu/2}$$

$$\times \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)} + i\langle \vec{U}, \vec{V}_{n+1} + \cdots + \vec{V}_0\rangle - \frac{\|\vec{U}\|^2}{2A} \right\} d\vec{U}$$

$$d(\mu\times\cdots\times\mu)\big(s_1,\vec{V}_1;\cdots;s_n,\vec{V}_n\big)\, d\phi\big(\vec{V}_0\big)\, d\sigma_s\big(\vec{V}_{n+1}\big)\, ds$$

$$= \int_0^t \lim_{A\to\infty} \lim_{N\to\infty} \sum_{n=0}^{N} \int_{\mathbf{R}^\nu} \int_{\mathbf{R}^\nu} \int_{\Delta_n(s)\times\mathbf{R}^{n\nu}}$$

$$\times \exp\left\{\frac{1}{2qi} \sum_{l=0}^{n} \sum_{j=0}^{l} (2-\delta_{j,l})(s-s_l)\langle \vec{V}_j, \vec{V}_l\rangle\right\}\left[\frac{-iqA}{(t-s)-iqA}\right]^{\nu/2}$$

$$\times \exp\left\{\frac{Aiq\langle \vec{\xi}, \vec{V}_{n+1}+\cdots+\vec{V}_0\rangle}{Aq+i(t-s)}\right.$$

$$\left. - \frac{A(t-s)\|\vec{V}_{n+1}+\cdots+\vec{V}_0\|^2}{2(t-s)-2iqA} + \frac{iq\|\vec{\xi}\|^2}{2(t-s)-2iqA}\right\}$$

$$d(\mu\times\cdots\times\mu)(s_1,\vec{V}_1;\cdots;s_n,\vec{V}_n)\, d\phi(\vec{V}_0)\, d\sigma_s(\vec{V}_{n+1})\, ds.$$

Next we want to interchange the order of the limits with respect to A and N. We can justify doing this by finding a series independent of A which is summable and dominates

$$(7.6)\quad \sum_{n=0}^{\infty} \left| \int_{\mathbf{R}^\nu} \int_{\mathbf{R}^\nu} \int_{\Delta_n(s)\times\mathbf{R}^{n\nu}} \exp\left\{\frac{1}{2qi} \sum_{l=0}^{n} \sum_{j=0}^{l} (2-\delta_{j,l})(s-s_l)\langle \vec{V}_j, \vec{V}_l\rangle\right\}\right.$$

$$\times\left[\frac{-iqA}{(t-s)-iqA}\right]^{\nu/2}$$

$$\times \exp\left\{\frac{Aiq\langle \vec{\xi}, \vec{V}_{n+1}+\cdots+\vec{V}_0\rangle}{Aq+i(t-s)} - \frac{A(t-s)\|\vec{V}_{n+1}+\cdots+\vec{V}_0\|^2}{2(t-s)-2iqA}\right.$$

$$\left. + \frac{iq\|\vec{\xi}\|^2}{2(t-s)-2iqA}\right\}$$

$$\left. d(\mu\times\cdots\times\mu)(s_1,\vec{V}_1;\cdots;s_n,\vec{V}_n)\, d\phi(\vec{V}_0)\, d\sigma_s(\vec{V}_{n+1})\right|.$$

But by (5.2) we can write (7.6) in the form

$$(7.7)\quad \left|\frac{-iqA}{(t-s)-iqA}\right|^{\nu/2}$$

$$\times \sum_{n=0}^{\infty}\left|\int_{\mathbf{R}^\nu}\int_{\Delta_n(s)}\int_{\mathbf{R}^{(n+1)\nu}}\exp\left\{\frac{1}{2qi}\sum_{l=0}^{n}\sum_{j=0}^{l}(2-\delta_{j,l})(s-\tau_l)\langle\vec{V}_j,\vec{V}_l\rangle\right\}\right.$$

$$\times\exp\left\{\frac{Aiq\langle\vec{\xi},\vec{V}_{n+1}+\cdots+\vec{V}_0\rangle}{A+i(t-s)}-\frac{A(t-s)\|\vec{V}_{n+1}+\cdots+\vec{V}_0\|^2}{2(t-s)-2iqA}\right.$$

$$\left.+\frac{iq\|\vec{\xi}\|^2}{2(t-s)-2iqA}\right\}$$

$$\left. d\sigma_{\tau_1}(\vec{V}_1)\cdots d\sigma_{\tau_n}(\vec{V}_n)\, d\sigma_s(\vec{V}_{n+1})\, d\vec{\tau}\, d\phi(\vec{V}_0)\right|$$

where, as before,

$$\Delta_n(s)=\{\vec{\tau}=(\tau_1,\ldots,\tau_n)\in[0,t]^n: 0<\tau_1<\tau_2<\cdots<\tau_n<s\le t\}.$$

Using Lemma 6.3 and the fact that $|-iqA/((t-s)-iqA)|\le 1$, we see that the series (7.7) is dominated by the series

$$\sum_{n=0}^{\infty}\|\phi\|\,\|\sigma_s\|\int_{\Delta_n(s)}\|\sigma_{\tau_1}\|\cdots\|\sigma_{\tau_n}\|d\vec{\tau}$$

$$=\sum_{n=0}^{\infty}\frac{\|\phi\|\,\|\sigma_s\|}{n!}\int_{[0,s]^n}\|\sigma_{\tau_1}\|\cdots\|\sigma_{\tau_n}\|d\tau_1\cdots d\tau_n$$

$$=\sum_{n=0}^{\infty}\frac{\|\phi\|\,\|\sigma_s\|}{n!}\left(\int_0^s\|\sigma_r\|dr\right)^n=\|\phi\|\,\|\sigma_s\|\exp\left(\int_0^t\|\sigma_r\|dr\right)<\infty.$$

Hence the interchange of limits is justified and so using Lemma 6.3 and the Dominated Convergence Theorem, equation (7.5) becomes

$$(7.8)\quad H(t,\vec{\xi})=\int_0^t\lim_{N\to\infty}\sum_{n=0}^{N}\int_{\mathbf{R}^\nu}\int_{\mathbf{R}^\nu}\int_{\Delta_n(s)\times\mathbf{R}^{n\nu}}$$

$$\times\exp\left\{\frac{1}{2qi}\sum_{l=0}^{n}\sum_{j=0}^{l}(2-\delta_{j,l})(s-s_l)\langle\vec{V}_j,\vec{V}_l\rangle\right.$$

$$\left.+i\langle\vec{\xi},\vec{V}_{n+1}+\cdots+\vec{V}_0\rangle+\frac{(t-s)}{2qi}\left\|\vec{V}_{n+1}+\cdots+\vec{V}_0\right\|^2\right\}$$

$$d(\mu\times\cdots\times\mu)(s_1,\vec{V}_1;\cdots;s_n,\vec{V}_n)\,d\phi(\vec{V}_0)\,d\sigma_s(\vec{V}_{n+1})\,ds.$$

Now relabeling s as s_{n+1} in (7.8) and doing some algebra we see that

$$(7.9)\quad H(t,\vec{\xi}) = \int_0^t \lim_{N\to\infty} \int_{\mathbf{R}^\nu}\int_{\mathbf{R}^\nu}\int_{\Delta_n(s_{n+1})\times\mathbf{R}^{n\nu}} \times\exp\left\{\frac{1}{2qi}\sum_{l=0}^{n+1}\sum_{j=0}^{l}(2-\delta_{j,l})(t-s_l)\langle\vec{V}_j,\vec{V}_l\rangle + i\langle\vec{\xi},\vec{V}_{n+1}+\cdots+\vec{V}_0\rangle\right\} d(\mu\times\cdots\times\mu)(s_1,\vec{V}_1;\cdots;s_n,\vec{V}_n)\,d\phi(\vec{V}_0)\,d\sigma_{s_{n+1}}(\vec{V}_{n+1})\,ds_{n+1}$$

$$= \int_0^t \lim_{N\to\infty}\left[\sum_{n=0}^{N}\int_{\mathbf{R}^\nu}\int_0^{s_{n+1}}\int_0^{s_n}\cdots\int_0^{s_2}\int_{\mathbf{R}^{(n+1)\nu}} \times\exp\left\{\frac{1}{2qi}\sum_{l=0}^{n+1}\sum_{j=0}^{l}(2-\delta_{j,l})(t-s_l)\langle\vec{V}_j,\vec{V}_l\rangle + i\langle\vec{\xi},\vec{V}_{n+1}+\cdots+\vec{V}_0\rangle\right\} d\sigma_{s_1}(\vec{V}_1)\cdots d\sigma_{s_n}(\vec{V}_n)\,d\phi(\vec{V}_0)\,ds_1\cdots ds_n\,d\sigma_{s_{n+1}}(\vec{V}_{n+1})\right] ds_{n+1}$$

where the last equality follows from equation (5.9).

Next we want to take the limit as $N\to\infty$ outside the integral with respect to s_{n+1}. The Dominated Convergence Theorem will allow us to do this if we can produce a dominating function independent of N which is integrable with respect to s_{n+1}. But an argument much like that used to establish (5.10) shows that

$$\|\phi\|\exp\left(\int_0^t \|\sigma_r\|\,dr\right)\|\sigma_{s_{n+1}}\|$$

is such a function. Hence by the argument just made and the Fubini Theorem

(7.10)

$$H(t,\vec{\xi}) = \sum_{n=0}^{\infty} \int_0^t \int_0^{s_{n+1}} \cdots \int_0^{s_2} \int_{\mathbf{R}^{(n+2)\nu}} \times \exp\left\{ \frac{1}{2qi} \sum_{l=0}^{n+1} \sum_{j=0}^{l} (2-\delta_{j,l})(t-s_l)\langle \vec{V}_j, \vec{V}_l\rangle + i\langle \vec{\xi}, \vec{V}_{n+1} + \cdots + \vec{V}_0\rangle \right\}$$
$$d\sigma_{s_1}(\vec{V}_1) \cdots d\sigma_{s_n}(\vec{V}_n)\, d\sigma_{s_{n+1}}(\vec{V}_{n+1})\, d\phi(\vec{V}_0)\, ds_1 \cdots ds_n\, ds_{n+1}.$$

But careful examination of the series (7.10) shows that it is just the series (5.8) (also see equation (5.9)) for $\Gamma(t,\vec{\xi})$ except with the first term missing. From (7.10) and the statement concerning (5.8) immediately following (7.10), the existence of $H(t,\vec{\xi})$ follows, and thus the earlier expressions for it also exist. Hence by (7.10), (4.9), and Lemma 7.1 we see that

$$H(t,\vec{\xi}) = \Gamma(t,\vec{\xi}) - \int_{\mathbf{R}^\nu} \exp\left\{ \frac{t\|\vec{V}_0\|^2}{2qi} + i\langle \vec{\xi}, \vec{V}_0\rangle \right\} d\phi(\vec{V}_0)$$
$$= \Gamma(t,\vec{\xi}) - (q/2\pi i t)^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \psi(\vec{U}) \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2t} \right\} d\vec{U}$$

as desired.

A careful examination of the proof of Theorem 7.1 shows that we have established the following useful corollary which will be needed in the next section to obtain a uniqueness result.

COROLLARY 7.1. *For* $n = 0, 1, 2, \ldots$ *and* $(t,\vec{\xi}) \in [0, t_0] \times \mathbf{R}^\nu$,

$$\Gamma^{(n+1)}(t,\vec{\xi}) = \int_0^t [q/2\pi i(t-s)]^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \theta(s,\vec{U})\Gamma^{(n)}(s,\vec{U}) \times \exp\left\{ \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)} \right\} d\vec{U}\, ds \tag{7.11}$$

where $\Gamma^{(n)}$ *is given by* (5.9).

8. A uniqueness result. In this section we will show that $\Gamma(t,\vec{\xi})$ is the unique solution of (7.1) in $\mathcal{G}_0$ (see Definition 3.1). We will first establish 2 lemmas.

LEMMA 8.1. *Let $\theta \in \mathcal{G}$ be given by* (3.3) *and let G be in $\mathcal{G}_0$; that is, G is of the form*

$$G(s,\vec{U}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{U},\vec{V}\rangle)\, dg_s(\vec{V})$$

where $\{g_s\colon 0 \le s \le t_0\}$ is a family from $M(\mathbf{R}^\nu)$ satisfying: (3.4a), i.e., *for every B in $\mathfrak{B}(\mathbf{R}^\nu)$, $g_s(B)$ is Borel measurable in s, and* (3.4c), i.e., *there exists $M_0 \ge 0$ such that $\|g_s\| \le M_0$ for all s in $[0, t_0]$. For $(t,\vec{\xi}) \in [0,t_0] \times \mathbf{R}^\nu$ let*

$$G^*(t,\vec{\xi}) \equiv \int_0^t [q/2\pi i(t-s)]^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \theta(s,\vec{U})G(s,\vec{U}) \times \exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)}\right\} d\vec{U}\, ds. \tag{8.1}$$

Then G^* is in $\mathcal{G}_0$. In fact the associated family $\{g_t^*\colon 0 \le t \le t_0\}$ from $M(\mathbf{R}^\nu)$ such that

$$G^*(t,\vec{\xi}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi},\vec{V}\rangle)\, dg_t^*(\vec{V}) \tag{8.2}$$

satisfies

$$\|g_t^*\| \le \int_0^t \|\sigma_s\|\, \|g_s\| ds \le M_0 \int_0^{t_0} \|\sigma_s\| ds. \tag{8.3}$$

Proof. We first claim that for every $B \in \mathfrak{B}(\mathbf{R}^\nu \times \mathbf{R}^\nu)$, $(\sigma_s \times g_s)(B)$ is Borel measurable in s. To see this, let $\mathcal{C} \equiv \{B \in \mathfrak{B}(\mathbf{R}^\nu \times \mathbf{R}^\nu)\colon (\sigma_s \times g_s)(B)$ is measurable in $s\}$. Quite clearly $\mathcal{C}$ contains the measurable rectangles and is closed under finite disjoint unions and is a monotone class. It follows from the Monotone Class Theorem [**8**, p. 27] that $\mathcal{C} = \mathfrak{B}(\mathbf{R}^\nu) \times \mathfrak{B}(\mathbf{R}^\nu) = \mathfrak{B}(\mathbf{R}^\nu \times \mathbf{R}^\nu)$.

Now let $\gamma_s \equiv \sigma_s * g_s$, $0 \le s \le t_0$. Given $B \in \mathfrak{B}(\mathbf{R}^\nu)$,

$$\gamma_s(B) = \int_{\mathbf{R}^\nu \times \mathbf{R}^\nu} \chi_B(\vec{U}+\vec{V})\, d(\sigma_s \times g_s)(\vec{U},\vec{V}).$$

We can now apply Lemma 3.2 to see that $\gamma_s(B)$ is a measurable function of s. In applying Lemma 3.2, let $(Y, \mathcal{Y}, \gamma) = ([0,t_0], \mathfrak{B}([0,t_0])$, Lebesgue measure) and let $(Z, \mathcal{Z}) = (\mathbf{R}^\nu \times \mathbf{R}^\nu, \mathfrak{B}(\mathbf{R}^\nu \times \mathbf{R}^\nu))$. Associate with every $s \in [0,t_0]$ the measure $\sigma_s \times g_s$ on Z. Let $\psi(s,\vec{U},\vec{V}) \equiv \chi_B(\vec{U}+\vec{V})$. It now follows from Lemma 3.2 and the formula for $\gamma_s(B)$ above, that $\gamma_s(B)$ is a measurable function of s. Of course $\|\gamma_s\| \le \|\sigma_s\|\,\|g_s\|$ and so $\|\gamma_s\| \in L_1[0,t_0]$.

Now note that $\theta(s,\vec{U})G(s,\vec{U}) = \int_{\mathbf{R}^\nu}\exp(i\langle\vec{U},\vec{V}\rangle)\,d\gamma_s(\vec{V})$. We substitute this expression into (8.1) and then use the Fubini Theorem, Lemmas 6.2 and 6.3, and the Dominated Convergence Theorem to obtain

$$G^*(t,\vec{\xi}) = \int_0^t [q/2\pi i(t-s)]^{\nu/2} \int_{\mathbf{R}^\nu}^{-} \int_{\mathbf{R}^\nu} \exp(i\langle\vec{U},\vec{V}\rangle)\,d\gamma_s(\vec{V}) \times\exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)}\right\} d\vec{U}\,ds$$

$$= \int_0^t \lim_{A\to\infty} [q/2\pi i(t-s)]^{\nu/2} \int_{\mathbf{R}^\nu} \int_{\mathbf{R}^\nu} \exp(i\langle\vec{U},\vec{V}\rangle)\,d\gamma_s(\vec{V}) \times\exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)} - \frac{\|\vec{U}\|^2}{2A}\right\} d\vec{U}\,ds$$

$$= \int_0^t \lim_{A\to\infty} [q/2\pi i(t-s)]^{\nu/2} \times\int_{\mathbf{R}^\nu}\int_{\mathbf{R}^\nu} \exp\left\{i\langle\vec{U},\vec{V}\rangle + \frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)} - \frac{\|\vec{U}\|^2}{2A}\right\} d\vec{U}\,d\gamma_s(\vec{V})\,ds$$

$$= \int_0^t \lim_{A\to\infty} \int_{\mathbf{R}^\nu} \left[\frac{-iqA}{(t-s)-iqA}\right]^{\nu/2} \times\exp\left\{\frac{iqA\langle\vec{\xi},\vec{V}\rangle}{i(t-s)+qA} - \frac{A(t-s)\|\vec{V}\|^2}{2(t-s)-2iqA} + \frac{iq\|\vec{\xi}\|^2}{2(t-s)-2iqA}\right\} d\gamma_s(\vec{V})\,ds$$

$$= \int_0^t \int_{\mathbf{R}^\nu} \exp\left\{i\langle\vec{\xi},\vec{V}\rangle + \frac{(t-s)\|\vec{V}\|^2}{2qi}\right\} d\gamma_s(\vec{V})\,ds.$$

Here the existence of each of the members of the above continued equation follows from the existence of the last member.

Now applying Theorem 3.1, the family $\{\gamma_s\colon 0\le s\le t_0\}$ can be combined with Lebesgue measure on $[0,t_0]$ to produce a measure μ on $[0,t_0]\times\mathbf{R}^\nu$:

$$\mu(E) = \int_0^{t_0}\gamma_s(E^{(s)})\,ds.$$

For each $t \in [0, t_0]$, the following formula clearly defines another measure on $[0, t_0] \times \mathbf{R}^\nu$:

$$d\lambda_t(s, \vec{V}) = \chi_{[0,t]}(s)\exp\left\{\frac{(t-s)\|\vec{V}\|^2}{2qi}\right\} d\mu(s, \vec{V}).$$

Let $g_t^*(B) \equiv \lambda_t([0, t_0] \times B)$ for $B \in \mathfrak{B}(\mathbf{R}^\nu)$. Of course $g_t^* \in M(\mathbf{R}^\nu)$ and it is easy to check that

$$\|g_t^*\| \leq \|\lambda_t\| \leq \int_0^t \|\gamma_s\| ds \leq \int_0^t \|\sigma_s\| \, \|g_s\| ds \leq M_0 \int_0^t \|\sigma_s\| ds. \tag{8.4}$$

Now for $B \in \mathfrak{B}(\mathbf{R}^\nu)$,

$$g_t^*(B) = \int_{[0,\, t_0]\times \mathbf{R}^\nu} \chi_{[0,t]}(s)\chi_B(\vec{V})\exp\left\{\frac{(t-s)\|\vec{V}\|^2}{2qi}\right\} d\mu(s, \vec{V}),$$

and since the integrand is a Borel measurable function of $(t, s, \vec{V})$, it follows from the Fubini Theorem that $g_t^*(B)$ is a Borel measurable function of t.

Now we have shown above that

$$G^*(t, \vec{\xi}) = \int_0^t \int_{\mathbf{R}^\nu} \exp\left\{i\langle \vec{\xi}, \vec{V}\rangle + \frac{(t-s)\|\vec{V}\|^2}{2qi}\right\} d\gamma_s(\vec{V})\, ds.$$

We will finish the proof by showing that this last expression equals $\int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V}\rangle) dg_t^*(\vec{V})$. Using the Change of Variable Theorem and Theorem 3.1 to justify respectively the first and third equalities below, we can write

$$\begin{aligned}
\int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V}\rangle)\, dg_t^*(\vec{V}) &= \int_{[0,\, t_0]\times \mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V}\rangle)\, d\lambda_t(s, V) \\
&= \int_{[0,\, t_0]\times \mathbf{R}^\nu} \chi_{[0,t]}(s)\exp\left\{i\langle \vec{\xi}, \vec{V}\rangle + \frac{(t-s)\|\vec{V}\|^2}{2qi}\right\} d\mu(s, \vec{V}) \\
&= \int_0^{t_0} \int_{\mathbf{R}^\nu} \chi_{[0,t]}(s)\exp\left\{i\langle \vec{\xi}, \vec{V}\rangle + \frac{(t-s)\|\vec{V}\|^2}{2qi}\right\} d\gamma_s(\vec{V})\, ds \\
&= \int_0^t \int_{\mathbf{R}^\nu} \exp\left\{i\langle \vec{\xi}, \vec{V}\rangle + \frac{(t-s)\|\vec{V}\|^2}{2qi}\right\} d\gamma_s(\vec{V})\, ds.
\end{aligned}$$

LEMMA 8.2. *Let G and θ be as in Lemma* 8.1. *Further assume that*

$$(8.5)\qquad G(t,\vec{\xi}) = \int_0^t [q/2\pi i(t-s)]^{\nu/2} \times \int_{\mathbf{R}^\nu}^{-} \theta(s,\vec{U})G(s,\vec{U})\exp\left\{\frac{iq\|\vec{U}-\vec{\xi}\|^2}{2(t-s)}\right\} d\vec{U}\,ds$$

for all $(t,\vec{\xi}) \in [0,t_0]\times\mathbf{R}^\nu$. Then $G(t,\vec{\xi}) \equiv 0$ on $[0,t_0]\times\mathbf{R}^\nu$.

Proof. Let $E = \int_0^{t_0} \|\sigma_s\|\,ds$. It will suffice to show that

$$\|g_t\| \le \frac{M_0E^n}{n!}$$

for $n = 0,1,2,\ldots,$ and all $t \in [0,t_0]$. For then $\|g_t\|$ will be zero identically on $[0,t_0]$ and hence $G(t,\vec{\xi})$ will vanish identically on $[0,t_0]\times\mathbf{R}^\nu$.

We first apply Lemma 8.1 to see that $g_t \equiv g_t^*$ and so for all $t\in[0,t_0]$, using (8.4), we have that

$$(8.6)\qquad \|g_t\| \le \int_0^t \|\sigma_s\|\,\|g_s\|\,ds \le M_0\int_0^t \|\sigma_s\|\,ds \le M_0\int_0^{t_0}\|\sigma_s\|\,ds = M_0E.$$

Now using (8.6) and substituting into (8.4) we see that for all $t\in[0,t_0]$,

$$\|g_t\| \le \int_0^t \|\sigma_{s_1}\|\,\|g_{s_1}\|\,ds_1 \le \int_0^t \|\sigma_{s_1}\| M_0\int_0^{s_1}\|\sigma_{s_2}\|\,ds_2\,ds_1$$
$$= M_0\int_0^t\int_0^{s_1}\|\sigma_{s_1}\|\,\|\sigma_{s_2}\|\,ds_2\,ds_1 = \frac{M_0}{2!}\left[\int_0^t\|\sigma_s\|\,ds\right]^2 \le \frac{M_0E^2}{2!}.$$

Continuing inductively we see that for any integer $n>2$,

$$\|g_t\| \le \int_0^t \|\sigma_{s_1}\|\,\|g_{s_1}\|\,ds_1$$
$$\le \int_0^t \|\sigma_{s_1}\| M_0\int_0^{s_1}\int_0^{s_2}\cdots\int_0^{s_n}\prod_{j=2}^n \|\sigma_{s_j}\|ds_n\,ds_{n-1}\cdots ds_1$$
$$= \frac{M_0}{n!}\left[\int_0^t\|\sigma_s\|ds\right]^n \le \frac{M_0E^n}{n!}.$$

THEOREM 8.1. *Under the hypotheses of Theorem* 5.1, $\Gamma(t,\vec{\xi})$ *is the unique solution of equation* (7.1) *in* $\mathcal{G}_0$.

Proof. What we need to establish is that Γ is indeed in $\mathcal{G}_0$. For then it follows that Γ is the only solution of (7.1) in $\mathcal{G}_0$ because if there were another solution, say Γ_1, their difference would satisfy equation (8.5) and would thus vanish.

By Theorem 7.1 we know that Γ exists and satisfies (7.1). In addition recall that $\Gamma(t, \vec{\xi}) = \sum_{n=0}^{\infty} \Gamma^{(n)}(t, \vec{\xi})$. We will first show that each $\Gamma^{(n)}$ is in $\mathcal{G}_0$. Clearly $\Gamma^{(0)}$ belongs to $\mathcal{G}_0$ because

$$\Gamma^{(0)}(t, \vec{\xi}) = \int_{\mathbf{R}^\nu} \exp\left\{\frac{t\|\vec{V}\|^2}{2qi} + \langle \vec{\xi}, \vec{V}\rangle\right\} d\phi(\vec{V})$$

$$= \int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V}\rangle)\, dg_t^{(0)}(\vec{V})$$

where $g_t^{(0)}$ is that element of $M(\mathbf{R}^\nu)$ such that for each $B \in \mathcal{B}(\mathbf{R}^\nu)$

$$g_t^{(0)}(B) \equiv \int_B \exp\left\{\frac{t\|\vec{V}\|^2}{2qi}\right\} d\phi(\vec{V}).$$

Clearly $g_t^{(0)}(B)$ is measurable as a function of t. Also $\|g_t^{(0)}\| \leq \|\phi\|$. Next using Corollary 7.1 and Lemma 8.1 we obtain that

$$\Gamma^{(1)}(t, \vec{\xi}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V}\rangle)\, dg_t^{(1)}(\vec{V})$$

where $g_t^{(1)}$ is in $M(\mathbf{R}^\nu)$ and satisfies

$$\|g_t^{(1)}\| \leq \int_0^t \|\sigma_s\|\, \|g_s^{(0)}\| ds \leq \|\phi\| \int_0^t \|\sigma_s\| ds.$$

Thus $\Gamma^{(1)}$ belongs to $\mathcal{G}_0$. Continuing on inductively we see that

$$\Gamma^{(n)}(t, \vec{\xi}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V}\rangle)\, dg_t^{(n)}(\vec{V})$$

where $g_t^{(n)}$ is in $M(\mathbf{R}^\nu)$ and satisfies the inequality

$$\begin{aligned}\|g_t^{(n)}\| &\leq \int_0^t \|\sigma_s\|\, \|g_s^{(n-1)}\| ds \\ &\leq \|\phi\| \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \prod_{j=1}^{n} \|\sigma_{s_j}\| ds_n \cdots ds_1 \\ &= \frac{\|\phi\|}{n!} \left[\int_0^t \|\sigma_s\|\, ds\right]^n.\end{aligned}$$

Thus $\Gamma^{(n)}$ is in $\mathcal{G}_0$ for $n = 0, 1, 2, \ldots$.

Next let

$$\eta_t \equiv \sum_{n=0}^{\infty} g_t^{(n)}$$

for $t \in [0, t_0]$. Note that for $t \in [0, t_0]$,

$$\begin{aligned}\|\eta_t\| &\leq \sum_{n=0}^{\infty} \|g_t^{(n)}\| \leq \|\phi\| \sum_{n=0}^{\infty} \frac{\left[\int_0^t \|\sigma_s\| ds\right]^n}{n!} \\ &= \|\phi\| \exp\left[\int_0^t \|\sigma_s\| ds\right] \leq \|\phi\| \exp\left[\int_0^{t_0} \|\sigma_s\| ds\right].\end{aligned}$$

In addition it now follows that for every $B \in \mathfrak{B}(\mathbf{R}^\nu)$, $\eta_t(B) = \sum_{n=0}^{\infty} g_t^{(n)}(B)$, and so $\eta_t(B)$ is measurable as a function of t. Finally

$$\Gamma(t, \vec{\xi}) = \int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V} \rangle)\, d\eta_t(\vec{V})$$

on $[0, t_0] \times \mathbf{R}^\nu$ since

$$\begin{aligned}&\left| \int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V} \rangle)\, d\eta_t(\vec{V}) - \sum_{n=0}^{N} \Gamma^{(n)}(t, \vec{\xi}) \right| \\ &\quad \leq \left| \int_{\mathbf{R}^\nu} \exp(i\langle \vec{\xi}, \vec{V} \rangle)\, d\left[\sum_{n=N+1}^{\infty} g_t^{(n)}(\vec{V}) \right] \right| \\ &\quad \leq \sum_{n=N+1}^{\infty} \|g_t^{(n)}\| \leq \|\phi\| \sum_{n=N+1}^{\infty} \frac{\left[\int_0^{t_0} \|\sigma_s\|\, ds\right]^n}{n!}\end{aligned}$$

which goes to zero as $N \to \infty$. Hence Γ is in $\mathcal{G}_0$.

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