PACIFIC JOURNAL OF MATHEMATICS
Vol. 119, No. 2, 1985

NON-COMPACT SETS WITH CONVEX SECTIONS

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Two further generalizations of Ky Fan's generalizations of his well-known intersection theorem concerning sets with convex sections are obtained.

1. Introduction. Let I be an index set; in the case when I is finite, it is always assumed that I contains at least two indices. Let $\{X_i\}_{i\in I}$ be a family of topological spaces and $X := \prod_{i\in I} X_i$. For each $i \in I$, set

$$X^i := \prod_{\substack{j\neq i\\ j\in I}} X_j \qquad \left(\text{so that } X = X_i \times X^i\right),$$

and let $p_i\colon X \to X_i$ and $p^i\colon X \to X^i$ be the projections. For each $x \in X$, we write $p_i(x) = x_i$ and $p^i(x) = x^i$. For any non-empty subset K of X, we let $p_i(K) = K_i$ and $p^i(K) = K^i$.

Our aim in this paper is to give two generalizations of the following intersection theorem of Ky Fan [2] concerning sets with convex sections.

THEOREM 1. (*Ky Fan.*) *Let $X_1, X_2, \ldots, X_n$ be n (≥ 2) non-empty compact convex sets each in a Hausdorff topological vector space. Let $X := \prod_{i=1}^n X_i$ and $A_1, A_2, \ldots, A_n$ be n subsets of X such that*

(a) *For each $i = 1, 2, \ldots, n$ and any $x_i \in X_i$, the section*

$$A_i(x_i) := \left\{x^i \in X^i\colon \left(x_i, x^i\right) \in A_i\right\}$$

is open in X^i.

(b) *For each $i = 1, 2, \ldots, n$ and any $x^i \in X^i$, the section*

$$A_i(x^i) := \left\{x_i \in X_i\colon \left(x_i, x^i\right) \in A_i\right\}$$

is convex and non-empty.

Then the intersection $\bigcap_{i=1}^n A_i$ is non-empty.

Theorem 1 is a unified account of game-theoretic results for arbitrary n-person games and has several applications [2], [3]. In particular, Tychonoff's fixed point theorem [11], Sion's generalization [10] of von Neumann's minimax principle [8] and Nash's equilibrium point theorem [7] are immediate consequences of Theorem 1.

2. Infinite system. Ma [**6**] extended Theorem 1 to an arbitrary system $\{X_i\}_{i\in I}$ of compact convex sets. In a recent paper, Ky Fan [**5**] extends Ma's result by introducing an auxiliary family $\{B_i\}_{i\in I}$. Ky Fan's theorem can be further generalized to non-compact convex sets as follows:

THEOREM 2. *Let $\{E_i\}_{i\in I}$ be a family of Hausdorff topological vector spaces. For each $i \in I$, let X_i be a non-empty convex set in E_i. Let $X := \prod_{i\in I} X_i$. Suppose $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ are two families of subsets of X satisfying the following conditions*:

(a) *For each $i \in I$ and any $x_i \in X_i$, the section*

$$A_i(x_i) := \{x^i \in X^i : (x_i, x^i) \in A_i\}$$

is open in X^i.

(b) *For each $i \in I$ and any $x^i \in X^i$, the section*

$$B_i(x^i) := \{x_i \in X_i : (x_i, x^i) \in B_i\}$$

contains the convex hull of the section

$$A_i(x^i) := \{x_i \in X_i : (x_i, x^i) \in A_i\}.$$

(c) *There exists a non-empty compact convex subset K of X such that*

(c$'$) *for each $i \in I$ and any $x^i \in K^i$, the section*

$$A_i(x^i) := \{x_i \in X_i : (x_i, x^i) \in A_i\} \neq \varnothing \text{ and}$$

(c$''$) $K \cap \prod_{i\in I} A_i(y^i) \neq \varnothing$ *for each $y \in X \setminus K$.*

Then the intersection $\bigcap_{i\in I} B_i$ is non-empty.

Proof. Let $i \in I$. For any $x^i \in K^i$, we can find $x_i \in X_i$ such that $x_i \in A_i(x^i)$ by (c$'$), so that $x^i \in A_i(x_i)$; thus $K^i \subset \bigcup_{x_i\in X_i} A_i(x_i)$. Since each $A_i(x_i)$ is open in X^i by (a), by the compactness of K^i (since each projection p^i is continuous), there is a finite subet $\{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$ of X_i such that

$$K^i \subset \bigcup_{k=1}^{n_i} A_i(x_{ik}). \tag{1}$$

Let Ω_i be the convex hull of $K_i \cup \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$. Define $\Omega := \prod_{i\in I} \Omega_i$ and $\tilde{A}_i := A_i \cap \Omega$ and $\tilde{B}_i := B_i \cap \Omega$ for each $i \in I$. Since the projection p_i is continuous and affine, K_i is compact convex for each $i \in I$; it follows that Ω_i is a nonempty compact convex set in E_i for each $i \in I$. Furthermore, we have:

(i) For each $i \in I$ and any $x_i \in \Omega_i$, the section

$$\tilde{A}_i(x_i) := \{x^i \in \Omega^i : (x_i, x^i) \in \tilde{A}_i\}$$

is open in Ω^i by (a).

(ii) For each $i \in I$ and any $x^i \in \Omega^i$, the section

$$\tilde{B}_i(x^i) := \{x_i \in \Omega_i : (x_i, x^i) \in \tilde{B}_i\}$$

contains the convex hull of the section

$$\tilde{A}_i(x^i) := \{x_i \in \Omega_i : (x_i, x^i) \in \tilde{A}_i\}$$

by (b).

(iii) For each $i \in I$ and any $x^i \in \Omega^i$, the section

$$\tilde{A}_i(x^i) := \{x_i \in \Omega_i : (x_i, x^i) \in \tilde{A}_i\} \neq \varnothing .$$

by (c′), (c″) and (1).

For each $i \in I$ and any $x^i \in \Omega^i$, we can find $x_i \in \Omega_i$ such that $x_i \in \tilde{A}_i(x^i)$ by (iii), so that $x^i \in \tilde{A}_i(x_i)$, it follows that $\Omega^i = \bigcup_{x_i \in \Omega_i} \tilde{A}_i(x_i)$; since each $\tilde{A}_i(x_i)$ is open in Ω^i by (i), by compactness of Ω^i, there is a finite subset $\{y_{i1}, y_{i2}, \ldots, y_{im_i}\}$ of Ω_i such that

$$\Omega^i = \bigcup_{k=1}^{m_i} \tilde{A}_i(y_{ik}).$$

Let $f_{i1}, f_{i2}, \ldots, f_{im_i}$ be a continuous partition of unity subordinated to the covering $\{\tilde{A}_i(y_{i1}), \tilde{A}_i(y_{i2}), \ldots, \tilde{A}_i(y_{im_i})\}$ of Ω^i. Then

$$\begin{cases} f_{ik}(x^i) = 0 & \text{for } x^i \in \Omega^i \setminus \tilde{A}_i(y_{ik}),\ k = 1, 2, \ldots, m_i, \\ \sum_{k=1}^{m_i} f_{ik}(x^i) = 1 & \text{for each } x^i \in \Omega^i. \end{cases}$$

Define a continuous map $\phi_i : \Omega^i \to \Omega_i$ by setting

$$\phi_i(x^i) = \sum_{k=1}^{m_i} f_{ik}(x^i) y_{ik} \quad \text{for } x^i \in \Omega^i.$$

Since $f_{ik}(x^i) \neq 0$ implies $x^i \in \tilde{A}_i(y_{ik})$, i.e. $y_{ik} \in \tilde{A}_i(x^i)$, and since $\tilde{B}_i(x^i)$ contains the convex hull of $\tilde{A}_i(x^i)$ by (ii), we have

$$\phi_i(x^i) \in \tilde{B}_i(x^i) \quad \text{for each } x^i \in \Omega^i. \tag{2}$$

Let C_i be the convex hull of $\{y_{i1}, y_{i2}, \ldots, y_{im_i}\}$; then $C_i \subset \Omega_i$. Denote by F_i the vector subspace of E_i generated by C_i; then F_i is locally convex since it is finite dimensional.

Now let $C = \prod_{i \in I} C_i$, then C is a non-empty compact convex subset in the Hausdorff locally convex space $\prod_{i \in I} F_i$. Note that for each $i \in I$, we have $C^i \subset \Omega^i$. Define $\psi : C \to C$ as follows: For each $x \in C$ and each $i \in I$, write $x = (x_i, x^i) \in C_i \times C^i$, then $\psi(x) := \{y_i\}_{i \in I}$ is determined by $y_i := \phi_i(x^i)$ for each $i \in I$. Clearly ψ is continuous. By Tychonoff's fixed point theorem [**11**], ψ has a fixed point $z := \{z_i\}_{i\in I}$ in C, so that for each $i \in I$, we have $z_i = \phi_i(z^i) \in \tilde{B}_i(z^i)$, by (2); it follows that $z = (z_i, z^i) \in \tilde{B}_i \subset B_i$ for each $i \in I$. Hence $z \in \bigcap_{i\in I} B_i$. This concludes the proof of our theorem. □

Similar to [**2**], Theorem 2 has the following analytic formulation:

THEOREM 3. *Let $\{E_i\}_{i\in I}$ be a family of Hausdorff topological vector spaces. For each $i \in I$, let X_i be a non-empty convex set in E_i. Let $X := \prod_{i\in I} X_i$ and $\{t_i\}_{i\in I}$ be a family of real numbers. Suppose that $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are two families of real-valued functions defined on X, satisfying the following conditions*:

(a) *For each $i \in I$ and any $x_i \in X_i$, $f_i(x_i, x^i)$ is a lower semi-continuous function of $x^i \in X^i$.*

(b) *For each $i \in I$ and any $x^i \in X^i$, the set*

$$\{x_i \in X_i: g_i(x_i, x^i) > t_i\}$$

contains the convex hull of the set

$$\{x_i \in X_i: f_i(x_i, x^i) > t_i\}.$$

(c) *There exists a non-empty compact convex subset K of X such that*

(c′) *for each $i \in I$ and any $x^i \in K^i$, there exists $x_i \in X_i$ with $f_i(x_i, x^i) > t_i$ and*

(c″) *for any $y \in X \setminus K$, there exists $x \in K$ with $f_i(x_i, y^i) > t_i$ for all $i \in I$.*

Then there exists a point $\hat{y} \in X$ such that $g_i(\hat{y}) > t_i$ for all $i \in I$.

3. Finite system. By relaxing the compactness condition for X_i's and the convexity conndition for the sections of the A_i's in Theorem 1, Ky Fan [**5**] generalizes Theorem 1 as follows:

THEOREM 4. (*Ky Fan*) *Let $X_1, X_2, \ldots, X_n$ be n (≥ 2) convex sets each in a Hausdorff topological vector space. Let $X := \prod_{i=1}^n X_i$ and $A_1, A_2, \ldots, A_n$ be n subsets of X such that*

(a) *For each $i = 1, 2, \ldots, n$ and any $x_i \in X_i$, the section*

$$A_i(x_i) := \{x^i \in X^i: (x_i, x^i) \in A_i\}$$

is open in X^i,

(b) *For each $i = 1, 2, \ldots, n$ and any $x^i \in X^i$, the section*

$$A_i(x^i) := \{x_i \in X_i: (x_i, x^i) \in A_i\}$$

is non-empty.

(c) *For any* $x \in X$, *at least* q *of the sections* $A_1(x^1)$, $A_2(x^2), \ldots, A_n(x^n)$ *are convex*; *where* q *is a given integer with* $2 \leq q \leq n$.

(d) *There exists a non-empty compact convex subset* K *of* X *such that*

$$K \cap \prod_{i=1}^{n} A_i(y^i) \neq \varnothing \quad \text{for each } y \in X \setminus K.$$

Then at least q *of the sets* $A_1, A_2, \ldots, A_n$ *have a non-empty intersection.*

Theorem 4 can be improved as follows:

THEOREM 5. *Let* $X_1, X_2, \ldots, X_n$ *be* n (≥ 2) *convex sets each in a Hausdorff topological vector space. Let* $X := \prod_{i=1}^{n} X_i$ *and* $A_1, A_2, \ldots, A_n$, $B_1, B_2, \ldots, B_n$ *be* $2n$ *subsets of* X *such that*

(a) $A_i \subset B_i$ *for* $i = 1, 2, \ldots, n$.

(b) *For each* $i = 1, 2, \ldots, n$ *and any* $x_i \in X_i$, *the section*

$$A_i(x_i) := \{x^i \in X^i : (x_i, x^i) \in A_i\}$$

is open in X^i.

(c) *For any* $x \in X$, *at least* q *of the sections* $B_1(x^1)$, $B_2(x^2), \ldots, B_n(x^n)$ *are convex*; *where* q *is a given integer with* $2 \leq q \leq n$.

(d) *There exists a non-empty compact convex subset* K *of* X *such that*

(d′) *For each* $i = 1, 2, \ldots, n$ *and for each* $x \in K$, *the section*

$$A_i(x^i) := \{x_i \in X_i : (x_i, x^i) \in A_i\}$$

is non-empty and

(d″) $K \cap \prod_{i=1}^{n} A_i(y^i) \neq \varnothing$ *for each* $y \in X \setminus K$.

Then at least q *of the sets* $B_1, B_2, \ldots, B_n$ *have a non-empty intersection.*

For $n = 2$, Theorem 5 was given in **[9]** together with an application to von Neumann type minimax inequalities. The proof of Theorem 5 is a slight modification of that in Ky Fan **[5]**, hence we need the following further generalization of the *KKM* mapping principle due to Ky Fan **[5]**:

THEOREM 6. (*Ky Fan*) *Let* Y *be a convex set in a Hausdorff topological vector space and let* X *be a non-empty subset of* Y. *For each* $x \in X$, *let* $F(x)$ *be a relatively closed subset of* Y *such that the convex hull of every finite subset* $\{x_1, x_2, \ldots, x_n\}$ *of* X *is contained in the corresponding union* $\bigcup_{i=1}^{n} F(x_i)$. *If there is a non-empty subset* X_0 *of* X *such that the intersection* $\bigcap_{x \in X_0} F(x)$ *is compact and* X_0 *is contained in a compact convex subset of* Y, *then* $\bigcap_{x \in X} F(x) \neq \varnothing$.

Proof of Theorem 5. For each $x \in X$, let

$$F(x) := \{ y \in X : (x_i, y^i) \notin A_i \text{ for at least one index } i \},$$

then $F(x)$ is relative closed in X by (b). By (d′), for each $y \in K$, for each $i = 1, 2, \ldots, n$, there exists $x_i \in A_i(y^i)$, so that by setting $x = (x_1, x_2, \ldots, x_n) \in X$, we have $y \notin F(x)$ and it follows that $K \cap \bigcap_{x \in X} F(x) = \varnothing$. On the other hand, by (d″), for each $y \in X \setminus K$, there exists $x \in K$ such that $(x_i, y^i) \in A_i$ for all $i = 1, 2, \ldots, n$, so that $y \notin F(x)$; it follows that $(X \setminus K) \cap \bigcap_{x \in K} F(x) = \varnothing$. Hence $\bigcap_{x \in X} F(x) = \varnothing$ and $\bigcap_{x \in K} F(x)$ is compact, being a closed subset of the compact set K.

According to Theorem 6, there exist $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in X$, and non-negative real numbers $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$ with $\sum_{k=1}^m \alpha^{(k)} = 1$ such that $\sum_{k=1}^m \alpha^{(k)} x^{(k)} \notin \bigcup_{k=1}^m F(x^{(k)})$. Let $z := \sum_{k=1}^m \alpha^{(k)} x^{(k)} := (z_i, z^i) \in X_i \times X^i$ and let $p_i(x^{(k)}) = x_i^{(k)}$. Then $(x_i^{(k)}, z^i) \in A_i$ for all $1 \le i \le n$ and $1 \le k \le m$, or $x_i^{(k)} \in A_i(z^i)$ for all $1 \le i \le n$ and $1 \le k \le m$. By (a), we have

$$x_i^{(k)} \in B_i(z^i) \quad \text{for all } 1 \le i \le n \text{ and } 1 \le k \le m. \tag{3}$$

By (c), at least q of the sections $B_1(z^1), B_2(z^2), \ldots, B_n(z_n)$ are convex. Since $z_i = \sum_{k=1}^m \alpha^{(k)} x_i^{(k)}$ for $i = 1, 2, \ldots, n$, (3) implies that $z_i \in B_i(z^i)$ holds for at least q indices i. Thus z is a point common to at least q of the sets $B_1, B_2, \ldots, B_n$. This completes the proof. □

The following is an analytic formulation of Theorem 5:

THEOREM 7. *Let $X_1, X_2, \ldots, X_n$ be n (≥ 2) convex sets each in a Hausdorff topological vector space. Let $X := \prod_{i=1}^n X_i$ and $\{t_i\}_{i=1}^n$ be a set of n real numbers. Let $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^n$ be $2n$ real-valued functions defined on X satisfying the following conditions*:

(a) *$f_i \le g_i$ on X for each $i = 1, 2, \ldots, n$.*

(b) *For each $i = 1, 2, \ldots, n$ and any $x_i \in X_i$, $f_i(x_i, x^i)$ is a lower semi-continuous function of $x^i \in X^i$.*

(c) *For any $x \in X$, at least q of the functions $g_i(y_i, x^i)$ are quasi-concave functions of $y_i \in X_i$.*

(d) *There exists a non-empty compact convex subset K of X such that*

(d′) *For each $i = 1, 2, \ldots, n$ and any $x^i \in K^i$, there exists $x_i \in X_i$ such that $f_i(x_i, x^i) > t_i$ and*

(d″) *for each $y \in X \setminus K$, there exists $x \in K$ such that $f_i(x_i, y^i) > t_i$ for all $i = 1, 2, \ldots, n$.*

Then there exists a point $\hat{y} \in X$ such that $g_i(\hat{y}) > t_i$ for at least q indices i in $\{1, 2, \ldots, n\}$.

Acknowledgment. The writing of this paper is inspired by Ky Fan's paper [**5**]. We would like to thank Professor Ky Fan for providing us with a preprint of his paper [**5**].

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Received January 12, 1984. This work was partially supported by NSERC of Canada under grant A-8096.

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