PACIFIC JOURNAL OF MATHEMATICS
Vol. 124, No. 1, 1986

RELATIVE NULLITY FOLIATIONS AND INDEFINITE ISOMETRIC IMMERSIONS

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This paper investigates the relative nullity distribution of an indefinite Riemannian manifold isometrically immersed into an indefinite space form.

Introduction. In this paper we investigate the relative nullity distribution of an indefinite Riemannian manifold isometrically immersed into an indefinite space form. This distribution is totally geodesic and gives rise to a Ricatti-type differential equation along a geodesic in a leaf of the distribution.

This differential equation is applied in several ways to estimate the index of relative nullity ν for geodesically complete, connected, Lorentzian submanifolds M_1^n of $\tilde{M}_1^{n+1}(c)$, the Lorentzian sphere. These applications extend the work of Abe [**1**], [**2**], [**3**], Ferus [**7**], [**8**], and others to the setting of indefinite manifolds. Some of the work in §2 was obtained previously by Graves [**10**] in the codimension one case and by M. Dajczer. In particular Theorem 2 was conjectured by Dajczer [**5**].

Sections 1 and 2 lay the groundwork and derive the Ricatti-type differential equation. In §3 an integer ν_n is defined and it is shown that if M_1^n is as above and if $\nu > \nu_n$ then M_1^n is totally geodesic. This integer is used to formulate a geometric condition which guarantees that a complete connected hypersurface of $S_1^n(c)$ is totally geodesic. We also estimate ν given a natural condition on the space-like Ricci curvature of the submanifold. In [**6**] other conditions on Ricci curvature are given.

The general scheme of our investigation is very similar to that of the Riemannian case as formulated in the papers mentioned above. However, there are a few basic and non-trivial differences from the Riemannian case. These differences are due to the indefinite metric and are to be overcome. Therefore, we think it worthwhile to include the details of the proofs for most of our results.

1. Preliminaries. An indefinite Riemannian manifold M_s^n of dimension n is a connected manifold with a non-degenerate metric in each tangent space. The metric can be written

$$ds^2 = -(dx^1)^2 - \cdots - (dx^s)^2 + (dx^{s+1})^2 + \cdots + (dx^n)^2$$

at every point of M^n. In this case, we say the signature of M^n is $(s, n-s)$ and write M_s^n.

If $f\colon M_s^n \to \tilde{M}_t^{n+k}$ is an immersion and the metric induced on M^n is non-degenerate, f is an isometric immersion of M_s^n with this metric into $\tilde{M}_t^{n+k}$. Denote by D the torsion-free metric connection on $\tilde{M}_t^{n+k}$. D induces a torsion-free, metric connection ∇ on M as follows:

$$D_X Y = f_*(\nabla_X Y) + \alpha(X, Y), \tag{1.1}$$

where X and Y are tangent vectors on M, $f_*(\nabla_X Y)$ is the tangential component and $\alpha(X, Y)$ is the normal component. α is called the second fundamental form of f.

Given a field of unit normal vectors ξ on $f(M_s^n)$ we can define a field of endomorphisms A_ξ on M by

$$D_X \xi = -f_*(A_\xi X) + \nabla_X^\perp \xi, \tag{1.2}$$

where $-f_*(A_\xi X)$ is the tangential component. A_ξ is called the shape operator associated to ξ.

We denote by $N(x)$ the set of all normal vectors to $f(M_s^n)$ at $f(x)$. The metric on $\tilde{M}_t^{n+k}$ and M_s^n is denoted by $\langle\ ,\ \rangle$. As usual, $R(\ ,\ )$ denotes the curvature tensor of M_s^n. For the sake of future use, we list the following:

$$\begin{aligned} R(X,Y)Z &= \tilde{R}(X,Y)Z + \sum_{p=1}^{k} \langle \xi_p, \xi_p \rangle \left[(A_{\xi_p} X) \wedge (A_{\xi_p} Y) \right] Z \\ &= \tilde{R}(X,Y)Z + A_{\alpha(Y,Z)} X - A_{\alpha(X,Z)} Y \\ &\qquad \text{(Gauss equation)}; \end{aligned} \tag{1.3}$$

$$S(X,Y) = \sum_{i=1}^{n} \langle X_i, X_i \rangle \langle R(X_i, X) Y, X_i \rangle \quad \text{(Ricci tensor)}; \tag{1.4}$$

$\tilde{R}(\ ,\ )$ is the curvature tensor of $\tilde{M}_t^{n+k}$; ξ_p's form an orthonormal base for $N(x)$; and X_i's form an orthonormal base for $T_x M$, the tangent space of M_s^n at x.

2. Relative nullity. If $f\colon M_s^n \to \tilde{M}_t^{n+k}$ is an isometric immersion between indefinite Riemannian manifolds we define the relative nullity space at x, $T^0(x)$, to be

$$T^0(x) = \{ X \in T_x(M) \colon A_\xi X = 0 \ \forall \xi \in N(x) \}. \tag{2.1}$$

The orthogonal complement $[T^0(x)]^\perp$ of $T^0(x)$ in T_xM is denoted by $T^1(x)$.

PROPOSITION 1. $T^0(x) = \{X \in T_x(M): \alpha(X, Y) = 0 \,\forall Y \in T_x(M)\}$.

Proof. It is obvious.

PROPOSITION 2. $T^1(x) = \mathrm{span}\{A_\xi Y\}$ *for* $\xi \in N(x)$, $Y \in T_x(M)$.

Proof. Given any $\xi \in N(x)$, $Y \in T_x(M)$ and $X \in T^0(x)$, $\langle X, A_\xi Y\rangle = 0$, *so* $A_\xi Y \in T^1(x)$.

On the other hand, suppose $Z \in T_x(M)$ satisfies $\langle Z, A_\xi Y\rangle = 0$ for all ξ, Y as above. Then $\alpha(Y, Z) = 0$ for all Y and $Z \in T^0(x)$. This means that $[\mathrm{span}\{A_\xi Y\}]^\perp \subset T^0(x)$ so that $[\mathrm{span}\{A_\xi Y\}] \supset T^1(x)$.

The dimension $\nu(x)$ of $T^0(x)$ is called the relative nullity of the immersion at x. The minimum value of $\nu(x)$ on M is called the index of relative nullity and is denoted by ν_0.

THEOREM 1. *Assume that* $\tilde{M}$ *is a space form and let* G *denote the set of points in* M *where* $\nu(x) = \nu_0$. *Then*

(1) *G is an open subset of M*;

(2) $x \to T^0(x)$, $x \in G$ *is a differentiable and involutive distribution in* G;

(3) *the foliation* T^0 *is totally geodesic in* M; *and*

(4) *each leaf of* T^0 *is immersed as a totally geodesic submanifold of* $\tilde{M}$.

Proof. (1) Pick a point $x_0 \in G$ and a basis $T_{x_0}(M)$, $\{Y_1, \dots, Y_{\nu_0}, Y_{\nu_0+1}, \dots, Y_n\}$, such that, for some $\xi_j(x_0)$, $\{A_{\xi_j} Y_{\nu_0+j}\}$ forms a basis of $T^1(x_0)$. Extend $Y_1, \dots, Y_n$ and $\xi_1, \dots, \xi_{n-\nu_0}$ smoothly in a neighborhood of x_0. The set $\{A_{\xi_j} Y_{\nu_0+j}\}$ remains linearly independent in a neighborhood of x_0. Therefore in a neighborhood of x_0 the dimension of $T^0(x_0)$ must be less than or equal to ν_0, and so equals ν_0.

(2) It can be shown that T^0 is a smooth distribution on G by noting that $T^0 = \bigcap_{j=1}^k \ker A_{\xi_j}$ and that this intersection has constant rank on G.

(3) We use Codazzi's equation to see that T^0 is totally geodesic. Let Y, Z be vector fields in T^0. For all X in TM and normal vectors ξ we have

$$\nabla_X(A_\xi Y) - A_\xi(\nabla_X Y) - A_{\nabla_X^\perp \xi} Y = \nabla_Y(A_\xi X) - A_\xi(\nabla_Y X) - A_{\nabla_Y^\perp \xi} X.$$

This reduces to

$$-A_\xi(\nabla_X Y) = \nabla_Y(A_\xi X) - A_\xi(\nabla_Y X) - A_{\nabla_Y^\perp \xi}X.$$

Taking the inner product of both sides with Z yields $0 = \langle \nabla_Y(A_\xi X), Z\rangle$. $\langle A_\xi X, Z\rangle = 0$ can be differentiated in the direction of Y.

$$0 = Y\langle A_\xi X, Z\rangle = \langle \nabla_Y(A_\xi X), Z\rangle + \langle A_\xi X, \nabla_Y Z\rangle.$$

This gives $\langle A_\xi X, \nabla_Y Z\rangle = 0$, i.e., $\nabla_Y Z \in T^0$. This shows T^0 is totally geodesic. Examining Codazzi's equation with Y in T^0 and ξ, X as above gives

$$-A_\xi(\nabla_X Y) = \nabla_Y(A_\xi X) - A_\xi(\nabla_Y X) - A_{\nabla_Y^\perp \xi}X$$

so that

$$\nabla_Y(A_\xi X) = A_\xi(\nabla_Y X) + A_{\nabla_Y^\perp \xi}X - A_\xi(\nabla_X Y).$$

This shows that T^1 is also parallel along T^0.

For (4), notice that

$$D_Y f_*(Z) = f_*(\nabla_Y Z) + \alpha(Y, Z) = f_*(\nabla_Y Z). \qquad \square$$

Next we define a complementary distribution $T^c(x)$ in a neighborhood of $y(t)$, $t \in [a, b)$ where $y(t)$ is a geodesic in a leaf of T^0. The distribution is complementary in the sense that $T^c(x) \oplus T^0(x) = T_x(M)$.

If, at a fixed point $y(0)$, $T^0(y(0))$ is non-degenerate then $T^0(x)$ is non-degenerate for all points x near $y(0)$. Along a geodesic $y(t)$ in a leaf of T^0, $T^0(y(t))$ remains non-degenerate since $T^0(y(t))$ is parallel along the geodesic. Therefore, $T^0(x)$ is non-degenerate in a neighborhood of the geodesic. In this case set $T^c(x) = T^1(x)$.

If $T^0(y(0))$ is degenerate we use the following procedure. At $y(0)$ choose a pseudo-orthonormal basis

$$\{L_1(0), \ldots, L_r(0), E_1(0), \ldots, E_{\nu_0 - r}(0)\} \text{ of } T^0(y(0))$$

and

$$\{L_1(0), \ldots, L_r(0), F_1(0), \ldots, F_{n-\nu_0-r}(0)\} \text{ of } T^1(y(0)),$$

so that

$$\langle L_i(0), L_j(0)\rangle = 0 = \langle L_i(0), E_k(0)\rangle = \langle L_i(0), F_l(0)\rangle$$

and the $E_k(0)$ and $F_l(0)$ form an orthonormal set. Add $\{\hat{L}_1(0), \ldots, \hat{L}_r(0)\}$ so that each $\hat{L}_i(0)$ is perpendicular to $E_k(0)$ and $F_l(0)$, $\langle \hat{L}_i(0), \hat{L}_j(0)\rangle = 0$ and $\langle L_i(0), \hat{L}_j(0)\rangle = -\delta_{ij}$.

Denote the parallel extension of this basis

$$\{L_1(0),\dots,L_r(0),E_1(0),\dots,E_{\nu_0-r}(0),\\ F_1(0),\dots,F_{n-\nu_0-r}(0),\hat{L}_1(0),\dots,\hat{L}_r(0)\}$$

along $y(t)$ by $\{L_1(t),\dots,\hat{L}_r(t)\}$.

Assume, without loss of generality, that $\vec{y}(0)$ is one of

$$\{L_1(0),\dots,L_r(0),E_1(0),\dots,E_{\nu_0-r}(0)\},$$

say $E_{\nu_0-r}(0)$. Generalizing an argument in **[10]**, define $h\colon \mathbf{R}^n \to M^n$ by

$$h(t,x_1,\dots,x_r,y_1,\dots,y_{\nu_0-r-1},u_1,\dots,u_{n-\nu_0-r},v_1,\dots,v_r)\\ =\exp_{y(t)}\big(\Sigma x_jL_j(t)+\Sigma y_kE_k(t)+\Sigma u_lF_l(t)+\Sigma v_j\hat{L}_j(t)\big).$$

Since

$$\begin{aligned}(h_*)_{(t,0)}(\partial/\partial t)&=\vec{y}(t)\\ (h_*)_{(t,0)}(\partial/\partial x_j)&=L_j(t)\\ (h_*)_{(t,0)}(\partial/\partial y_k)&=E_k(t)\\ (h_*)_{(t,0)}(\partial/\partial u_l)&=F_l(t)\\ (h_*)_{(t,0)}(\partial/\partial v_j)&=\hat{L}_j(t)\end{aligned}$$

for each t, there is a neighborhood U of $(t,\vec{0})$ such that h is an imbedding on U. By shrinking, if necessary, we can find a neighborhood V of $\{(t,\vec{0})\mid t\in\mathbf{R}\}$ such that $h_*(\partial/\partial u_l)$ and $h_*(\partial/\partial v_j)$ are extensions of $F_l(t)$ and $\hat{L}_j(t)$ respectively to $h(V)$. By making V smaller and restricting t to $[a,b)$ we can assume $h_*(\partial/\partial u_l)$ and $h_*(\partial/\partial v_j)$ span a complement in a neighborhood of $y(t)$, $t\in[a,b)$.

In this neighborhood we let Q be the projection defined by the decomposition $T_x(M)=T^0(x)\oplus T^c(x)$

(2.2) $$Q\colon T_x(M)\to T^c(x).$$

For any $Y\in T^0$ and $X\in TM$ we can define

(2.3) $$C_YX=-Q(\nabla_XY).$$

C is called the conullity operator.

We need the following simple, technical lemma.

LEMMA 1. *Let C and Q be defined by* (2.2) *and* (2.3). *If Y is in T^0 and U and V are in TM then*

(1) $Q(\nabla_YU)=Q(\nabla_Y(QU))$

(2) $Q(\nabla_{U-QU}Y)=0$

(3) $\alpha(U,V)=\alpha(QU,V)$

(4) *C is a tensor.*

Proof (1) Suppose $U = U_0 + U_c$, where $U_0 \in T^0$ and $U_c \in T^c$. Then $Q(\nabla_Y U) = Q(\nabla_Y(U_0 + U_c)) = Q(\nabla_Y U_c)$, since $\nabla_Y U_0$ is in T^0. This means that $Q(\nabla_Y U) = Q(\nabla_Y(QU))$.

(2) If $U = U_0 + U_c$ then $QU = U_c$. This says that $U - QU \in T^0$, so that $Q(\nabla_{U-QU} Y) = 0$.

(3) Again $\alpha(U, V) = \alpha(U_0 + QU, V) = \alpha(QU, V)$.

(4) It is sufficient to show that $C_{\varphi Y} = \varphi C_Y U$ for $\varphi\colon M \to \mathbf{R}$.

$$\begin{aligned} C_{\varphi Y} U &= -Q(\nabla_U \varphi Y) = -Q\left[(U\varphi) Y + \varphi(\nabla_U Y)\right] \\ &= \varphi\left[-Q(\nabla_U Y)\right] = \varphi C_Y U. \end{aligned}$$ □

We now define a connection ∇' in the complementary local bundle T^c. If $U \in TM$, $V \in T^c$, then

$$\nabla'_U V = Q(\nabla_U V). \tag{2.4}$$

Using this connection we can differentiate C.

$$(\nabla'_Y C_Y) X = \nabla'_Y (C_Y X) - C_Y(\nabla'_Y X). \tag{2.5}$$

Another expression can be found for $(\nabla'_Y C_Y) X$. The first term is

$$\nabla'_Y (C_Y X) = Q(\nabla_Y (C_Y X)) = -Q(\nabla_Y (Q(\nabla_X Y))) = -Q(\nabla_Y \nabla_X Y).$$

The second term is, by Lemma (1.2),

$$-C_Y(\nabla'_Y X) = Q(\nabla_{\nabla'_Y X} Y) = Q(\nabla_{Q(\nabla_Y X)} Y) = Q(\nabla_{\nabla_Y X} Y).$$

Combining both terms gives

$$\nabla'_Y (C_Y X) - C_Y(\nabla'_Y X) = -Q\left(R(Y, X) Y + \nabla_X \nabla_Y Y - \nabla_{\nabla_X Y} Y\right).$$

If $W \in T^1$ then $\langle \nabla_Y Y, W\rangle = 0$ and so $0 = X\langle \nabla_Y Y, W\rangle = \langle \nabla_X \nabla_Y Y, W\rangle + \langle \nabla_Y Y, \nabla_X W\rangle$. Along a geodesic y_t in T^0 let $Y = \vec{y}_t$. Then $\nabla_Y Y = 0$, and along y_t we have $\langle \nabla_X \nabla_Y Y, W\rangle = 0$ and $Q(\nabla_X \nabla_Y Y) = 0$.

Next we claim that $Q(\nabla_{\nabla_X Y} Y) = C_Y(C_Y X)$. In fact $C_Y(C_Y X) = -Q(\nabla_{C_Y X} Y) = Q(\nabla_{Q(\nabla_X Y)} Y) = Q(\nabla_{\nabla_X Y} Y)$ by Lemma 1.

Finally then, if Y is an extension of the tangent vectors $\vec{y}_t$ along a geodesic in T^0 then

$$(\nabla'_Y C_Y) X = Q(R(X, Y) Y) + C_Y^2 X. \tag{2.6}$$

THEOREM 2. *If $f\colon M_s^n \to M_t^{n+k}(c)$ is an isometric immersion and M^n is complete, then the relative nullity foliation is geodesically complete.*

We first sketch a proof of Theorem 2.

Let y_t be a geodesic in a leaf L of T^0. It must be shown that y_t can be extended indefinitely in L. Since L is totally geodesic in M we know that y_t is a geodesic in M which can be extended indefinitely in M because M is geodesically complete. It must be proven to lie entirely in L. Assume that y_t is in L for t in $[a, b)$. If we can show that y_b is in G, then we can take a coordinate system $\{y^1, \ldots, y^n\}$ adapted to the foliation with origin y_b, that is, with the property that the integral manifolds of T^0 are given by $y^{\nu_0+j} = c_j$. Now all points y_t, for t less than and close to b, belong to one slice. As t approaches b, y_t approaches y_b with coordinates $(0, \ldots, 0)$, so that $c_1, \ldots, c_{n-\nu_0}$ are all zero. Thus, $y_b \in L$ and we are done.

To show that $y_b \in G$ we need the following lemma, which will be proved after the proof of the theorem.

LEMMA 2. *For any Z in $T_{y_a}(M)$ there exist $Z_t \in T_{y_t}(M)$, $a \le t < b$, such that $Z_a = Z$ and*

$$\nabla_t'(QZ_t) + C_{\vec{y}_t}(QZ_t) = 0 \quad \text{for } a \le t < b. \tag{2.7}$$

Moreover, Z_t can be extended differentiably to $t = b$. Here ∇_t' stands for $\nabla'_{\vec{y}_t}$.

The extension part of Lemma 2 will be proved using 2.6. Let X_t be a parallel vector field along y_t, $a \le t < b$, such that $X_b \in T^0(y_b)$. We will prove that $X_a \in T^0(y_a)$ so $\nu(y_a) \ge \upsilon(y_b) \ge \nu_0$ and $\nu(y_b) = \nu_0$. Take Z_t as in Lemma 2. For each point y_t, $t < b$, extend $Y_t = \vec{y}_t$, X_t and Z_t to vector fields Y, X and Z with Y in T^0.

Examine Codazzi's equation with Y, X and Z.

$$\begin{aligned} &\nabla_Y^{\perp}\alpha(Z, X) - \alpha(\nabla_Y Z, X) - \alpha(Z, \nabla_Y X) \\ &= \nabla_Z^{\perp}\alpha(Y, X) - \alpha(\nabla_Z Y, X) - \alpha(Y, \nabla_Z X) \end{aligned}$$

Along y_t $\nabla_Y X = 0$ and $\alpha(Y, X) = 0$ and $\alpha(Y, \nabla_Z X) = 0$. The equation reduces to

$$\nabla_Y^{\perp}\alpha(Z, X) - \alpha(\nabla_Y Z, X) = -\alpha(\nabla_Z Y, X) = \alpha(C_Y Z, X)$$

by Lemma (1.3). This gives

$$\begin{aligned} \nabla_Y^{\perp}\alpha(Z, X) &= \alpha(\nabla_Y Z, X) + \alpha(C_Y Z, X) \\ &= \alpha\big(Q(\nabla_Y(QZ)), X\big) + \alpha(C_Y Z, X) \\ &= \alpha\big(\nabla_t' QZ, X\big) + \alpha\big(C_{\vec{y}_t} Z, X\big) \end{aligned}$$

along y_t.

This is true for $t < b$, so by continuity it holds for $t = b$. That is to say that $\alpha(Z, X)$ is parallel along y_t, $a \le t \le b$. $\alpha(Z, X) = 0$ at $t = b$

which means $\alpha(Z, X) = 0$ at y_a. Since Z_a is arbitrary, $X_a \in T^0(y_a)$, as desired.

Proof of Lemma 2. In what follows we use the same notation that was introduced in the definition of $T^c(x)$. Along y_t write

$$QZ_t = \Sigma\varphi_j(t)\hat{L}_j(t) + \Sigma\psi_p(t)F_p(t).$$

We want to solve $Q(\nabla_t QZ_t) + C_{\vec{y}_t}(QZ_t) = 0$. Rewriting, this is

$$Q\big(\nabla_t\big(\Sigma\varphi_j(t)\hat{L}_j(t) + \Sigma\psi_p(t)F_p(t)\big)\big) + C_{\vec{y}_t}(QZ_t) = 0;$$

or

$$Q\big(\Sigma\varphi_j'(t)\hat{L}_j(t) + \Sigma\psi_p'(t)F_p(t)\big) + C_{\vec{y}_t}(QZ_t) = 0;$$

or

$$\Sigma\varphi_j'(t)\hat{L}_j(t) + \Sigma\psi_p'(t)F_p(t) + C_{\vec{y}_t}(QZ_t) = 0.$$

Since $C_{\vec{y}_t}(QZ_t)$ can be written in terms of $\hat{L}_j(t)$ and $F_p(t)$ this is a system of ordinary differential equations which can be solved for $a \le t < b$. To see that the solution can be extended: By (2.6)

$$(\nabla_Y' C_Y)X = C_Y^2 X + Q(R(X,Y)Y) \quad \text{for all } X \text{ in } TM.$$

$Q(R(X,Y)Y) = c\langle Y, Y\rangle QX$, so that along y_t, $a \le t < b$,

$$(\nabla_Y' C_Y) = C_Y^2 + c\langle Y, Y\rangle Q. \tag{2.8}$$

If Z_t satisfies (2.7), then by differentiating once more we have:

$$\nabla_t'^2(QZ_t) + \nabla_t' C_{\vec{y}_t}(QZ_t) = 0$$

or

$$\nabla_t'^2(QZ_t) + \big(\nabla_t' C_{\vec{y}_t}\big)QZ_t + C_Y\big(\nabla_t'(QZ_t)\big) = 0.$$

This yields, using (2.8),

$$\nabla_t'^2(QZ_t) + C_Y^2(QZ_t) + c\langle Y, Y\rangle QZ_t + C_Y\big(\nabla_t'(QZ_t)\big) = 0.$$

Plugging in (2.7) gives

$$\nabla_t'^2(QZ_t) + C_Y^2(QZ_t) + c\langle Y, Y\rangle QZ_t - C_Y^2(QZ_t) = 0,$$

i.e.,

$$\nabla_t'(QZ_t) - c\langle Y, Y\rangle QZ_t = 0 \quad \text{for } a \le t < b. \tag{2.9}$$

In terms of the parallel basis for T^c along y_t (2.9) can be written

$$\frac{d^2\varphi_j(t)}{dt^2} + c\langle Y, Y\rangle\varphi_j = 0, \qquad \frac{d^2\psi_p(t)}{dt^2} + c\langle Y, Y\rangle\psi_p = 0.$$

Solutions to these equations can be extended differentiably beyond b. □

To see that (2.6) is an equation of Ricattti-type write

$$C_Y\hat{L}_j(t) = \sum_{i=1}^{r} C_{ij}^L(t)\hat{L}_i(t) + \sum_{p=1}^{n-\nu_0-r} C_{pj}^F(t)F_p(t)$$

$$C_Y F_q(t) = \sum_{i=1}^{r} D_{iq}^L(t)\hat{L}_i(t) + \sum_{p=1}^{n-\nu_0-r} D_{pq}^F(t)F_p(t).$$

Then

$$\begin{aligned}(\nabla'_Y C_Y)\hat{L}_j(t) &= \nabla'_Y(C_Y\hat{L}_j(t)) - C_Y(\nabla'_Y\hat{L}_j(t))\\ &= \nabla'_Y\left[\sum_i C_{ij}^L(t)\hat{L}_i(t) + \sum_p C_{pj}^F(t)F_p(t)\right]\\ &= \sum_i \frac{dC_{ij}^L(t)}{dt}\hat{L}_i(t) + \sum_p \frac{dC_{pj}^F}{dt}F_p(t).\end{aligned}$$

Similarly,

$$(\nabla'_Y C_Y)F_q(t) = \sum_i \frac{dD_{iq}^L(t)}{dt}\hat{L}_i(t) + \sum_p \frac{dD_{pq}^F(t)}{dt}F_p(t).$$

$$Q\big(R(\hat{L}_j(t), Y)Y\big) = c\langle Y, Y\rangle \hat{L}_j(t),$$

$$Q\big(R(F_q(t), Y)Y\big) = c\langle Y, Y\rangle F_q(t).$$

Let

$$C(t) = \begin{bmatrix} C_{ij}^L(t) & D_{iq}^L(t)\\ C_{pj}^F(t) & D_{pq}^F(t)\end{bmatrix} \quad \text{and} \quad K(t) = c\langle Y, Y\rangle I_n.$$

Then, (2.6) can be written as

$$\frac{dC(t)}{dt} = C^2(t) + K(t). \tag{2.10}$$

3. Applications of the Ricatti-type differential equation. Let M_1^n be a geodesically complete Lorentzian submanifold of $\tilde{M}_1^{n+k}(c)$ $(c > 0)$, where $\tilde{M}_1^{n+k}(c)$ is the Lorentzian space form of constant curvature c. Let ν_0 be the index of relative nullity of M_1^n in $\tilde{M}_1^{n+k}(c)$.

It is well-known that for $K(t) > 0$, the equation (2.10) has no global solution with an initial condition $C(0)$ which has a real eigenvalue. This implies, in our case, that there is no global solution of (2.10) if $\langle Y, Y\rangle > 0$. If $\langle Y, Y\rangle = 0$, 0 is the only global solution under an initial condition $C(0)$

with a real eigenvalue. In particular, if $C(t)$ has a real eigenvalue for some t, Y must be 0, provided that Y is space-like.

Here let us state an elementary fact on a Lorentzian vector space.

PROPOSITION 3. *Let L^n be a Lorentzian inner product space with the inner product $\langle\ ,\ \rangle$. Let V^ν be a linear subspace of dimension ν. Then,*

(i) $V^\nu = L^\nu$ and $\langle\ \rangle|_{V^\nu}$ is non-degenerate, or

(ii) $V^\nu = E^\nu$ and $\langle\ \rangle|_{V^\nu}$ is positive definite, or

(iii) $\langle\ ,\ \rangle|_{V^\nu}$ is degenerate and $V^\nu = E^{\nu-1} + \mathrm{span}\{\xi\}$, where $\langle\xi, \xi\rangle = 0$ and $\zeta \perp E^{\nu-1}$.

(*Proof*). See Graves [**9**].

Now let $Y_1, \ldots, Y_{\nu_0-1}$ be a set of $(\nu_0 - 1)$, linearly independent space-like vectors in $T^0(x)$ such that $\mathrm{span}\{Y_1, \ldots, Y_{\nu_0-1}\}$ is a positive definite subspace of $T^0(x)$. Proposition 1 tells us the choice of $Y_1, \ldots, Y_{\nu_0-1}$ is possible. Denote $C_i = C_{Y_i}$ $(i = 1, \ldots, \nu_0 - 1)$ for simplicity.

LEMMA 3. *The set of vectors $\{X, C_1(X), \ldots, C_{\nu_0-1}(X)\}$ forms a ν_0-frame in $T^c(x)$ for $X \neq 0 \in T^c(x)$.*

Proof. Let $\alpha X + \alpha_1 C_1(X) + \cdots + \alpha_{\nu_0-1} C_{\nu_0-1}(X) = 0$. Then

$$\alpha_1 C_1(X) + \cdots + \alpha_{\nu_0-1} C_{\nu_0-1}(X) = C_{\alpha_1 Y_1 + \cdots + \alpha_{\nu_0-1} Y_{\nu_0-1}}(X) = (-\alpha) X.$$

Hence, $-\alpha$ is a real eigenvalue of $C_{\alpha_1 Y_1 + \cdots + \alpha_{\nu_0-1} Y_{\nu_0-1}}$. Since $\alpha_1 Y_1 + \cdots + \alpha_{\nu_0-1} Y_{\nu_0-1}$ is a space-like vector, $\alpha_1 Y_1 + \cdots + \alpha_{\nu_0-1} Y_{\nu_0-1} = 0$ from the above remark. This implies $\alpha_1 = \cdots = \alpha_{\nu_0-1} = 0$ and $\alpha = 0$.

As was done in [7], denote by $V_{n,r}$ the Stiefel manifold of ordered r-frames in E^n. It is well known that $V_{n,r} \to V_{n,1}$ is a principal fiber bundle in a natural way. Denote by $\rho(n)$ the largest integer such that the fibration $V_{n,\rho(n)} \to V_{n,1}$ has a global cross-section. Define by ν_n the largest integer such that $\rho(n - \nu_n) \geq \nu_n$.

THEOREM 3. *Let M_1^n be a geodesically complete, connected submanifold of $\tilde{M}_1^{n+k}(c)$, $c > 0$. If the index of relative nullity $\nu_0 > \nu_n$, then M_1^n is totally geodesic in $\tilde{M}_1^{n+k}(c)$ and $\nu_0 = n$.*

Proof. For any $x \in G$, $T^0(x)$ always contains a copy of E^{ν_0-1}. By Lemma 3, $V_{n-\nu_0,\nu_0} \to V_{n-\nu_0,1}$ has a global cross-section φ defined by $\varphi(X) = (X, C_1 X, \ldots, C_{\nu_0-1} X)$ for $\forall X \neq 0 \in T^c(x)$. Hence, $\rho(n - \nu_0) \geq \nu_0$; therefore, $\nu_0 \leq \nu_n$.

REMARK. If one of the relative nullity leaves is a Riemannian manifold relative to the induced metric, the above ν_n can be improved by defining $\bar{\nu}_n$ to be the largest integer such that $\rho(n - \bar{\nu}_n) \geq \bar{\nu}_n + 1$. Clearly $\bar{\nu}_n \leq \nu_n$.

Some of the numerical values for ν_n are as follows: $\nu_1 = 0$, $\nu_2 = 1$, $\nu_3 = 1$, $\nu_4 = 2$, $\nu_5 = 1$, $\nu_6 = 2$, $\nu_7 = 3$, $\nu_8 = 4$, $\nu_9 = 1$.

The argument used here can be applied to obtain a similar result for more general indefinite metrics. Unlike the Riemannian case, it is known that the above ν_n's are often the best possible value. For example, Graves and Nomizu [**11**] constructed an isometric immersion of S_1^2 into S_1^3 with the index of relative nullity 1.

Our next result states:

THEOREM 4. *Let f: $M_1^n \to \tilde{M}_1^{n+p}(c)$ be an isometric immersion between two Lorentzian manifolds, where $\tilde{M}_1^{n+p}(c)$ is the Lorentzian space form of positive curvature c. Suppose that the Ricci curvature S of M_1^n satisfies $S(X, X) \geq (n-1)c\langle X, X\rangle$ for all space-like vectors X.*

(1) *If $T^0(x)$ is Lorentzian for some $x \in G$, then the index of relative nullity is either* 0 *or n.*

(2) *If $T^0(x)$ is degenerate for some $x \in G$, then the index of relative nullity is* 0, 1, $n-1$ *or n.*

(3) *If $T^0(x)$ is Riemannian for all $x \in G$ and if $p = 1$, then the index of relative nullity is* 0, $n-2$, $n-1$, *or n.*

We will prove Theorem 4 after a sequence of propositions and lemmas.

PROPOSITION 4. *Let* (,) *be a symmetric bilinear form on an $n - \nu_0$ dimensional vector space V over* **R** *with signature (m_1, m_2, m_3). If $m_1 \neq m_2$ and T: $V \to V$ is a symmetric linear operator with respect to* (,), *then T has a real eigenvalue. Note a symmetric bilinear form of signature (m_1, m_2, m_3) has m_1* (–1)'s, m_2 (+1)'s *and m_3* 0's *in the canonical form.*

Proof. Choose a canonical basis $\{e_1, \ldots, e_{m_1}, f_1, \ldots, f_{m_2}, g_1, \ldots, g_{m_3}\}$ of V for the symmetric bilinear form, so that $(e_\alpha, e_{\alpha'}) = -\delta_{\alpha\alpha'}$, $(f_\beta, f_{\beta'}) = \delta_{\beta\beta'}$, $(g_\gamma, g_{\gamma'}) = 0$ and all other products are zero. The matrix of the symmetric operator T with respect to this basis has the form:

$$\begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} \left[\begin{array}{c|c|c} A & -B^t & 0 \\ \hline B & C & 0 \\ \hline D & E & F \end{array}\right] \quad \begin{array}{ccc} m_1 & m_2 & m_3 \end{array}.$$

Here $A^t = A$ and $C^t = C$. Denote this matrix by M_T. The matrix $M_s = \left[\begin{smallmatrix} A & -B^t \\ B & C \end{smallmatrix}\right]$ represents a symmetric linear transformation on the span of $\{e_1, \ldots, e_{m_1}, f_1, \ldots, f_{m_2}\}$. By a change of basis, M_s can be put into a standard form [15]. If $m_1 \neq m_2$, there is one block of the form:

$$\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ 0 & & \cdots & \lambda \end{bmatrix}.$$

Thus, M_s has an eigenvalue λ. Noting that

$$\det[\lambda I - M_T] = \det[\lambda I - M_s]\det[\lambda I - F] = 0,$$

we see that T has a real eigenvalue λ. □

Now let $\eta = \sum_{i=1}^{n} \langle X_i, X_i \rangle \alpha(X_i, X_i)$, where $\{X_1, \ldots, X_n\}$ is an orthonormal basis of $T_x(M)$. Define a bilinear form $(\ ,\ )$: $T^c(x) \times T^c(x) \to \mathbf{R}$ by, for any X and $Y \in T^c(x)$,

$$(X, Y) = \langle \alpha(X, Y), \eta \rangle.$$

It is clear that $(\ ,\ )$ is symmetric, since α is symmetric. We may find an alternative expression for $(\ ,\ )$ using the Gauss equation and the Ricci curvature of M_1^n.

$$\begin{aligned} (X, Y) &= \langle \alpha(X, Y), \eta \rangle \\ &= \sum_{i=1}^{n} \langle X_i, X_i \rangle \langle \alpha(X, Y), \alpha(X_i, X_i) \rangle \\ &= S(X, Y) - nc\langle X, Y \rangle + c\langle X, Y \rangle \\ &\quad + \sum_{i=1}^{n} \langle X_i, X_i \rangle \langle \alpha(X_i, X), \alpha(X_i, Y) \rangle. \end{aligned}$$

Clearly, the definition of $(\ ,\ )$ and the above expression for (X, Y) do not depend on the choice of the orthonormal basis $\{X_1, \ldots, X_n\}$.

If $\{\xi_1, \ldots, \xi_p\}$ is an orthonormal basis for $N(x)$, then the last term of the above expression can be rewritten

$$\begin{aligned}
&\sum_{i,j} \langle X_i, X_i\rangle\langle \xi_j, \xi_j\rangle\langle \alpha(X_i, X), \xi_j\rangle\langle \alpha(X_i, Y), \xi_j\rangle \\
&= \sum_{i,j} \langle X_i, X_i\rangle\langle \xi_j, \xi_j\rangle\langle A_{\xi_j} X, X_i\rangle\langle A_{\xi_j} Y, X_i\rangle \\
&= \sum_{j} \langle \xi_j, \xi_j\rangle\langle A_{\xi_j} X, A_{\xi_j} Y\rangle.
\end{aligned}$$

Since the normal space is positive definite,

$$(X, Y) = S(X, Y) - c(n-1)\langle X, Y\rangle + \sum_{j=1}^{p} \langle A^2_{\xi_j} X, Y\rangle.$$

For convenience, let $\tau = c(n-1)$, and A_{ξ_j} by A_j. We examine (,) by looking at

$$h(X, Y) := S(X, Y) - \tau\langle X, Y\rangle \quad \text{and} \quad k(X, Y) := \sum_{j=1}^{p} \langle A_j^2 X, Y\rangle.$$

LEMMA 4. *If the Ricci curvature satisfies the hypothesis in the theorem, then $h(X, Y)$ is positive semi-definite or Lorentzian.*

Proof. We show that h cannot have two (–1)'s in its signature. If it did, we could find a pair of linearly independent vectors e and f in T_xM such that $h(e, e) = -1$, $h(f, f) = -1$ and $h(e, f) = 0$ i.e., $S(e, e) = -1 + \tau\langle e, e\rangle$, $S(f, f) = -1 + \tau\langle f, f\rangle$ and$S(e, f) = \tau\langle e, f\rangle$. By the hypothesis, it would then be the case that $\langle e, e\rangle \leq 0$ and $\langle f, f\rangle \leq 0$.

We now examine the various possibilities for the lengths of e and f. In each case, we will find a space-like vector which violates the condition on the Ricci curvature. We will use the reverse Cauchy-Schwarz inequality, i.e, $0 \leq \langle e, e\rangle\langle f, f\rangle \leq \langle e, f\rangle^2$ in this case. In the following argument, denote $\langle e, f\rangle$ by a for brevity. $\langle e, f\rangle = a$ can be positive.

If both e and f were light-like, $e + f$ would be space-like and $\langle e + f, e + f\rangle = 2a$.

$$\begin{aligned}
S(e+f, e+f) &= S(e, e) + 2S(e, f) + S(f, f) \\
&= (-1) + 2\tau\langle e, f\rangle + (-1) < 2\tau\langle e, f\rangle = 2\tau a \\
&= \tau\langle e+f, e+f\rangle.
\end{aligned}$$

This contradicts the hypothesis.

If e were light-like and f were time-like, we could also assume that $\langle f, f\rangle = -1$. Then $e + af$ would be space-like, for $\langle e + af, e + af\rangle = a^2$. But

$$S(e + af, e + af) = S(e, e) + 2aS(e, f) + a^2S\langle f, f\rangle$$
$$= -1 + 2a\tau a + a^2(-1 - \tau) < \tau a^2,$$

a contradiction.

If e and f are both time-like, we could assume that $\langle e, e\rangle = -1 = \langle f, f\rangle$. Then, $e + af$ is space-like with $\langle e + af, e + af\rangle = -1 + a^2 > 0$ by the reverse Cauchy-Schwarz inequality.

$$S(e + af, e + af) = -1 - \tau + 2a^2\tau - a^2 - a^2\tau < (-1 + a^2)\tau,$$

a contradiction.

Next we show that if there are any vectors which are time-like with respect to h, then h is non-degenerate and therefore Lorentzian. If this is not the case, there are linearly independent e and g such that $h(e, e) = -1$, $h(g, g) = 0$ and $h(e, g) = 0$, i.e., $S(e, e) = -1 + \tau\langle e, e\rangle$, $S(g, g) = \tau\langle g, g\rangle$ and $S(e, g) = \tau\langle e, g\rangle$. We know that $\langle e, e\rangle \le 0$ and can assume that $\langle e, g\rangle \ge 0$. If $\langle g, g\rangle > 0$, then $\langle g + e, g + e\rangle = \langle g, g\rangle + 2\langle e, g\rangle > 0$, so that $g + e$ is space-like. However,

$$S(g + e, g + e) = \tau\langle g, g\rangle + 2\tau\langle e, g\rangle - 1$$
$$+\tau\langle e, e\rangle < \tau[\langle g, g\rangle + 2\langle e, g\rangle],$$

a contradiction. The only remaining possibility is that $\langle e, e\rangle \le 0$ and $\langle g, g\rangle \le 0$. The span of e and g is non-degenerate, so for some $k \in \mathbf{R}$, $e + kg$ is space-like.

$$S(e + kg, e + kg) = S(e, e) + 2kS(e, g) + k^2S(g, g)$$
$$= -1 + \tau\langle e, e\rangle + 2k\tau\langle e, g\rangle + k^2\tau\langle g, g\rangle$$
$$< \tau[\langle e, e\rangle - 2k\langle e, g\rangle + k^2\langle g, g\rangle],$$

a contradiction.

This completes the proof of Lemma 4.

We now turn to $k(X, Y) = \sum_{j=1}^{p}\langle A_j^2X, Y\rangle$ on $T^c(x)$.

LEMMA 5. *If $T^c(x)$ is positive definite, then k restricted to $T^c(x)$ is positive definite.*

Proof.

$$k(X, X) = \sum_{j=1}^{p}\langle A_j^2X, X\rangle = \sum_{j=1}^{p}\langle A_jX, A_jX\rangle.$$

If $k(X,X) = 0$, then $A_jX = 0$ for $j = 1,\dots,p$, since $T^c(x) = T^1(x) = \operatorname{span}\{A_jY\}$ is positive definite. This means that $X \in T^0(x) \cap T^1(x)$; therefore, $X = 0$. Hence, k is positive definite.

Proposition 5. *If $T^c(x)$ is positive definite, then $(\ ,\ )$ is positive definite on $T^c(x)$.*

Proof. For no non-zero $e \in T^c(x)$ is $h(e,e) = -1$, since this implies $\langle e,e\rangle \le 0$. Thus, the form $(\ ,\ )$ is the sum of a positive semi-definite form and a positive definite form.

Lemma 6. *If $T^c(x)$ is a degenerate subspace with respect to $\langle\ ,\ \rangle$, then h is positive semi-definite on $T^c(x)$ for* $\dim T^c(x) > 1$.

Proof. $T^c(x)$ is a positive semi-definite subspace with respect to $\langle\ ,\ \rangle$ by hypothesis. If there were an $e \in T^c(x)$ with $h(e,e) = -1$, then $\langle e,e\rangle = 0$. By Lemma 4, there would be $g \in T^c(x)$ such that $h(g,g) = 1$ and $h(e,g) = 0$. By hypothesis we would have $\langle g,g\rangle > 0$ and $\langle e,g\rangle = 0$. For all $t \in \mathbf{R}$ $\langle g+te, g+te\rangle = \langle g,g\rangle > 0$, but

$$S(g+te, g+te) = 1 + \tau\langle g,g\rangle - t^2 < \tau\langle g,g\rangle.$$

This is a contradiction.

Lemma 7. *If $T^c(x)$ is a degenerate subspace with respect to $\langle\ ,\ \rangle$ and if* $\dim T^c(x) > 1$, *then k is positive semi-definite on $T^c(x)$ and for some $Y \in T^c(x)$ $k(Y,Y) > 0$.*

Proof. If $T^0(x)$ is degenerate, so is $T^1(x) = \operatorname{span}[A_\xi Y]$. Thus, there is a light-like vector L such that, for any normal vector ξ, $\operatorname{Im} A_\xi \subseteq [L]^\perp$. The metric on $[L]^\perp$ is positive semi-definite, so $\langle A_\xi Y, A_\xi Y\rangle \ge 0$ for all ξ, Y. This implies that k is positive semi-definite. If $k(Y,Y) = 0$ for all Y, then each $A_\xi Y$ is light-like. We also have $\langle A_\xi Y, L\rangle = 0$ for ξ, Y. Recalling that perpendicular light-like vectors are linearly dependent, we see that $T^1(x)$ would be one-dimensional, which is not the case.

Proposition 6. *If $T^c(x)$ is a degenerate subspace, then $(\ ,\ )$ is positive semi-definite on $T^c(x)$ and for some $Y \in T^c(x)$, $(Y,Y) > 0$.*

Finally, we assume M_1^n is a hypersurface in $\tilde{M}_1^{n+1}$ and that $T^c = T^1$ is Lorentzian. Here $k(X,Y) = \langle AX, AY\rangle$ where A is a transformation of T^1 which is one-to-one.

Proposition 7. *If M_1^n is a hypersurface in $\tilde{M}_1^{n+1}(c)$, T^1 is Lorentzian and $n - \nu_0 > 2$, then the signature (m_1, m_2, m_3) of $(\ ,\ )$ has $m_2 > m_1$.*

Proof. We know that h is positive semi-definite or Lorentzian. The shape operator A can be put into one of four canonical forms [15] and k can be explicitly calculated using these forms.

$$\text{(i)} \qquad A = \begin{bmatrix} \lambda & 0 & & & \\ 1 & \lambda & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-\nu_0-2} \end{bmatrix},$$

where A is given with respect to a pseudo-orthonormal basis $\{\hat{L}, L, f_1, \ldots, f_{n-\nu_0-2}\}$. k then has the following matrix with respect to this basis:

$$\begin{bmatrix} -2\lambda & -\lambda^2 & & & \\ -\lambda^2 & 0 & & & \\ & & \lambda_1^2 & & \\ & & & \ddots & \\ & & & & \lambda_{n-\nu_0-2}^2 \end{bmatrix}.$$

Note that $\lambda\lambda_1 \cdots \lambda_{n-\nu_0-2} \neq 0$ and the signature of k is $(1, n - \nu_0 - 1)$. We can find a space-like vector U in $\operatorname{span}\{\hat{L}, L\}$ such that $k(U,U) > 0$. In fact, there are choices for $b \in \mathbf{R}$ such that $\langle \hat{L} + bL, \hat{L} + bL\rangle = -2b$ and $k(\hat{L} + bL, \hat{L} + bL) = -2\lambda(1 + b\lambda)$ are both positive. Set $U = \hat{L} + bL$. We claim that $(\ ,\ )$ is positive definite on $\operatorname{span}\{U, f_1, \ldots, f_{n-\nu_0-2}\}$.

$$h\left(cU + \Sigma c_j f_j, cU + \Sigma c_j f_j\right) \geq 0$$

since

$$\langle cU + \Sigma c_j f_j, cU + \Sigma c_j f_j\rangle = c^2\langle U, U\rangle + \Sigma c_j^2 \geq 0.$$

It is also clear that $k(cU + \Sigma c_j f_j, cU + c_j f_j) \geq 0$ and is equal to 0 if and only if $c = 0 = c_j$ for $j = 1, \ldots, n - \nu_0 - 2$. Hence, if $n - \nu_0 > 2$, the signature of $(\ ,\ )$ will have more plus signs than minus signs.

$$\text{(ii)} \qquad A = \begin{bmatrix} \lambda & 0 & 0 & & & \\ 0 & \lambda & 1 & & & \\ -1 & 0 & \lambda & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-\nu_0-3} \end{bmatrix}$$

with respect to a pseudo-orthonormal basis $\{\hat{L}, L, f, f_1, \ldots, f_{n-\nu_0-3}\}$. k has the following form with respect to the above basis:

$$\begin{bmatrix} 1 & -\lambda^2 & -2\lambda & & & \\ -\lambda^2 & 0 & 0 & & & \\ -2\lambda & 0 & \lambda^2 & & & \\ & & & \lambda_1^2 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-\nu_0-3}^2 \end{bmatrix}$$

In this case, we construct a space-like vector $U = \hat{L} + bL$ such that $k(U, U) > 4$. That is, we want $\langle \hat{L} + bL, \hat{L} + bL \rangle = -2b > 0$ and $k(\hat{L} + bL, \hat{L} + bL) = 1 - 2b\lambda^2 > 4$.

Here, too, $\langle\ ,\ \rangle$ is positive definite on span $\{U, f, f_1, \ldots, f_{n-\nu_0-3}\}$. For any vector V in this span, $\langle V, V \rangle \geq 0$. Therefore, $h(V, V) \geq 0$ and the fact that $k(U, U) > 4$ guarantees that $k(V, V) \geq 0$ with $k(V, V) = 0$ if and only if $V = 0$.

(iii)
$$A = \begin{bmatrix} a & -b & & & \\ b & a & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-\nu_0-2} \end{bmatrix}$$

with respect to orthonormal basis $\{e, f, f_1, \ldots, f_{n-\nu_0-2}\}$. We can find a space-like vector U in span $\{e, f\}$ with $k(U, U) > 0$. As above, the signature of $(\ ,\ )$ has more plus signs than minus signs.

(iv) A is diagonalizable with respect to an orthonormal basis. Then, it is easy to see that the conclusion of the proposition is satisfied. This completes the proof.

We are now in the position of proving Theorem 4. We follow the argument in [7]. If $n > \nu_0 > 0$, choose any non-zero $Z \in T^0(x)$. Then, C_Z: $T^c(x) \to T^c(x)$ is a symmetric operator with respect to $(\ ,\ )$. In (1), by Propositions 4 and 5, C_Z would have a real eigenvalue. Also Z can be chosen as a space-like vector. But the equation (2.10) has no global solution in which $k(t) > 0$ and C_Z has a real eigenvalue. This is a contradiction. In case (2), if $n - \nu_0 > 1$, by Propositions 4 and 6, C_Z would have a real eigenvalue. Clearly, Z can be chosen to be a space-like vector; hence, a contradiction as before. Finally, in case (3), Propositions 4 and 7 assure us a real eigenvalue of C_Z. Since $n - \nu_0 > 2$ and since T^1 is Lorentzian, Z can be chosen to be space-like. The same argument as above completes the proof.

The following result concerns the over simplified "The axiom of sphere" for the hypersurfaces in the Lorentzian space form of positive constant curvature c.

Let $f\colon M_1^n \to \tilde{M}_1^{n+1}(c)$ be a geodesically complete Lorentzian hypersurface, where $\tilde{M}_1^{n+1}(c)$ is the Lorentzian space form of positive curvature c. Let us assume:

($*$) Through each point x of M exists a $k(x)$-dimensional local submanifold S_x of M which is mapped under f isometrically into a $k(x)$-dimensional totally geodesic submanifold of $\tilde{M}_1^{n+1}(c)$.

THEOREM 5. *If* $2k(x) - n \geq \nu_n$, M_1^n *is totally geodesic in* $\tilde{M}_1^{n+1}(c)$. *Here* ν_n *is the numerical value determined in Theorem* 3.

LEMMA 8. *If* $2k(x) - n > 0$, *the relative nullity* $\nu(x)$ *at* $x \geq 2k(x) - n$. *In particular, the index of relative nullity* $\nu_0 \geq 2k - n$, *where* $k = \min_{x \in M_1^n} k(x)$.

Proof. If S_x is non-degenerate, the result of Lemma 8 will be obtained in the same manner as in **[3]**. Now let us assume that S_x is degenerate. Let $\hat{e}_1(x), \dots, \hat{e}_{n-k-1}(x)$, L form a pseudo-orthonormal basis for $(TS_x)^{\perp}$, which is the orthogonal complement of TS_x in $T_xM_1^n$. Extend it to a basis of T_xM by adding $\{e_1(x), \dots, e_{k-1}(x), \hat{L}(x)\}$, which is also pseudo-orthonormal. Then, $e_1(x), \dots, e_{k-1}(x), L(x)$ form a basis for TS_x. Since S_x is totally geodesic, $\alpha(e_i(x), e_j(x)) = \alpha(e_i(x), L) = 0$, where α is the second fundamental form. Set

$$A_\xi(e_i) = \sum_j a_{ji} e_j + b_i L + \text{a linear combination of } \hat{e}_k \text{ and } \hat{L}$$

and

$$A_\xi(L) = \sum_j c_j e_j + bL + \text{a linear combination of } \hat{e}_k \text{ and } \hat{L}.$$

Similarly, $A_\xi(\hat{e}_k)$ and $A_\xi(L)$ can be given as a linear combination of the basis elements $e_1, \dots, e_{k-1}, L, \hat{L}, \hat{e}_1, \dots, \hat{e}_{n-k-1}$. With respect to this basis, A_ξ is represented by an $n \times n$-matrix, which we also denote by the same symbol. In fact, we have

$$A_\xi(e_i) = -\langle A_\xi(e_i), \hat{L}\rangle L + \text{a linear combination of } \hat{e}_k\text{'s}$$

and

$$A_\xi(L) = -\langle A_\xi(L), \hat{L}\rangle L + \text{a linear combination of } \hat{e}_k\text{'s}.$$

Thus,

$$A_\xi = \begin{array}{c} k \\ \\ \\ \\ n-k \end{array} \overset{\qquad k \qquad\qquad\qquad n-k}{\begin{bmatrix} 0 & \cdots & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & & & \\ * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * \\ * & \cdots & * & & & \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & *. \end{bmatrix}}$$

Thus, by interchanging the kth and $(k+1)$st rows, A_ξ has the form:

$$A_\xi = \begin{array}{c} k \\ \\ n-k \end{array} \overset{\quad k \qquad n-k}{\begin{bmatrix} O & B \\ & \\ C & D. \end{bmatrix}}$$

Here, O is the $k \times k$-zero matrix and B, C and D are $k \times (n-k)$, $(n-k) \times k$ and $(n-k) \times (n-k)$-matrix, respectively. Let RB be the row-reduced echelon matrix of B. Then, at least $k - (n-k) = 2k - n$ rows from the bottom of RB must be the zero rows. Similarly, the column-reduced echelon matrix CC of C must have at least $2k - n$ zero columns on the right side C. Denote by $r(x)$ the smaller between the number of the zero rows of RB and the number of the zero columns of CC. Applying an appropriate sequence of row operations and column operations, we finally get an $n \times n$-matrix of the following form:

$$\begin{bmatrix} O_{r \times r} & O_{r \times (n-r)} \\ O_{(n-r) \times r} & * \end{bmatrix}.$$

Here $O_{p \times q}$ is the $p \times q$-zero matrix.

Since $\nu(x)$ is the multiplicity of zero as an eigenvalue of A_ξ and since the multiplicity of zero is invariant under row and column operations, A_ξ must have at least $r(x)$ as its nullity, i.e., $\nu(x) \geq r(x) \geq 2k - n$.

Theorem 5 is then obtained immediately from Theorem 3. This result may be regarded as an oversimplified version of "the axiom of sphere" for hypersurfaces.

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Received April 2, 1985.

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