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POLYNOMIAL EQUATIONS OF IMMERSED SURFACES

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If V is a nonsingular real algebraic set we say $H_i(V;\mathbf{Z}_2)$ is algebraic if it is generated by nonsingular algebraic subsets of V.

Let V^3 be a 3-dimensional nonsingular real algebraic set. Then, we prove that any immersed surface in V^3 can be isotoped to an algebraic subset if and only if $H_i(V:\mathbf{Z}_2)$ $i = 1, 2$ are algebraic. This isotopy above carries the natural stratification of the immersed surface to the algebraic stratification of the algebraic set. Along the way we prove that if V is any nonsingular algebraic set then any simple closed curve in V is ε-isotopic to a nonsingular algebraic curve if and only if $H_1(V:\mathbf{Z}_2)$ is algebraic.

Let V^3 be a 3-dimensional nonsingular real algebraic set. We call a homology group of V algebraic if it is generated by nonsingular algebraic subsets. In this paper we prove:

THEOREM. *The following are equivalent:*

(a) *If $f: M^2 \looparrowright V^3$ is any immersion of a closed smooth surface in general position, then $f(M^2)$ is isotopic to an algebraic subset Z of V^3 by an arbitrarily small isotopy. This isotopy carries the natural stratification of $f(M^2)$ to the algebraic stratification of Z.*

(b) *$H_1(V;\mathbf{Z}_2)$ and $H_2(V;\mathbf{Z}_2)$ are algebraic.*

To be more precise for $i = 1, 2$ let $AH_i(V^3;\mathbf{Z}_2)$ be the subgroup of $H_i(V^3;\mathbf{Z}_2)$ generated by nonsingular algebraic subsets. Then $H_i(V;\mathbf{Z}_2)$ is algebraic if it is equal to $AH_i(V;\mathbf{Z}_2)$. In particular zero homology groups are algebraic. We will refer to elements of $AH_i(V^3;\mathbf{Z}_2)$ as algebraic homology classes. This definition is consistent with the conventions of [$\mathbf{AK}_1$].

In case f is an imbedding this theorem reduces to a special case of Proposition 1 below, which is Theorem 4.1 and Remark 4.2 of [$\mathbf{AK}_1$]. Recall, if W^n is a nonsingular algebraic set of dimension n, then $AH_{n-1}(W;\mathbf{Z}_2)$ is the subgroup of $H_{n-1}(W;\mathbf{Z}_2)$ generated by nonsingular algebraic subsets. Also if $M \subset W$ is a closed submanifold, denote the

$\mathbf{Z}_2$-homology class in W induced by the fundamental class of M by $[M]_2$. Then

PROPOSITION 1. *A codimension one closed smooth submanifold M of W is ε-isotopic to a nonsingular real algebraic subset if and only if $[M]_2 \in AH_{n-1}(W; \mathbf{Z}_2)$. Furthermore, this isotopy can fix any smooth submanifold L of M which is already a nonsingular algebraic set.*

REMARK. Proposition 1 remains true if L is a union of nonsingular algebraic sets in M ([**T**]).

We first prove a codimension two version of this proposition for V^3, which is an interesting result in itself.

PROPOSITION 2. *A simple closed curve $C \subset V^3$ is ε-isotopic to a nonsingular algebraic curve if and only if $[C]_2 \in AH_1(V; \mathbf{Z}_2)$. Furthermore this isotopy can fix any collection of points in C.*

REMARK. This proposition remains true if V^3 is replaced by a nonsingular algebraic set of any dimension. The proof is essentially the same.

LEMMA 3. *Let $C \subset V^3$ be a nonsingular algebraic curve and $L \subset V^3$ be a smooth manifold. Then C can be moved by an ε-isotopy to a nonsingular algebraic curve C' which is transversal to L.*

Proof. Let F^2 be the boundary of a small closed tubular neighborhood of C in V. F is a circle bundle over C and hence has a section, so after a small isotopy of F we can assume that $C \subset F$. Since F is null homologous, by Proposition 1, it is ε-isotopic to a nonsingular algebraic surface Z with $C \subset Z$. By the terminology of [$\mathbf{AK}_1$] C is a stable algebraic set. Stable algebraic sets have the required property (Proposition 4.3 of [$\mathbf{AK}_1$]).

LEMMA 4. *If V^3 is orientable and $F^2 \subset V^3$ is a compact orientable surface with $\partial F^2 = C \cup A$ where A is a nonsingular algebraic curve, then C is ε-isotopic to a nonsingular algebraic curve.*

Proof. Since V is orientable F has a trivial normal bundle in V. Let $F' = \partial(F \times I) \subset V^3$ corners smoothed, and $C \cup A = \partial(F \times 0) \subset F'$. $C \cup A$ separates F'. Since $[F']_2 = 0$ by Proposition 1 F' is ε-isotopic to a nonsingular algebraic surface Z with $A \subset Z$. After a small isotopy of C

we can assume $C \subset Z$. Then $C \cup A$ separates Z; this means $[C]_2 = [A]_2 \in AH_1(Z; \mathbf{Z}_2)$. Hence by Proposition 1 C is ε-isotopic to a nonsingular algebraic curve C^* in Z. C^* is the required algebraic curve. □

REMARK. We can assume that the isotopy $C \rightsquigarrow C^*$ fixes any finite number of points of C. This is because by Proposition 1 we can arrange that Z and C^* fix these points.

LEMMA 5. *If $S \subset V^3$ is an orientable surface and*

$$i_*: H_1(V - S; \mathbf{Z}_2) \to H_1(V; \mathbf{Z}_2)$$

is the map induced by the inclusion, then $\ker(i_*) \subset AH_1(V - S; \mathbf{Z}_2)$.

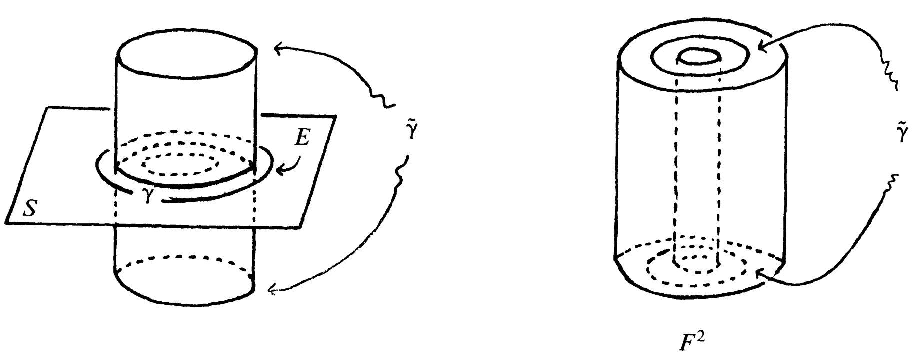
Proof. From the homology exact sequence

$$H_2(V, V - S; \mathbf{Z}_2) \xrightarrow{\partial} H_1(V - S; \mathbf{Z}_2) \xrightarrow{i_*} H_1(V; \mathbf{Z}_2) \quad \operatorname{im}(\partial) = \ker(i_*).$$

Also we have isomorphisms

$$H_2(V, V - S; \mathbf{Z}_2) \underset{\cong}{\overset{\text{excision}}{\leftarrow}} H_2(N, \partial N; \mathbf{Z}_2) \underset{\cong}{\overset{\text{Thom}}{\rightarrow}} H_1(S; \mathbf{Z}_2)$$

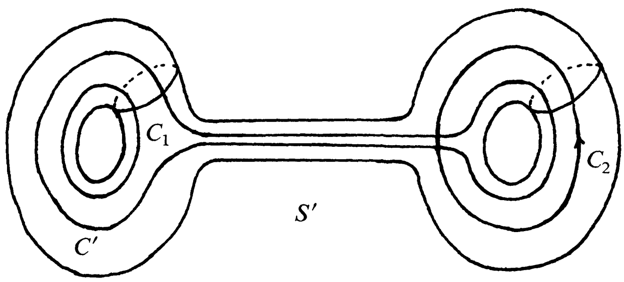
where N is a small closed tubular neighborhood of S in V. In particular N is an I-bundle over S, and ∂N is an $\dot{I}$-bundle over S ($\dot{I} = S^0$). From the above isomorphism we see that elements of $\operatorname{im}(\partial)$ are represented by the induced $\dot{I}$-bundles $\tilde{\gamma}$ over the curves γ of S



Let E be a small closed tubular neighborhood of γ in S, since S orientable $E \approx \gamma \times I$. Let E' be the induced I-bundle over E. Let $F^2 = \partial E'$. Clearly F^2 is a null homologous surface in V containing $\tilde{\gamma}$. Furthermore $\tilde{\gamma}$ separates F^2. By Proposition 1 F^2 can be ε-isotoped to a nonsingular algebraic surface Z. After a small isotopy of $\tilde{\gamma}$ we can assume that $\tilde{\gamma} \subset Z$. Since $\tilde{\gamma}$ separates Z, by Proposition 1 $\tilde{\gamma}$ is ε-isotopic to a nonsingular algebraic curve γ^* in Z. By construction $\gamma^* \subset V - S$ and $[\tilde{\gamma}]_2 = [\gamma^*]_2 \in AH_1(V - S; \mathbf{Z}_2)$. □

LEMMA 6. *Every element of $AH_1(V; \mathbf{Z}_2)$ can be represented by a connected nonsingular algebraic curve.*

Proof. Let $\alpha \in AH_1(V; \mathbf{Z}_2)$ then α is represented by a union of nonsingular algebraic curves $C = C_1 \cup \cdots \cup C_k$. By Lemma 3 we can assume that they are disjoint. Let S be the boundary of a closed tubular neighborhood of C. Since the normal bundle of C has nowhere zero section, after an ε-isotopy of S we can assume that $C \subset S$. Then by tubing the components of S we get a connected surface S' with $C \subset S'$. Let C_i' be ε-isotopic copies of C_i on S' which are in general position with C_i. Connect C_i', $i = 1, \ldots, k$, by tubes in S' to get a connected curve $C' = C_1' \# \cdots \# C_k'$ such that C' is homologous to C in S'



By construction $[S']_2 = 0$ in $H_2(V; \mathbf{Z}_2)$, so by Proposition 1 we can ε-isotop S' to a nonsingular algebraic surface Z with $C \subset Z$. Continue to denote the isotopic copy of C' in Z by C'. Again since $[C']_2 = [C]_2 \in AH_1(Z; \mathbf{Z}_2)$ by Proposition 1, C' is ε-isotopic to a nonsingular algebraic curve C^* in Z. C^* is connected and $\alpha = [C]_2 = [C^*]_2 \in AH_1(V; \mathbf{Z}_2)$. □

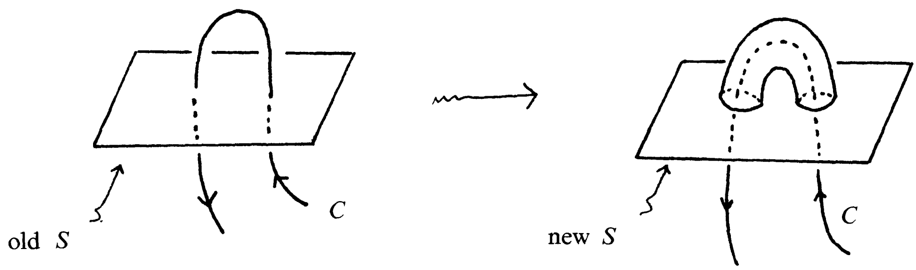
Proof of Proposition 2. We will prove this in three steps,

Case 1. *V^3 is orientable.*

Let $c = [C] \in H_1(V; \mathbf{Z})$. Since $[C]_2$ is algebraic there is a nonsingular algebraic curve $A \subset V$ such that $[C] = [A] + 2b$ for some $b \in H_1(V; \mathbf{Z})$. This means if $B \subset V$ is a simple closed curve with $b = [B]$, then $A \cup 2B \cup C$ bounds an orientable surface. Here $2B$ denotes the link $B \cup B'$ where B' is a parallel copy of B, so $2B$ is a boundary of an orientable surface $B \times I$ in V. By Lemma 4 we can assume that $2B$ is a nonsingular algebraic curve. Again by Lemma 4 C is ε-isotopic to a nonsingular algebraic curve. By the Remark following Lemma 4 we can assume that this isotopy fixes any finite number of points of C.

Case 2. $[C]_2 = 0$ *in* $H_1(V; \mathbf{Z}_2)$

Let $S \subset V$ be a surface representing the dual of the first Steifel-Whitney class $w_1(V)$ of V. We can assume that $C \cap S = \varnothing$. This is because by homological reasons $C \cap S$ must be an even number of points, and we can modify S as in the picture below without affecting its homology class.



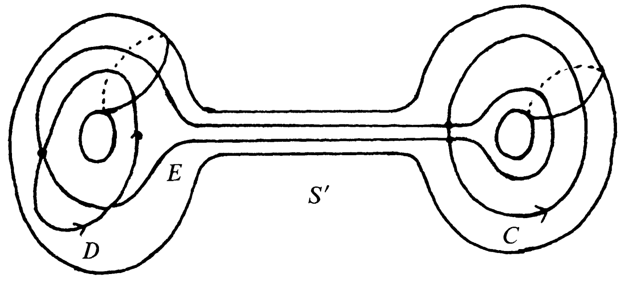
Hence $C \subset V - S$, and by assumption $[C]_2 \in \ker(i_*)$ where

$$i_* : H_2(V - S; \mathbf{Z}_2) \to H_2(V; \mathbf{Z}_2)$$

is the induced map by inclusion. Since $[S]_2 = w_1(V)$, S is orientable (exercise), so by Lemma 5 $[C]_2 \in AH_1(V - S;\ \mathbf{Z}_2)$. Since $V - S$ is orientable, by Case 1 C is ε-isotopic to a nonsingular algebraic curve in $V - S$, fixing any finite number of points of C.

Case 3. *The general case.*

We choose a connected nonsingular algebraic curve D disjoint from C so that $[C]_2 = [D]_2$. Let S be the boundary of a closed tubular neighborhood of $C \cup D$. As in the proof of Lemma 6 after a small isotopy of S we can assume that $C \cup D \subset S$, and let S' be the connected surface obtained by tubing the two components of S. By construction $C \cup D \subset S'$. Let C' and D' be ε-isotopic transverse copies of C and D in S'. Then by tubing C' and D' in S' we get a curve $E = C'\#D'$ as in the picture



By construction we have

(a) $[S']_2 = 0$ in $H_2(V; \mathbf{Z}_2)$

(b) $[E]_2 = [C \cup D]_2$ in $H_1(S'; \mathbf{Z}_2)$

(c) $[E]_2 = 0$ in $H_1(V; \mathbf{Z}_2)$

By Case 2 E is ε-isotopic to a nonsingular algebraic curve E^* in V fixing the points $E \cap (C \cup D)$. After an ε-isotopy of S' we may assume $C \cup D \cup E^* \subset S'$. By Proposition 1 (and by the remark following it) we can ε-isotop S' to a nonsingular algebraic surface Z with $D \cup E^* \subset Z$. Let C' be the corresponding ε-isotopic copy of C in Z. Since $[C']_2 = [D \cup E^*]_2 \in AH_1(Z; \mathbf{Z}_2)$ by Proposition 1. C' is ε-isotopic to a nonsingular algebraic curve C^* in Z. Furthermore given any finite number of points on C_1 by Proposition 1 we can require that all these isotopies fix these points. $\square$

Proof of the Theorem. First we show (b) $\Rightarrow$ (a). For every $y \in f(M^2)$ consider $n(y) = \max\{n \mid$ there are n distinct points $x_1, \ldots, x_n \in M$ with $f(x_i) = y$ for $i = 1, 2, \ldots, n\} =$ the cardinality of $f^{-1}(y)$. $f(M)$ is a stratified set with strata $\{L_i\}_{i=1}^3$ where L_i are the i-fold point sets, $L_i = \{y \in f(M) \mid n(y) = i\}$. Call $d(f) = \max\{i \mid L_i \neq \varnothing\}$, then $d(f) \leq 3$ and if $d(f) = 3$, L_3 is a collection of points (the triple points). Let $M_3 = f^{-1}(L_3)$. By ([**AK**$_1$], Lemma 2.3) there is a unique immersion f' with $d(f') = 2$ making the following commute

$$\begin{array}{ccc} M' = B(M, M_3) & \overset{f'}{\looparrowright} & B(V, L_3) = V' \\ \downarrow p' & & \downarrow \pi' \\ M & \overset{f}{\looparrowright} & V \end{array}$$

where the vertical maps are the blowing up maps along the centers M_3, L_3. Since the points are algebraic, we can assume that $V' \xrightarrow{\pi'} V$ is the algebraic blow up of V along L_3.

Since $d(f') = 2$ the 2-fold point set $L_2 \subset V'$ of the map f' is a smooth manifold (i.e., collection of smooth circles). Let $M_2 = (f')^{-1} L_2$. Once again by [**AK**$_1$] there is a unique immersion f'' with $d(f'') = 1$ (i.e., it is an imbedding) making the following commute

$$\begin{array}{ccc} M'' = B(M', M_2) & \overset{f''}{\hookrightarrow} & B(V', L_2) = V'' \\ \downarrow p'' & & \downarrow \pi'' \\ M' & \overset{f'}{\looparrowright} & V' \end{array}$$

where the vertical maps are the blowing up maps. In particular $M'' = M'$ and p'' = identity, since $M_2 \subset M'$ is codimension one.

$V' = V \overset{k}{\#} \mathbf{R}P^3$ so $H_i(V') = H_i(V) \oplus H_i(\overset{k}{\#} \mathbf{R}P^3)$ for $i = 1, 2$; in particular $H_1(V'; \mathbf{Z}_2)$ and $H_2(V'; \mathbf{Z}_2)$ are algebraic. By Proposition 2 the curve L_2 is ε-isotopic to a nonsingular algebraic set. We can change $f(M)$ by a small isotopy in V keeping L_3 fixed so that the corresponding double point set L_2 in V' is this nonsingular algebraic set. Therefore we can take π'' to be the algebraic blow up along L_2, in particular V'' is a nonsingular algebraic set.

We claim that $H_2(V''; \mathbf{Z}_2)$ is algebraic. This can be seen by the homology exact sequences

$$\begin{array}{ccccccccc}
\cdots & \to & H_2(C'') & \xrightarrow{i_*} & H_2(V'') & \to & H_2(V'', C'') & \to & \cdots \\
 & & \downarrow \pi''_* & & \downarrow \pi''_* & & \cong \downarrow \text{excision} & & \\
\cdots & \to & H_2(C') & \to & H_2(V') & \to & H_2(V', C') & \to & \cdots
\end{array}$$

where all the homology groups have coefficient $\mathbf{Z}_2$, and C', C'' are closed tubular neighborhoods of L_2, $(\pi'')^{-1}(L_2)$ respectively. Since π'' is degree 1 π''_* is onto, and by the above diagram $\ker \pi''_* = \operatorname{im}(i_*)$ where i is the inclusion $C'' \hookrightarrow V''$. So $H_2(V''; \mathbf{Z}_2)$ is generated by the nonsingular algebraic sets $(\pi'')^{-1}(L_2)$, and $(\pi'')^{-1}(S_i)$ where S_i are surfaces in V'. By Proposition 1 we can assume S_i are nonsingular algebraic surfaces. By ([**AK$_1$**] Proposition 4.3) we can assume S_i are transverse to L_2. Hence $H_2(V''; \mathbf{Z}_2)$ is generated by nonsingular algebraic sets.

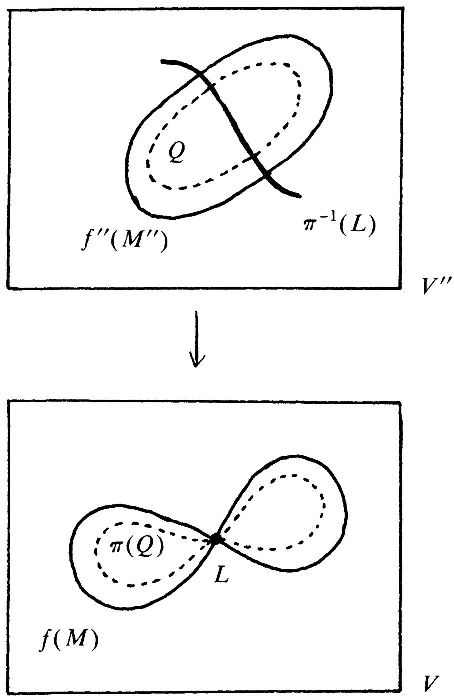
By Proposition 1 we can ε-isotop the smooth submanifold $f''(M'')$ to a nonsingular algebraic subset Q of V'' by a smooth isotopy. By ([**AK$_1$**] Lemma 2.5) $\pi' \circ \pi''(Q)$ is an algebraic set. $\pi' \circ \pi''(Q)$ is isotopic to $f(M)$ by a small isotopy. More precisely, the last remark can be seen by applying ([**AK$_2$**] Proposition 5.5). Namely [**AK$_2$**] gives an isotopy h_t: $V'' \to V''$ such that

(1) $h_0 = \mathrm{Id}$,
(2) $h_1(f''(M'')) = Q$,
(3) $h_t^{-1}(\pi^{-1}(x)) = \pi^{-1}(x)$ for all $x \in L \subset V$, where $\pi = \pi' \circ \pi''$, $L = L_3 \cup \pi'(L_2)$.

Then we can define an isotopy

$$g_t : V \to V \text{ by } g_t(x) = \pi h_t(y) \text{ for } \begin{cases} y = \pi^{-1}(x), & \text{if } x \notin L, \\ y \in \pi^{-1}(x), & \text{if } x \in L. \end{cases}$$

(Notice π is a diffeomorphism over the complement of L.) g_t gives an isotopy of $f(M)$ to $\pi(Q)$ fixing L pointwise. Also g_t is smooth in the complement of L.



It remains to show (a) $\Rightarrow$ (b). Clearly (a) implies $H_2(V; \mathbf{Z}_2)$ algebraic. To see $H_1(V; \mathbf{Z}_2)$ algebraic we write every simple closed curve $C \subset V^3$ as the double point of an immersion. C has a normal bundle $C \tilde{\times} D^2 \subset V$. Then $C \tilde{\times} X \subset V$ where X is the figure eight, so $C \tilde{\times} X = f(S^1 \tilde{\times} S^1)$ where $f: S^1 \tilde{\times} S^1 \to V$ is the obvious immersion. Hence by (a) $f(S^1 \tilde{\times} S^1)$ can be made algebraic and C is the singular set of this algebraic set. $\square$

Note added in proof. After writing this paper we have been informed by W. Kucharz that he had proved a special case of Proposition 2 when V is orientable in "Topology of Real Algebraic Threefolds" Duke Math. Journal, vol. 53, No. 4, Dec. 1986.

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