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THE INTRINSIC GROUP OF MAJID'S BICROSSPRODUCT KAC ALGEBRA

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A precise description of the intrinsic group of a Kac algebra considered in the recent work of Majid associated with a modular matched pair is given. By using the result, a detailed computation is done to produce an interesting pair of nonisomorphic Kac algebras.

0. Introduction. In [M1], [M2] and [M3], Majid studied the notion of a matched pair of locally compact groups and their actions. He exhibited plenty of examples of such pairs, relating them to solutions to the classical Yang-Baxter equations. Among other things, he showed in [M3] that every matched pair gives rise to two involutive Hopf-von Neumann algebras which are not commutative or cocommutative except in the trivial case. Moreover, he proved that, if a matched pair is modular in his sense, then the resulting von Neumann algebras turn out to be Kac algebras, dual to each other. (See [E&S] and §1 for definitons of an involutive Hopf-van Neumann algebra and a Kac algebra.) He called these algebras bicrossproduct Kac algebras. Thus his result furnishes abundant examples of nontrivial Kac algebras. It should be noted that, through his construction, one can even obtain a noncommutative, noncocommutative, self-dual Kac algebra. All of these would suggest that matched pairs of groups and bicrossproduct algebras deserve a further detailed investigation.

In the meantime, the notion of the intrinsic group $G(\mathbf{K})$ of a Kac algebra $\mathbf{K}$ was introduced by Schwartz in [S]. Roughly speaking, it consists of "group-like" elements of the given Kac algebra. Thus the group $G(\mathbf{K})$ can be considered as a natural kind of invariant attached to each Kac algebra $\mathbf{K}$. In fact, if a Kac algebra $\mathbf{K}$ is either commutative or cocommutative, then the intrinsic group of $\mathbf{K}$ (or that of the dual Kac algebra $\widehat{\mathbf{K}}$) completely determines the structure of the given algebra $\mathbf{K}$ (see [Ta]). So it is one of the important things in the theory of Kac algebras to know the intrinsic group, once one is given a Kac algebra, although it is known (see [DeC1]) that the intrinsic group is not a complete invariant for general Kac algebras. In this direction, De Canière's work [DeC1] should be noted as one of the significant achievements.

The purpose of this paper is to give a precise description of the intrinsic group of a bicrossproduct Kac algebra associated with a modular matched pair, in terms of the given data. Our main tool is De Canière's characterization of an intrinsic group as a certain group of automorphisms of the dual Kac algebra.

The organization of the paper is the following. In §1, we recall some facts on the theory of Kac algebras, relevant to our later discussion. We then review the notion of a matched pair of locally compact groups and their actions, following **[M3]**. Section 2 is devoted to investigating the intrinsic group of a bicrossproduct Kac algebra of a matched pair $(G_1, G_2, \alpha, \beta)$. We shall completely describe it in terms of the system $(G_1, G_2, \alpha, \beta)$. It turns out to be the semi-direct product of the character group of G_2 by the action β of a subgroup G_1^α of G_1. In the final section, as an application of the result of the preceding section, we carry out a detailed computation of the intrinsic group of a bicrossproduct Kac algebra that arises from Majid's example of a modular matched pair. As a result, we obtain an interesting example of nonisomorphic Kac algebras $\mathbf{K}$ and $\mathbf{K}_1$ for which $G(\mathbf{K}) \cong G(\mathbf{K}_1)$, $G(\widehat{\mathbf{K}}) \cong G(\widehat{\mathbf{K}}_1)$ and the associated bicharacters do not coincide. It should be remarked that the above Kac algebra $\mathbf{K}$ is a noncommutative, noncocomuutative, self-dual one.

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1. Preliminaries. In this section, we first recall some of the most important facts on the theory of Kac algebras, introducing notations used in our later discussion. For the general theory of Kac algebras, we refer the reader to **[E&S]** and **[S]**. Our notations will be mainly adopted from these literatures. Secondly, we review relevant results concerning matched pairs of locally compact groups due to Takeuchi **[T]** and Majid **[M1]**, **[M2]**, **[M3]**. Then we recall Majid's bicrossproduct construction from a modular matched pair.

A Kac algebra $\mathbf{K}$ is a quadruple $(\mathscr{M}, \Gamma, \kappa, \varphi)$ in which

(Ki) $(\mathscr{M}, \Gamma, \kappa)$ is an involutive Hopf-von Neumann algebra (Def. 1.2.1 of **[E&S]**);

(Kii) φ is a faithful, normal, semifinite weight on $\mathscr{M}$;

(Kiii) $(\iota_{\mathscr{M}} \otimes \varphi)(\Gamma(x)) = \varphi(x) \cdot 1$ for all $x \in \mathscr{M}_+$;

(Kiv) $(\iota_{\mathscr{M}} \otimes \varphi)((1 \otimes y^*)\Gamma(x)) = \kappa((\iota_M \otimes \varphi)(\Gamma(y^*)(1 \otimes x)))$ for all $x, y \in N_\varphi$;

(Kv) $\sigma_t^\varphi \circ \kappa = \kappa \circ \sigma_{-t}^\varphi$ for all $t \in \mathbf{R}$.

Here $N_\varphi = \{x \in \mathscr{M} : \varphi(x^*x) < \infty\}$, and σ^φ is the modular automorphism of φ. The symbol $\iota_{\mathscr{M}}$ is the identity morphism of $\mathscr{M}$. We will always think of $\mathscr{M}$ as represented in a standard form on the Hilbert space $\mathscr{H}_\varphi$ associated with φ. Given a Kac algebra $\mathbf{K} = (\mathscr{M}, \Gamma, \kappa, \varphi)$, there canonically exists another Kac algebra $\widehat{\mathbf{K}} = (\widehat{\mathscr{M}}, \widehat{\Gamma}, \widehat{\kappa}, \widehat{\varphi})$, called the dual Kac algebra of $\mathbf{K}$ **[E&S]**. The pair $\{\widehat{\mathscr{M}}, \mathscr{H}_\varphi\}$ is again a standard representation. The intrinsic group, denoted by $G(\mathbf{K})$, of the Kac algebra $\mathbf{K}$ consists of all non-zero solutions to the equation $\Gamma(x) = x \otimes x$ $(x \in \mathscr{M})$ (see **[S]** for the details). Every member in $G(\mathbf{K})$ is automatically a unitary operator. Thus $G(\mathbf{K})$ is a closed subgroup of the unitary group of $\mathscr{M}$, when equipped with the weak topology. It was shown in **[S]** that, if $w \in G(\widehat{\mathbf{K}})$, then $\operatorname{Ad} w$ induces an automorphism of $\mathbf{K}$. De Canière obtained in **[DeC1]** the complete characterization of the automorphism of $\mathbf{K}$ that arises in this way. Since we will make use of this characterization in a crucial manner later, we state it here.

THEOREM 1.1 (*Theorem* 2.3 *of* **[DeC1]**). *Let v be a unitary operator on $\mathscr{H}_\varphi$. Then v belongs to the intrinsic group $G(\widehat{\mathbf{K}})$ if and only if $\beta_v = \operatorname{Ad} v$ induces an automorphism of $\mathbf{K}$ in such a way that the unitary operator v is the canonical implementation of β_v in the sense of Haagerup* **[H]**, *and that β_v satisfies*

$$(\beta_v \otimes \iota_{\mathscr{M}}) \circ \Gamma = \Gamma \circ \beta_v.$$

Thanks to the above theorem, the group $\mathscr{G}(\mathbf{K})$ of all automorphisms γ of $\mathscr{M}$ satisfying the identity

$$(\gamma \otimes \iota_{\mathscr{M}}) \circ \Gamma = \Gamma \circ \gamma$$

is topologically isomorphic to the intrinsic group $G(\widehat{\mathbf{K}})$.

To any locally compact group G, one can associate two canonical Kac algebras. One is the commutative Kac algebra $\mathbf{KA}(G) = (L^\infty(G), \Gamma_G, j_G, \tau_G)$ in which

$$\Gamma_G(f)(s, t) = f(st), \quad j_G(f)(s) = f(s^{-1}),$$
$$\tau_G(f) = \int f(s)\, ds, \quad (f \in L^\infty(G), s, t \in G),$$

where ds is a left Haar measure of G, and $L^\infty(G)$ is the algebra of all (equivalence classes of) essentially bounded measurable functions on G with respect to the left Haar measure class. The other is the system $\mathbf{KS}(G) = (\mathscr{R}(G), \delta_G, \kappa_G, \varphi_G)$, where $\mathscr{R}(G)$ is the group von Neumann algebra of G. The morphisms δ_G and κ_G are characterized by the following identities:

$$\delta_G(\lambda(s)) = \lambda(s) \otimes \lambda(s), \quad \kappa_G(\lambda(s)) = \lambda(s^{-1}) \qquad (s \in G).$$

Here λ denotes the left regular representation of G. The weight φ_G is the so-called Plancherel weight of G that is derived from the left Hilbert algebra $\mathscr{K}(G)$, the set of all continuous functions on G with compact support, with the usual convolution as its product. The Kac algebra $\mathbf{KS}(G)$ is cocommutative in the sense that the comultiplication δ_G is symemtric. $\mathbf{KA}(G)$ and $\mathbf{KS}(G)$ are dual to each other. The intrinsic group $G(\mathbf{KA}(G))$ of $\mathbf{KA}(G)$ is the set of all continuous homomorphisms (i.e. characters) from G into the unit circle, with the topology of compact convergence. The intrinsic group $G(\mathbf{KS}(G))$ of $\mathbf{KS}(G)$ is precisely the set $\lambda(G)$ (see **[Ta]** for example). Hence it is topologically isomorphic to the original group G. This is why $G(\mathbf{K})$ is called the intrinsic group.

We now review the notion of a matched pair of locally compact groups. For the details of this concept, we refer readers to **[M1,2,3]**.

Let G_1 and G_2 be locally compact groups with left Haar measures μ_1 and μ_2, respectively. The first assumption is that G_1 acts on, and is at the same time acted on by, G_2 continuously and nonsingularly. By nonsingularity of a group action, we mean that the action preserves the measure class in question. We denote by α (resp. β) the action of G_1 (resp. G_2). We shall still use the letters α and β for the induced actions of G_1 and G_2 on algebras $L^\infty(G_2)$ and $L^\infty(G_1)$, respectively. We put

$$\chi(g, s) = \frac{d\mu_2 \circ \alpha_g}{d\mu_2}(s), \quad \Psi(s, g) = \frac{d\mu_1 \circ \beta_s}{d\mu_1}(g) \qquad (g \in G_1,\, s \in G_2).$$

The Radon-Nikodym derivatives χ and Ψ are cocycles on $G_1 \times G_2$, and are assumed to be jointly continuous. Following Takeuchi's terminology **[T]**, we say that such a system $(G_1, G_2, \alpha, \beta)$ is a matched pair if the actions α and β satisfy the following compatibility conditions:

$$\alpha_g(e) = e, \quad \beta_s(e) = e,$$
$$\alpha_g(st) = \alpha_{\beta_t(g)}(s)\alpha_g(t), \quad \beta_s(gh) = \beta_{\alpha_h(s)}(g)\beta_s(h)$$

for any $g, h \in G_1$ and $s, t \in G_2$. Majid showed in [M3] that, if the system $(G_1, G_2, \alpha, \beta)$ is a matched pair, then the ordinary crossed products $L^\infty(G_2) \times_\alpha G_1$ and $L^\infty(G_1) \times_\beta G_2$ both come equipped with a structure of an involutive Hopf-von Neumann algebra. He called these algebras bicrossproduct Hopf-von Neumann algebras associated to $(G_1, G_2, \alpha, \beta)$. With the additional condition that the matched pair is modular (see Definition 2.3 of [M3]), he also proved that the bicrossproduct algebras are in fact Kac algebras, and that they are dual to each other. What one should note is that bicrossprodcut Kac algebras are noncommutative and noncocommutative, except in the trivial case.

2. The intrinic group of a bicrossproduct Kac algebra. This section is concerned with investigation of the intrinsic group of the bicrossproduct Kac algebra associated with a modular matched pair. We shall show that the intrinsic group can be completely described by the given system.

In what follows, we fix a modular matched pair $(G_1, G_2, \alpha, \beta)$. We shall retain all the notations introduced in the preceding section. Let $\mathbf{K} = (\mathscr{M}, \Gamma, \kappa, \varphi)$ be the associated bicrossproduct Kac algebra in which $\mathscr{M} = L^\infty(G_2) \times_\alpha G_1$. By Majid's result mentioned in the previous section, the dual Kac algebra $\widehat{\mathbf{K}} = (\widehat{\mathscr{M}}, \widehat{\Gamma}, \widehat{\kappa}, \widehat{\varphi})$ is the other bicrossproduct Kac algebra, where $\widehat{\mathscr{M}} = L^\infty(G_1) \times_\beta G_2$. Let $\mathscr{H}_i = L^2(G_i)$ $(i = 1, 2)$. Then put $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, which is regarded as the set of all L^2-functions on $G_1 \times G_2$. Note that both $\mathscr{M}$ and $\widehat{\mathscr{M}}$ act on $\mathscr{H}$ in a standard form. Abusing notations, we still employ the letters α and β for the imbeddings of $L^\infty(G_2)$ and $L^\infty(G_1)$ into $\mathscr{M}$ and $\widehat{\mathscr{M}}$, respectively. Namely, α (resp. β) are injective $*$-homomorphisms of $L^\infty(G_2)$ (resp. $L^\infty(G_1)$) into $L^\infty(G_1 \times G_2)$ defined by

$$\{\alpha(k)\eta\}(g, s) = k(\alpha_g(s))\eta(g, s), \quad \{\beta(f)\eta\}(g, s) = f(\beta_s(g))\eta(g, s),$$

where $k \in L^\infty(G_2)$, $f \in L^\infty(G_1)$, $\eta \in \mathscr{H}$ and $(g, s) \in G_1 \times G_2$. Let λ_i $(i = 1, 2)$ denote the left regular representations of G_i. We now introduce a unitary operator W on $\mathscr{H} \otimes \mathscr{H}$, which we regard as the set of L^2-functions on $G_1 \times G_2 \times G_1 \times G_2$, given by

$$\{W\xi\}(g, s, h, t) = \xi(\beta_t(h)^{-1}g, s, h, \alpha_{\beta_t(h)^{-1}g}(s)t) \qquad (\xi \in \mathscr{H} \otimes \mathscr{H}).$$

See the proof of Theorem 2.6 of [M3] for this operator W. The inverse W^* is given by

$$\{W^*\xi\}(g, s, h, t) = \xi(\beta_{\alpha_g(s)^{-1}t}(h)g, s, h, \alpha_g(s)^{-1}t).$$

The coproduct Γ of $\mathbf{K}$ is then defined by the equation

$$\Gamma(x) = W(1 \otimes x)W^* \qquad (x \in \mathscr{M}).$$

We now observe what the morphism Γ does to generators of $\mathscr{M}$.

LEMMA 2.1. *With the notations as above, we have*

$$\{\Gamma(\alpha(k))\xi\}(g, s, h, t) = k(\alpha_g(s)\alpha_h(t))\xi(g, s, h, t),$$
$$\{\Gamma(\lambda_1(p) \otimes 1)\xi\}(g, s, h, t) = \xi(\beta_{\alpha_h(t)}(p^{-1})g, s, p^{-1}h, t)$$

for any $k \in L^\infty(G_2)$, $p \in G_1$ *and* $\xi \in \mathscr{H} \otimes \mathscr{H}$. *The first identity can be summarized to the following*:

$$\Gamma \circ \alpha(k) = (\alpha \otimes \alpha) \circ \Gamma_{G_2}(k).$$

Similar identities hold for $\widehat{\Gamma}$, β *and* λ_2 *after an appropriate change.*

Proof. The proof of the first two assertions is implicit in that of Theorem 2.6 of **[M3]**. Thus we leave the verification to the reader. For the third assertion, we first note that α is "implemented" by a unitary operator U on $\mathscr{H}$ as follows:

$$\alpha(k) = U^*(1_{\mathscr{H}_1} \otimes k)U \qquad (k \in L^\infty(G_2)),$$

where U is defined by

$$\{U\eta\}(g, s) = \chi(g^{-1}, s)^{1/2}\eta(g, \alpha_{g^{-1}}(s)) \qquad (\eta \in \mathscr{H}).$$

In fact, we have

$$\begin{aligned}\{U^*(1_{\mathscr{H}_1} \otimes k)U\eta\}(g, s) &= \chi(g, s)^{1/2}\{(1_{\mathscr{H}_1} \otimes k)U\eta\}(g, \alpha_g(s)) \\ &= \chi(g, s)^{1/2}k(\alpha_g(s))\{U\eta\}(g, \alpha_g(s)) \\ &= \chi(g, s)^{1/2}k(\alpha_g(s))\chi(g^{-1}, \alpha_g(s))^{1/2}\eta(g, s) \\ &= \{\alpha(k)\eta\}(g, s).\end{aligned}$$

Here we used the cocycle identity: $\chi(g, s)\chi(g^{-1}, \alpha_g(s)) = 1$. Thus the operator $(\alpha \otimes \alpha) \circ \Gamma_{G_2}(k)$ can be expressed as

$$(\alpha \otimes \alpha)(\Gamma_{G_2}(k)) = (U \otimes U)^*\Gamma_{G_2}(k)_{1,4}(U \otimes U),$$

where $\Gamma_{G_2}(k)_{1,4}$ is the operator on $\mathscr{H} \otimes \mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \mathscr{H}_1 \otimes \mathscr{H}_2$ given by

$$\Gamma_{G_2}(k)_{1,4} = (1_{\mathscr{H}_1} \otimes \sigma \otimes 1_{\mathscr{H}_2})\Gamma_{G_2}(k)(1_{\mathscr{H}_1} \otimes \sigma \otimes 1_{\mathscr{H}_2}).$$

Here σ in general denotes the unitary between the tensor products $\mathscr{K}_1 \otimes \mathscr{K}_2$ and $\mathscr{K}_2 \otimes \mathscr{K}_1$ of Hilbert spaces $\mathscr{K}_1$ and $\mathscr{K}_2$, given by flipping vectors: $\sigma(\eta_1 \otimes \eta_2) = \eta_2 \otimes \eta_1$ $(\eta_i \in \mathscr{K}_i \ (i = 1, 2))$. Recall that $\Gamma_{G_2}(k)(s, t) = k(st)$ $((s, t) \in G_2 \times G_2)$. Thus, for any $\xi \in \mathscr{H} \otimes \mathscr{H}$, we have

$$\begin{aligned}
&\{(\alpha \otimes \alpha)(\Gamma_{G_2}(k))\xi\}(g, s, h, t) \\
&\quad = \{(U \otimes U)^* \Gamma_{G_2}(k)_{1,4}(U \otimes U)\xi\}(g, s, h, t) \\
&\quad = \chi(g, s)^{1/2}\chi(h, t)^{1/2}\{\Gamma_{G_2}(k)_{1,4}(U \otimes U)\xi\}(g, \alpha_g(s), h, \alpha_h(t)) \\
&\quad = \chi(g, s)^{1/2}\chi(h, t)^{1/2} k(\alpha_g(s)\alpha_h(t))\{(U \otimes U)\xi\} \\
&\qquad \cdot (g, \alpha_g(s), h, \alpha_h(t)) \\
&\quad = \chi(g, s)^{1/2}\chi(h, t)^{1/2}\chi(g^{-1}, \alpha_g(s))^{1/2}\chi(h^{-1}, \alpha_h(t))^{1/2} \\
&\qquad \cdot k(\alpha_g(s)\alpha_h(t))\xi(g, s, h, t) \\
&\quad = k(\alpha_g(s)\alpha_h(t))\xi(g, s, h, t).
\end{aligned}$$

It follows from the first assertion that $\Gamma(\alpha(k)) = (\alpha \otimes \alpha)(\Gamma_{G_2}(k))$. By symmetry, we may obtain similar identities for $\widehat{\Gamma}$, β and λ_2. □

We now introduce two closed subsets G_1^β and G_2^α of G_1 and G_2, respectively, as follows.

$$\begin{aligned}
G_1^\beta &= \{g \in G_1 \colon \beta_s(g) = g \text{ for any } s \in G_2\}, \\
G_2^\alpha &= \{s \in G_2 \colon \alpha_g(s) = s \text{ for any } g \in G_1\}.
\end{aligned}$$

They are in fact subgroups of relevant groups. Indeed, if $g, h \in G_1^\beta$, then, for any $s \in G_2$, we have

$$\beta_s(gh) = \beta_{\alpha_h(s)}(g)\beta_s(h) = gh,$$

which implies that $gh \in G_1^\beta$. Moreover, the identity $\beta_s(gg^{-1}) = e$ yields

$$\beta_s(g^{-1}) = \beta_{\alpha_{g^{-1}}(s)}(g)^{-1} = g^{-1}.$$

Thus $g^{-1} \in G_1^\beta$. Similarly, one can see that G_2^α is a subgroup of G_2.

LEMMA 2.2. *The subsets $\alpha(G(\mathbf{KA}(G_2)))$ and $\{\lambda_1(g) \otimes 1 \colon g \in G_1^\beta\}$ are contained in the intrinsic group $G(\mathbf{K})$ of $\mathbf{K} = (\mathscr{M}, \Gamma, \kappa, \varphi)$. By symmetry, the subsets $\beta(G(\mathbf{KA}(G_2)))$ and $\{1 \otimes \lambda_2(s) \colon s \in G_2^\alpha\}$ are included in $G(\widehat{\mathbf{K}})$.*

Proof. It suffices to prove the first assertion.
Let $k \in G(\mathbf{KA}(G_2))$. Then, by Lemma 2.1, we have

$$\Gamma(\alpha(k)) = (\alpha \otimes \alpha)(\Gamma_{G_2}(k)) = (\alpha \otimes \alpha)(k \otimes k) = \alpha(k) \otimes \alpha(k).$$

Thus $\alpha(k)$ belongs to $G(\mathbf{K})$. Let $p \in G_1^{\beta}$. Then Lemma 2.1 shows that, for any $\xi \in \mathscr{H} \otimes \mathscr{H}$,

$$\begin{aligned}
&\{\Gamma(\lambda_1(p) \otimes 1)\xi\}(g, s, h, t) \\
&\quad = \xi(\beta_{\alpha_h(t)}(p^{-1})g, s, p^{-1}h, t) \\
&\quad = \xi(p^{-1}g, s, p^{-1}h, t) \\
&\quad = \{(\lambda_1(p) \otimes 1 \otimes \lambda_1(p) \otimes 1)\xi\}(g, s, h, t).
\end{aligned}$$

Thus $\Gamma(\lambda_1(p) \otimes 1) = \lambda_1(p) \otimes 1 \otimes \lambda_1(p) \otimes 1$, from which it follows that $\lambda_1(p) \otimes 1$ lies in $G(\mathbf{K})$. □

In the next lemma, recall that $\mathscr{G}(\mathbf{K})$ is the group of all automorphisms γ of $\mathscr{M}$ satisfying the condition: $(\gamma \otimes \iota_{\mathscr{M}}) \circ \Gamma = \Gamma \circ \gamma$.

LEMMA 2.3. *Let γ be in $\mathscr{G}(\mathbf{K})$. Then γ leaves $\alpha(L^{\infty}(G_2))$ globally invariant. Moreover, there uniquely exists an element s_0 in G_2^{α} such that*

$$\gamma|_{\alpha(L^{\infty}(G_2))} = \mathrm{Ad}(1 \otimes \lambda_2(s_0))|_{\alpha(L^{\infty}(G_2))}.$$

Here $\alpha|_{\mathscr{A}}$ denotes the restriction of α to a subset $\mathscr{A}$.

Proof. By Theorem 1.1, there exists a unique element w in $G(\widehat{\mathbf{K}})$ which gives the canonical implementation of γ, i.e. $\gamma = \mathrm{Ad}\, w$. Let $k \in L^{\infty}(G_2)$. Then the element $\alpha(k)$, by definition, belongs to $L^{\infty}(G_1)\overline{\otimes}L^{\infty}(G_2)$. Since $w \in G(\widehat{\mathbf{K}})$ and $\widehat{\mathscr{M}} = L^{\infty}(G_1) \times_{\beta} G_2$, the unitary w lies particularly in $L^{\infty}(G_1)\overline{\otimes}\mathscr{L}(L^2(G_2))$, where $\mathscr{L}(\mathscr{K})$ stands for the algebra of all bounded operators on a Hilbert space $\mathscr{K}$. It follows that

$$\gamma(\alpha(k)) = w\alpha(k)w^{*} \in L^{\infty}(G_1)\overline{\otimes}\mathscr{L}(L^2(G_2)).$$

By Lemma 3.5 (a) of **[DeC1]**, the operator $\gamma(\alpha(k))$ belongs to the algebra $\alpha(L^{\infty}(G_2))$. Thus we get an inclusion $\gamma(\alpha(L^{\infty}(G_2)))\subseteq\alpha(L^{\infty}(G_2))$. By applying the same argument as above to the automorphism $\gamma^{-1} \in \mathscr{G}(\mathbf{K})$, we obtain the reverse inclusion. Hence γ leaves $\alpha(L^{\infty}(G_2))$ globally invariant.

By the first paragraph, γ induces an automorphism $\tilde{\gamma}$ of $L^{\infty}(G_2)$ so that $\gamma \circ \alpha = \alpha \circ \tilde{\gamma}$. Then, due to Lemma 2.1, we get

$$\begin{aligned}
(\alpha \otimes \alpha) \circ \Gamma_{G_2} \circ \tilde{\gamma} &= \Gamma \circ \alpha \circ \tilde{\gamma} = \Gamma \circ \gamma \circ \alpha \\
&= (\gamma \otimes \iota_{\mathscr{M}}) \circ \Gamma \circ \alpha = (\gamma \otimes \iota_{\mathscr{M}}) \circ (\alpha \otimes \alpha) \circ \Gamma_{G_2} \\
&= (\gamma \circ \alpha \otimes \alpha) \circ \Gamma_{G_2} = (\alpha \otimes \alpha) \circ (\tilde{\gamma} \otimes \iota_{L^{\infty}(G_2)}) \circ \Gamma_{G_2}.
\end{aligned}$$

Since α is injective, it follows that

$$(\tilde{\gamma} \otimes \iota_{L^\infty(G_2)}) \circ \Gamma_{G_2} = \Gamma_{G_2} \circ \tilde{\gamma}.$$

From Theorem 1.1 and the fact (mentioned in the last section) that $\mathscr{G}(\mathbf{KA}(G_2)) \cong G(\mathbf{KS}(G_2)) = \lambda_2(G_2)$, there exists a unique element s_0 in G_2 such that

$$\tilde{\gamma} = \operatorname{Ad} \lambda_2(s_0).$$

Our aim is to show that the element s_0 really belongs to G_2^α. For this, we first note that $\lambda_1(g) L^\infty(G_1) \lambda_1(g)^* = L^\infty(G_1)$ for any $g \in G_1$; so the fact that $w \in L^\infty(G_1) \overline{\otimes} \mathscr{L}(L^2(G_2))$ implies that

$$(\lambda_1(g)^* \otimes 1) w (\lambda_1(g) \otimes 1) \in L^\infty(G_1) \overline{\otimes} \mathscr{L}(L^2(G_2)) \qquad (g \in G_1).$$

Accordingly, the element

$$(\lambda_1(g)^* \otimes 1) w (\lambda_1(g) \otimes 1) \cdot w^* = (\lambda_1(g)^* \otimes 1) \gamma(\lambda_1(g) \otimes 1)$$

lies in $L^\infty(G_1) \overline{\otimes} \mathscr{L}(L^2(G_2))$. Since $(\lambda_1(g)^* \otimes 1) \gamma(\lambda_1(g) \otimes 1)$ is, at the same time, a member of $\mathscr{M} = L^\infty(G_2) \times_\alpha G_1$, it follows from Lemma 3.5 (a) of **[DeC1]** that there exists an element $k_g \in L^\infty(G_2)$, depending upon $g \in G_1$, such that

$$(\lambda_1(g)^* \otimes 1) \gamma(\lambda_1(g) \otimes 1) = \alpha(k_g).$$

Since the left-hand side is a unitary operator, we have that $|k_g| = 1$. Then, for any $k \in L^\infty(G_2)$, we calculate

$$\begin{aligned}
\alpha \circ \tilde{\gamma} \circ \alpha_g(k) &= \gamma \circ \alpha \circ \alpha_g(k) \\
&= \gamma((\lambda_1(g) \otimes 1) \alpha(k) (\lambda_1(g)^* \otimes 1)) \\
&= \gamma(\lambda_1(g) \otimes 1) \gamma(\alpha(k)) \gamma(\lambda_1(g)^* \otimes 1) \\
&= (\lambda_1(g) \otimes 1) \alpha(k_g) \cdot \alpha(\tilde{\gamma}(k)) \cdot \alpha(k_g^*) (\lambda_1(g)^* \otimes 1) \\
&= (\lambda_1(g) \otimes 1) \alpha(k_g \tilde{\gamma}(k) k_g^*) (\lambda_1(g)^* \otimes 1) \\
&= (\lambda_1(g) \otimes 1) \alpha(\tilde{\gamma}(k)) (\lambda_1(g)^* \otimes 1) \\
&= \alpha \circ \alpha_g \circ \tilde{\gamma}(k).
\end{aligned}$$

Since α is injective, it follows that $\tilde{\gamma} \circ \alpha_g(k) = \alpha_g \circ \tilde{\gamma}(k)$ for any $g \in G_1$ and $k \in L^\infty(G_2)$. This is, in turn, equivalent to

$$k(\alpha_g(s_0^{-1} s)) = k(s_0^{-1} \alpha_g(s)) \qquad (g \in G_1,\, s \in G_2).$$

Here one can take k to be an arbitrary continuous function with compact support. This means that

$$\alpha_g(s_0^{-1} s) = s_0^{-1} \alpha_g(s) \qquad (g \in G_1,\, s \in G_2).$$

In particular, we have that $\alpha_g(s_0^{-1}) = s_0^{-1}$, which implies that $s_0 \in G_2^\alpha$. Moreover,

$$\begin{aligned}
\{(1 \otimes \lambda_2(s_0))\alpha(k)(1 \otimes \lambda_2(s_0)^*)\eta\}(g, s) \\
&= \{\alpha(k)(1 \otimes \lambda_2(s_0)^*)\eta\}(g, s_0^{-1}s) \\
&= k(\alpha_g(s_0^{-1}s))\{(1 \otimes \lambda_2(s_0)^*)\eta\}(g, s_0^{-1}s) \\
&= k(s_0^{-1}\alpha_g(s))\eta(g, s) = \tilde{\gamma}(k)(\alpha_g(s))\eta(g, s) \\
&= \{\alpha(\tilde{\gamma}(k))\eta\}(g, s) = \{\gamma(\alpha(k))\eta\}(g, s)
\end{aligned}$$

for any $k \in L^\infty(G_2)$ and $\eta \in \mathscr{H}$. Therefore, we conclude that

$$\gamma|_{\alpha(L^\infty(G_2))} = \mathrm{Ad}(1 \otimes \lambda_2(s_0))|_{\alpha(L^\infty(G_2))}.$$

This completes the proof. $\square$

LEMMA 2.4. *Let γ be in $\mathscr{G}(\mathbf{K})$ with $\gamma|_{\alpha(L^\infty(G_2))} = \mathrm{id}$. (This assumption makes sense due to Lemma 2.3.) Then there uniquely exists an element $f \in G(\mathbf{KA}(G_1))$ such that $\gamma = \mathrm{Ad}\,\beta(f)$.*

Proof. Uniqueness follows from Theorem 1.1 and Lemma 2.2.

By Theorem 1.1, there is a unique element w in $G(\widehat{\mathbf{K}})$ such that $\gamma = \mathrm{Ad}\,w$. Let U be the unitary operator introduced before. Then, by assumption, we have

$$U^*(1 \otimes k)U = \alpha(k) = \gamma(\alpha(k)) = wU^*(1 \otimes k)Uw^*$$

for any $k \in L^\infty(G_2)$. The computation shows that the operator UwU^* belongs to $\mathscr{L}(L^2(G_1))\overline{\otimes}L^\infty(G_2)$. It is easy to see, by definition, that $[U, a \otimes 1] = 0$ for any $a \in L^\infty(G_1)$, where the symbol $[p, q]$ stands for the commutator: $[p, q] = pq - qp$. Thus $U \in L^\infty(G_1)\overline{\otimes}\mathscr{L}(L^2(G_2))$. Since w is in $\widehat{\mathscr{M}} = L^\infty(G_1)\times_\beta G_2$, it lies also in $L^\infty(G_1)\overline{\otimes}\mathscr{L}(L^2(G_2))$. Accordingly, UwU^* belongs to $L^\infty(G_1)\overline{\otimes}\mathscr{L}(L^2(G_2))$. It follows that

$$\begin{aligned}
UwU^* &\in L^\infty(G_1)\overline{\otimes}\mathscr{L}(L^2(G_2)) \cap \mathscr{L}(L^2(G_1))\overline{\otimes}L^\infty(G_2) \\
&= L^\infty(G_1)\overline{\otimes}L^\infty(G_2).
\end{aligned}$$

Since the fact that $[U, a\otimes 1] = 0$ $(a \in L^\infty(G_1))$ and $U(\mathbf{C}\otimes L^\infty(G_2))U = \alpha(L^\infty(G_2))$ implies that

$$U^*(L^\infty(G_1)\overline{\otimes}L^2(G_2))U \subseteq L^\infty(G_1)\overline{\otimes}L^\infty(G_2),$$

the operator w itself belongs to $L^\infty(G_1)\overline{\otimes}L^\infty(G_2)$. By Lemma 3.5 (a) of **[DeC1]** again, there exists an element f in $L^\infty(G_1)$ such that

$$|f| = 1, \quad w = \beta(f).$$

Due to Lemma 2.1, we have that $\widehat{\Gamma} \circ \beta = (\beta \otimes \beta) \circ \Gamma_{G_1}$. Hence we obtain

$$(\beta \otimes \beta) \circ \Gamma_{G_1}(f) = \widehat{\Gamma}(\beta(f)) = \widehat{\Gamma}(w)$$
$$= w \otimes w = (\beta \otimes \beta)(f \otimes f).$$

Since β is injective, we have $\Gamma_{G_1}(f) = f \otimes f$. Therefore, $f \in G(\mathbf{KA}(G_1))$. □

We are now in a position to prove our main theorem.

THEOREM 2.5. *The intrinsic group $G(\mathbf{K})$ of the bicrossproduct Kac algebra $\mathbf{K} = (\mathscr{M}, \Gamma, \kappa, \varphi)$, where $\mathscr{M} = L^\infty(G_2) \times_\alpha G_1$, associated with a modular matched pair $(G_1, G_2, \alpha, \beta)$ is topologically isomorphic to the semi-direct product $G(\mathbf{KA}(G_2)) \times_\alpha G_1^\beta$. Here the product of the semi-direct product is given by*

$$(k_1, g_1) \cdot (k_2, g_2) = (\alpha_{g_2^{-1}}(k_1)k_2, g_1 g_2).$$

Similarly, the intrinsic group $G(\widehat{\mathbf{K}})$ of the other bicrossproduct Kac algebra $\widehat{\mathbf{K}}$ is topologically isomorphic to the semi-direct product $G(\mathbf{KA}(G_1)) \times_\beta G_2^\alpha$.

Proof. By symmetry, it suffices to prove the last half assertion.

Let $w \in G(\widehat{\mathbf{K}})$. We put $\gamma = \operatorname{Ad} w \in \mathscr{G}(\mathbf{K})$. By Lemma 2.3, there exists a unique element s_0 in G_2^α such that

$$\gamma|_{\alpha(L^\infty(G_2))} = \operatorname{Ad}(1 \otimes \lambda_2(s_0))|_{\alpha(L^\infty(G_2))}.$$

We set $\gamma_1 = \operatorname{Ad}(1 \otimes \lambda_2(s_0)^*)$. By Lemma 2.2 and Theorem 1.1, the morphism γ_1 belongs to $\mathscr{G}(\mathbf{K})$; so $\gamma_1 \circ \gamma$ also lies in $\mathscr{G}(\mathbf{K})$. By construction, the restriction of $\gamma_1 \circ \gamma$ to $\alpha(L^\infty(G_2))$ is the identity. Thus, by Lemma 2.4, there is a unique element f in $G(\mathbf{KA}(G_1))$ such that

$$\gamma_1 \circ \gamma = \operatorname{Ad} \beta(f).$$

It follows that

$$\operatorname{Ad} w = \gamma = \gamma_1^{-1} \circ \operatorname{Ad} \beta(f) = \operatorname{Ad}(1 \otimes \lambda_2(s_0))\beta(f).$$

Lemma 2.2 ensures that $(1 \otimes \lambda_2(s_0))\beta(f)$ is in $G(\widehat{\mathbf{K}})$. From Theorem 1.1 and the uniqueness of canonical implementation, it results that

$$w = (1 \otimes \lambda_2(s_0))\beta(f).$$

This shows that the map Φ from $G(\mathbf{KA}(G_1)) \times_\beta G_2^\alpha$ into $G(\widehat{\mathbf{K}})$ which sends (f, s) to $(1 \otimes \lambda_2(s))\beta(f)$ is bijective. Since

$$(1 \otimes \lambda_2(s_1))\beta(f_1)(1 \otimes \lambda_2(s_2))\beta(f_2) = (1 \otimes \lambda_2(s_1 s_2))\beta(\beta_{s_2^{-1}}(f_1)f_2)$$

for any $f_i \in G(\mathbf{KA}(G_1))$ and $s_i \in G_2^\alpha$ $(i = 1, 2)$, the map Φ is a group isomorphism as well, when $G(\mathbf{KA}(G_1)) \times_\beta G_2^\alpha$ is endowed with the product

$$(f_1, s_1) \cdot (f_2, s_2) = (\beta_{s_2^{-1}}(f_1)f_2, s_1 s_2).$$

Bicontinuity of Φ can be proven exactly by the same argument as in Proposition 3.4 of **[DeC1]**. □

REMARK 2.6. Let us consider the special case in which $\beta = \mathrm{id}$ (or $\alpha = \mathrm{id}$). Namely, α is an action of G_1 on G_2 by automorphisms. Such an example was treated in Example 5.3 of **[DeC2]**. Due to Theorem 2.5, the intrinsic groups $G(\mathbf{K})$ and $G(\widehat{\mathbf{K}})$ are $G(\mathbf{KA}(G_2)) \times_\alpha G_1$, $G(\mathbf{KA}(G_1)) \times G_2^\alpha$, respectively. Thus we can recover Proposition 3.4 and 3.6 of **[DeC1]** as a special case of ours, where $\mathbf{K}$ in these propositions should be taken as $\mathbf{K} = (L^\infty(G_2), \Gamma_{G_2}, j_{G_2}, \tau_{G_2})$.

Before we state the following corollary to Theorem 2.5, we recall the definition of the bicharacter of a Kac algebra $\mathbf{K}$. ($\mathbf{K}$ is a general Kac algebra for the moment.) For any $u \in G(\mathbf{K})$ and $v \in G(\widehat{\mathbf{K}})$, there exists a complex number $\Omega(u, v)$ of modulus 1 such that

$$uv = \Omega(u, v)vu.$$

(See **[S]** for properties that Ω enjoys.) The map Ω is called the bicharacter of the Kac algebra $\mathbf{K}$.

COROLLARY 2.7. *Let* $\mathbf{K}$ *be as in Theorem* 2.5. *Then the bicharacter* Ω *of* $\mathbf{K}$ *is given by*

$$\Omega((k, g), (f, s)) = k(s)\overline{f(g)}$$

for any $(k, g) \in G(\mathbf{K})$ *and* $(f, s) \in G(\widehat{\mathbf{K}})$.

Proof. This is easily verified by direct computations of

$$(\lambda_1(g) \otimes 1)\alpha(k)(1 \otimes \lambda_2(s))\beta(f)$$

and

$$(1 \otimes \lambda_2(s))\beta(f)(\lambda_1(g) \otimes 1)\alpha(k). \quad \square$$

3. Calculation. In **[M3]** (see also **[M1,2]**), Majid constructed a concrete example of a modular matched pair of Lie groups. His method shows that, to each group $T_1(n, \mathbf{R})$ of $n \times n$ upper triangular matrices in $\mathbf{R}$ with 1 on the diagonal, there corresponds a modular matched pair $(G_1, G_2, \alpha, \beta)$, where $G_1 = G_2 = T_1(n, \mathbf{R})$ and $\alpha = \beta$. In particular, the resulting bicrossproduct Kac algebra $\mathbf{K}$ is self-dual. As an application of the preceding section, we shall compute the intrinsic group $G(\mathbf{K})$ of $\mathbf{K}$ that arises in the case of $T_1(3, \mathbf{R})$, the Heisenberg group.

We let $G = T_1(3, \mathbf{R})$. We define an action α of G on itself by

$$\alpha_g(s) = (1 + g(s^{-1} - 1))^{-1} \qquad (g, s \in G).$$

In terms of a matrix form, this is equivalent to

$$\alpha_g(s) = \begin{pmatrix} 1 & x & y + az \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

where

$$g = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

By Theorem 2.5, the intrinsic group $G(\mathbf{K})$ where $\mathbf{K} = (L^\infty(G) \times_\alpha G, \Gamma, \kappa, \varphi)$ as in the previous section, is isomorphic to $G(\mathbf{KA}(G)) \times_\alpha G^\alpha$. Thus we need to investigate what $G(\mathbf{KA}(G))$, G^α and the action α really are. First, by (3.1), it is not difficult to check that

$$G^\alpha = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbf{R} \right\},$$

which is known to be a closed, normal, maximal abelian subgroup of G. It is clear that G^α is isomorphic to the additive group $(\mathbf{R}^2, +)$. Next we look at $G(\mathbf{KA}(G))$. It is known in general (see §3 of **[DeC1]**) that $G(\mathbf{KA}(G))$ = the set of "group characters" of G can be identified with the Pontryagin dual of the abelian group $G/[G, G]$, where $[G, G]$ is the closed commutator subgroup of G. This identification is a topological isomorphism in our case. So we first look at $[G, G]$. It can be verified that

$$[G, G] = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathbf{R} \right\},$$

which is the center of G. Thus the quotient group $G/[G, G]$ is isomorphic to the additive group $(\mathbf{R}^2, +)$ by the correspondence:

$$\left[\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\right] \in G/[G, G] \leftrightarrow (a, c) \in \mathbf{R}^2.$$

Hence $G(\mathbf{KA}(G))$ consists of functions $A_{\lambda, \mu}$ $(\lambda, \mu \in \mathbf{R})$ on G given by

$$A_{\lambda, \mu}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = e^{i(\lambda a + \mu c)}.$$

It follows that $G(\mathbf{KA}(G))$ is isomorphic to the additive group $(\mathbf{R}^2, +)$. Next we examine the action α of G^α on $G(\mathbf{KA}(G))$. By (3.1), we have

$$\begin{aligned} \alpha_{g^{-1}}(A_{\lambda, \mu})\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} &= A_{\lambda, \mu}\left(\alpha_g\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= A_{\lambda, \mu}\begin{pmatrix} 1 & x & y + az \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \\ &= e^{i(\lambda x + \mu z)} \\ &= A_{\lambda, \mu}\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $g \in G^\alpha$ and $A_{\lambda, \mu}$ is as before. This means that the action α on $G(\mathbf{KA}(G))$ is trivial; so the semi-direct product $G(\mathbf{KA}(G)) \times_\alpha G^\alpha$ is in fact the direct product. Consequently, the intrinsic group $G(\mathbf{K})$ is isomorphic to $(\mathbf{R}^2, +) \times (\mathbf{R}^2, +) = (\mathbf{R}^4, +)$. Since $\mathbf{K}$ is self-dual, $G(\widehat{\mathbf{K}}) \cong (\mathbf{R}^4, +)$.

Now it is obvious that, with $\mathbf{K}_1 = (\mathscr{R}(\mathbf{R}^4), \delta_{\mathbf{R}^4}, \kappa_{\mathbf{R}^4}, \varphi_{\mathbf{R}^4})$, $G(\mathbf{K}_1) = G(\widehat{\mathbf{K}}_1) \cong (\mathbf{R}^4, +)$. Thus we have $G(\mathbf{K}) \cong G(\mathbf{K}_1)$, $G(\widehat{\mathbf{K}}) \cong G(\widehat{\mathbf{K}}_1)$. We now look at the bicharacters Ω and Ω_1 of $\mathbf{K}$ and $\mathbf{K}_1$, respectively. Let

$$u = \left(A_{\lambda_1, \lambda_2}, \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \in G(\mathbf{K}),$$

$$v = \left(A_{\mu_1, \mu_2}, \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \in G(\widehat{\mathbf{K}}) = G(\mathbf{K}),$$

where A_{λ_1,λ_2}, A_{μ_1,μ_2} $(\lambda_i, \mu_i \in \mathbf{R}$ $(i = 1, 2))$ are as before. Then, by Corollary 2.7, we have

$$\Omega(u, v) = A_{\lambda_1,\lambda_2}\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \overline{A_{\mu_1,\mu_2}\begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = e^{i\lambda_1 a} \cdot e^{-i\mu_1 x}.$$

In the meantime, under the isomorphisms $G(\mathbf{K}) \cong G(\mathbf{K}_1)$, $G(\widehat{\mathbf{K}}) \cong G(\widehat{\mathbf{K}}_1)$, we have

$$\Omega_1(u, v) = e^{-i(\lambda_1\mu_1 + \lambda_2\mu_2 + xa + yb)}.$$

Hence the bicharacters Ω and Ω_1 do not coincide. Therefore we have established an interesting example of nonisomorphic Kac algebras $\mathbf{K}$ and $\mathbf{K}_1$ for which $G(\mathbf{K}) \cong G(\mathbf{K}_1)$, $G(\widehat{\mathbf{K}}) \cong G(\widehat{\mathbf{K}}_1)$ and the associated bicharacters do not coincide.

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