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TOPOLOGICAL FULL GROUPS AND STRUCTURE OF NORMALIZERS IN TRANSFORMATION GROUP C^*-ALGEBRAS

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In memory of late Henry A. Dye.

Given a topological dynamical system $\Sigma = (X, \sigma)$ for a homeomorphism σ on a compact space, we define the topological full group $[Z_\sigma]$ with respect to the action σ. We then clarify the relations between normalizers in the transformation group C^*-algebra $A(\Sigma)$ and those homeomorphisms in the group $[Z_\sigma]$. The result implies the general isomorphism theorem between transformation group C^*-algebras keeping their subalgebras of continuous functions.

Introduction.

Let $A(\Sigma)$ be the transformation group C^*-algebra associated to a topological dynamical system $\Sigma = (X, \sigma)$ on a compact Hausdorff space X with a homeomorphism σ. In the present paper, under a relatively mild condition for the dynamical system Σ, topological freeness, we shall clarify the structure of normalizers in $A(\Sigma)$, that is, of those unitary elements of $A(\Sigma)$ whose adjoint automorphisms leave $C(X)$, the algebra of continuous functions on X, invariant. As in the case of measurable dynamical systems for non-singular free actions, the analysis invokes the suitable definition of the topological full group $[Z_\sigma]$ with respect to the given homeomorphism σ. With this notion, we shall show (Theorem 1) the C^*-version of the correspondence between normalizers and those homeomorphisms belonging to the group $[Z_\sigma]$, together with the result to show when the C^*-subalgebra generated by $C(X)$ and a normalizer coincides with the whole algebra $A(\Sigma)$. The theorem then naturally induces an isomorphism theorem for transformation group C^*-algebras keeping their subalgebras of continuous functions.

From the point of view of operator algebras one would be tempted to discuss the problem under more general actions such as actions of countable discrete groups, but we are mainly interested in the interplay between topological dynamics and theory of C^*-algebras. Since then essential ingredients of the problem are used to be appeared in the present context, we restrict our discussion to the case of the topological dynamical system for a single homeomorphism.

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1. Preliminaries.

For the topological dynamical system $\Sigma = (X, \sigma)$, we impose no countability condition on the space X. Let α be the automorphism of $C(X)$, defined by $\alpha(f)(x) = f(\sigma^{-1}(x))$. We regard α also as an action of the group of all integers Z and denote sometimes by $\{C(X), Z, \alpha\}$, the C^*-dynamical system. The transformation group C^*-algebra $A(\Sigma)$ is then defined as the C^*-crossed product $C(X) \times_\alpha Z$ with respect to the action α, and since we are treating a Z-action, it coincides with the reduced crossed product $C(x) \times_{\alpha r} Z$. Write δ and E as the generating unitary element of $A(\Sigma)$ and the canonical faithful projection of norm one of $A(\Sigma)$ to $C(X)$ respectively. We write a representation of $A(\Sigma)$ always as $\tilde{\pi} = \pi \times u$ where $\pi = \tilde{\pi}|C(X)$, the restriction of $\tilde{\pi}$ to $C(X)$, and $u = \tilde{\pi}(\delta)$. It is then not so much confusing, as in the case of the action α, to mean u also a unitary representation of the group Z. For a point $x \in X$, we denote $O_\sigma(x)$ the orbit of x with respect to σ. By the sets $Per(\sigma)$ and $Aper(\sigma)$ we mean the set of all periodic points and that of all aperiodic points for Σ. We call a system $\Sigma = (X, \sigma)$ topologically free when the set $Per(\sigma)$ is of the first category, or equivalently $Aper(\sigma)$ is dense in X. In a dynamical system the set $Per(\sigma)$ plays an important role but in usual dynamical systems such as the ones on compact manifolds those sets of periodic points are often at most countable. Thus, they are all topologically free systems in our sense. Since however this concept may not be widely recognized, we should collect here those relevant results that we need for our discussions. We refer the article [**8**] or [**9**] for them.

Recall that a dynamical system σ is topologically transitive whenever any pair of open sets U, V in X meets each other after some iteration of the map σ, that is, $\sigma^n U \cap V \neq \phi$ for some integer n. It then can be shown that a topologically transitive system on an infinite compact space is necessarily topologically free.

Theorem A ([**8**, Theorem 4.3.5]). *For a dynamical system $\Sigma = (X, \sigma)$ the following three assertions are equivalent:*

(1) *Σ is topologically free,*

(2) *A closed ideal I of $A(\Sigma)$ is nonzero if and only if $I \bigcap C(X) \neq \{0\}$,*

(3) *$C(X)$ is a maximum abelian subalgebra of $A(\Sigma)$.*

Next, consider a representation $\tilde{\pi} = \pi \times u$ and let X_π be the spectrum of the C^*-subalgebra $\pi(C(X))$. As $\{\pi, u\}$ is a covariant representation of $\{C(X), Z, \alpha\}$, the automorphism Adu on $\pi(C(X))$ induces a homeomorphism σ_π and we obtain a dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$. We call this system Σ_π the derived dynamical system by the representation $\tilde{\pi} = \pi \times u$ or by the covariant representation $\{\pi, u\}$. It is then worth to notice that the system Σ_π can be identified with the restricted dynamical system $\{X'_\pi, \sigma'_\pi\}$ for an invariant closed subset X'_π of X defined by $\pi^{-1}(0) = k(X'_\pi)$, the kernel of X'_π, and the restriction of σ to X'_π, σ'_π, through the isomorphism

$$C(X'_\pi) \cong C(X)/\pi^{-1}(0) \cong \pi(C(X)) = C(X_\pi).$$

Theorem B ([**9**, Theorem 5.1] or [**8**, Theorem 4.3.1]). *Let $\tilde{\pi} = \pi \times u$ be a representation of $A(\Sigma)$ and suppose that the dynamical system $\Sigma_\pi = (X_\pi, \sigma_\pi)$ is topologically free. There exists then a projection of norm one ε_π from $\tilde{\pi}(A(\Sigma))$ to $\pi(C(X))$ such that*

$$\varepsilon_\pi \circ \tilde{\pi} = \pi \circ E.$$

As immediate consequences of this result we can conclude that in this situation:

(a) $\tilde{\pi}$ is an isomorphism if and only if π is an isomorphism,

(b) $\tilde{\pi}(A(\Sigma))$ is canonically isomorphic to the transformation group C^*-algebra $A(\Sigma_\pi)$.

We notice that every infinite dimensional factor representation (hence in particular irreducible representation) satisfies the above condition (cf. [**T1**, Proposition 4.3.2]). In fact, in that case the system Σ_π turns out to be topologically transitive and moreover the space X_π is an infinite set.

Now we call a unitary element v of $A(\Sigma)$ a normalizer (of $C(X)$) if the automorphism Adv leaves $C(X)$ invariant. Let τ be the homeomorphism of X induced by Adv and write $\Sigma_\tau = (X, \tau)$. We denote by $A(v)$ the C^*-subalgebra of $A(\Sigma)$ generated by $C(X)$ and v. Denote by $O_\sigma(x)$ and $O_\tau(x)$ the orbits of a point x for σ and τ respectively.

Lemma 1. *For every point x, we have that $\tau(O_\sigma(x)) = O_\sigma(x)$.*

Proof. Let $A(\Sigma_\tau)$ be the associated transformation group C^*-algebra for the dynamical system Σ_τ with the generating unitary element δ_τ. There exists then the canonical homomorphism q from $A(\Sigma_\tau)$ to the C^*-subalgebra $A(v)$ for which $q(\delta_\tau) = v$ and q brings the embedded image of $C(X)$ into $A(\Sigma_\tau)$ to $C(X)$ in $A(\Sigma)$. For the evaluation functional μ_x on $C(X)$ at the point

x, let ϕ be a pure state extension of μ_x to $A(v)$ and $\tilde{\phi}$ a further pure state extension of ϕ to $A(\Sigma)$. Put $\varphi = \phi \circ q$, then it may also be regarded as a pure state extension of μ_x to the C^*-algebra $A(\Sigma_\tau)$. Let $\{H, \tilde{\pi} = \pi \times u, \eta\}$ be the GNS-representation of $A(\Sigma)$ by the pure state $\tilde{\phi}$ with the canonical map η of $A(\Sigma)$ into H. The GNS-representation of $A(\Sigma_\tau)$ by φ is then compatible with this structure. Namely, denoting by $\{H_\varphi, \tilde{\pi_\varphi}, \eta_\varphi\}$ and $\{H_\phi, \pi_\phi, \eta_\phi\}$ the GNS-representations of $A(\Sigma_\tau)$ and $A(v)$ by φ and ϕ respectively we have that $\eta_\varphi = \eta_\phi \circ q$ and $\eta_\varphi(A(\Sigma_\tau))$ is considered as a subspace of $\eta(A(\Sigma))$ as well as its completion $H_\varphi = H_\phi$ in the space H. As it is then shown by [**9**, Proposition 4.3] or [**8**, p. 86], the unit vectors

$$e_n = \eta(\delta^n) \quad \text{and} \quad f_n = \eta_\varphi(\delta_\tau^n) = \eta_\phi(v^n) = \eta(v^n)$$

form complete orthonormal bases in H and H_ϕ where n is ranging over all integers if x is aperiodic for both maps σ and τ, $0 \leq n < p$ for e_n if x is p-periodic for σ and $0 \leq n < p'$ for f_n if it is p'-periodic for τ. Moreover for a continuous function g, we have that

$$\pi(g)e_n = g(\sigma^n(x))e_n \quad \text{and} \quad \pi_\varphi(g)f_n = \pi_\phi(g)f_n = g(\tau^n(x))f_n.$$

Now suppose that $\tau(x)$ does not belong to $O_\sigma(x)$, then for any n ($0 \leq n < p$ when x is p-periodic) $\tau(x) \neq \sigma^n(x)$ and we get a continuous function g such that $g(\tau(x)) = 1$ and $g(\sigma^n(x)) = 0$. It follows that

$$\begin{aligned}(f_1, e_n) = (\pi_\varphi(g)f_1, e_n) = (\pi(g)f_1, e_n) = \\ = (f_1, \pi(\bar{g})e_n) = (f_1, \overline{g(\sigma^n(x))}e_n) = 0,\end{aligned}$$

that is, the vector f_1 is orthogonal to the complete orthogonal basis $\{e_n\}$ in H, a contradiction. Namely, $\tau(O_\sigma(x)) \subset O_\sigma(x)$ and similarly we get the inclusion $\tau^{-1}(O_\sigma(x)) \subset O_\sigma(x)$. Hence, $\tau(O_\sigma(x)) = O_\sigma(x)$. □

2. Topological Full Groups and Structure of Normalizers.

Lemma 2. *Let x be an aperiodic point for σ . Then the following assertions are equivalent, where we keep the same assumptions for v and τ as above;*

(1) $\tau^i(x) = \sigma^n(x)$,

(2) $\eta(v^i) = \lambda(x)\eta(\delta^n)$ *for some complex number* $\lambda(x)$ *with* $|\lambda(x)| = 1$ *(and consequently* $\eta(v^j)$ *and* $\eta(\delta^m)$ *are orthogonal to* $\eta(\delta^n)$ *and* $\eta(v^i)$ *respectively for any integers* $j \neq i$ *and* $m \neq n$*).*

(3) $|v^{*i}(-n)(x)| = 1$ *where* $v^{*i}(-n)$ *is the* $(-n)$*-th Fourier coefficient of* v^{*i} *in* $A(\Sigma)$ *(and consequently* $v^{*j}(-n)(x) = 0$ *and* $v^{*i}(-m)(x) = 0$ *for any integers* $j \neq i$ *and* $m \neq n$ *respectively).*

Proof. Note first that in this case the pure state expression of μ_x to $A(\Sigma)$ is unique, and the extension $\tilde{\phi}$ in the proof of Lemma 1 must coincide with the state $\mu_x \circ E$ (cf. [**8**, Theorem 3.3.7]). Thus we can use all relations appeared in the above proof together with the equality $\tilde{\phi} = \mu_x \circ E$. We shall prove the Lemma in a way; (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Assume now that $\tau^i(x) = \sigma^n(x)$ and write

$$\eta(v^i) = \sum_{n \in Z} \lambda_n e_n \quad \text{for} \quad e_n = \eta(\delta^n).$$

If $k \neq n$, then $\sigma^k(x) \neq \sigma^n(x)$ and there exists a continuous function g such that $g(\tau^i(x)) = g(\sigma^n(x)) = 1$ and $g(\sigma^k(x)) = 0$. It follows that

$$\begin{aligned}(\eta(v^i), e_k) &= (g(\tau^i(x))\eta(v^i), e_k) = (\pi_\varphi(g)\eta(v^i), e_k) \\ &= (\pi(g)\eta(v^i), e_k) = (\eta(v^i), \pi(\bar{g})e_k) \\ &= (\eta(v^i), g(\sigma^k(x))e_k) = 0.\end{aligned}$$

Hence,

$$\eta(v^i) = \lambda_n e_n = \lambda_n \eta(\delta^n) \quad \text{and} \quad |\lambda_n| = 1.$$

Next, assume the assertion (2), then

$$\begin{aligned}&\tilde{\phi}((v^i - \lambda(x)\delta^n)^*(v^i - \lambda(x)\delta^n)) \\ &= \tilde{\phi}(2 - \lambda(x)v^{*i}\delta^n - \overline{\lambda(x)}\delta^{*n}v^i) \\ &= 2 - \lambda(x)E(v^{*i}\delta^n)(x) - \overline{\lambda(x)}E(\delta^{*n}v^i)(x) \\ &= 0.\end{aligned}$$

Hence

$$\lambda(x)v^{*i}(-n)(x) + \overline{\lambda(x)}v^{*i}(-n)^*(x) = 2$$

and since $|v^{*j}(-n)(x)| \leq 1$ we have that $|v^{*i}(-n)(x)| = 1$.

On the other hand, since the vector $\eta(v^j)$ (resp. $\eta(\delta^m)$) is orthogonal to $\eta(\delta^n)$ (resp. $\eta(v^i)$) if $j \neq i$ (resp. $m \neq n$)

$$v^{*j}(-n)(x) = (\eta(\delta^n), \eta(v^j)) = \bar{\lambda}(x)(\eta(v^i), \eta(v^j)) = 0,$$

$$(\text{resp.} v^{*i}(-m)(x) = (\eta(\delta^m), \eta(v^i)) = 0).$$

For the implication (3) $\Rightarrow$ (1) we have any way an equality, $\tau^i(x) = \sigma^k(x)$ for some k by Lemma 1. If $k \neq n$ here, taking the continuous function g such that $g(\tau^i(x)) = g(\sigma^k(x)) = 1$ and $g(\sigma^n(x)) = 0$, we easily reach a contradiction as in the same way as the implication (1) $\Rightarrow$ (2). Namely,

$$(\eta(v^i), e_n) = (\pi(g)\eta(v^i), e_n) = (\eta(v^i), \pi(\bar{g})e_n) = 0$$

whereas

$$|(\eta(v^i), e_n)| = |E(\delta^{*n}v^i)(x)| = |v^{*i}(-n)(x)| = 1.$$

This completes all proofs. □

Now we define the topological full group with respect to the homeomorphism σ.

Definition. For a dynamical system $\Sigma = (X, \sigma)$ the topological full group $[Z_\sigma]$ with respect to σ is the group of homeomorphisms τ such that there exists a continuous function $n : X \to Z$ defined as $\tau(x) = \sigma^{n(x)}(x)$.

Note that if $\tau(O_\sigma(x)) = O_\sigma(x)$ for all $x \in X$, the above function $n(x)$ is defined uniquely on the set $Aper(\sigma)$. Hence, when the system is topologically free this continuous function $n(x)$, if it exists, has to be unique for each homeomorphism $\tau \in [Z_\sigma]$.

We denote by $U(C(X))$ the group of all unitary elements in $C(X)$.

Lemma 3. *Let $\Sigma = (X, \sigma)$ be a topologically free dynamical system, then any homeomorphism τ in $[Z_\sigma]$ defines a normalizer v of $C(X)$ in $A(\Sigma)$ up to $U(C(X))$, whose adjoint automorphism Adv coincides with the one α_τ induced by τ on $C(X)$.*

Proof. Let $\{n_k | 1 \leq k \leq k(\tau)\}$ be the range of the continuous function $n(x)$ for τ and define the sets

$$X(\tau, n) = \{x \in X | n(x) = n\}.$$

The family $\{X(\tau, n_k)\}_{k=1}^{k(\tau)}$ then turns out to be a finite family of disjoint open and closed sets. Let $p(\tau, k)$ be the characteristic function of $X(\tau, n_k)$ and put

$$v = \sum_{k=1}^{k(\tau)} \delta^{n_k} p(\tau, k) = \sum_{k=1}^{k(\tau)} \alpha^{n_k}(p(\tau, k))\delta^{n_k}.$$

We assert that v is a normalizer of $C(X)$ in $A(\Sigma)$ satisfying $Adv = \alpha_\tau$ on $C(X)$. In fact, consider the sets

$$Y(\tau, n) = \{x \in X | n(\tau^{-1}(x)) = n\}.$$

We see then

$$\sigma^{n_k}(X(\tau, n_k)) = \tau(X(\tau, n_k)) = Y(\tau, n_k),$$

and $\alpha^{n_k}(p(\tau, k))$ is the characteristic function of $Y(\tau, n_k)$. Hence

$$\sum_{k=1}^{k(\tau)} p(\tau, k) = \sum_{k=1}^{k(\tau)} \alpha^{n_k}(p(\tau, k)) = 1.$$

On the other hand, one can verify that

$$p(\tau,k)\alpha^{n_l-n_k}(p(\tau,l))=\delta_{kl}p(\tau,k)$$

because there exists no aperiodic point x in $X(\tau,n_k)$ such that $\sigma^{n_k-nl}(x)\in X(\tau,n_l)$ if $k\neq l$ and the set $X(\tau,n_k)\cap Aper(\sigma)$ is dense in $X(\tau,n_k)$ by our assumption for Σ. Therefore,

$$vv^*=\sum_{k=1}^{k(\tau)}\delta^{n_k}p(\tau,k)\delta^{n_k^*}=\sum_{k=1}^{k(\tau)}\alpha^{n_k}(p(\tau,k))=1,$$

and

$$\begin{aligned}v^*v&=\sum_{k,l}p(\tau,k)\delta^{n_k^*}\delta^{n_l}p(\tau,l)\\&=\sum_{k,l}p(\tau,k)\alpha^{n_l-n_k}(p(\tau,l))\delta^{n_l-n_k}=\sum_{k=1}^{k(\tau)}p(\tau,k)=1.\end{aligned}$$

Finally for any function g in $C(X)$,

$$\begin{aligned}vgv^*&=\sum_{k=1}^{k(\tau)}\delta^{n_k}p(\tau,k)g\delta^{n_k^*}\\&=\sum_{k=1}^{k(\tau)}\alpha^{n_k}(p(\tau,k))\alpha^{n_k}(g)\\&=\sum_{k=1}^{k(\tau)}\alpha^{n_k}(p(\tau,k))\alpha_\tau(g)\\&=\alpha_\tau(g).\end{aligned}$$

The fact that v is determined up to $U(C(X))$ follows from part (3) of Theorem A. □

For the rest of the discussion we provide the following observation.

Lemma 4. *Let X be a compact Hausdorff space. If $\{F_n\}_{n\in Z}$ is a closed covering of X, then*

$$X=\overline{\bigcup_{n\in Z}F_n^\circ}$$

where F_n° means the interior of F_n.

Proof. Suppose that X does not coincide with the closure of $\bigcup_{n\in Z}F_n^\circ$. There exists an open set U whose closure is still contained in the complement of

$\overline{\bigcup_{n\in Z} F_n^\circ}$. We have that

$$\overline{U} = \bigcup_n (F_n \cap \overline{U})$$

hence by category theorem there exists a set $F_{n_0} \cap \overline{U}$ with nonempty interior in the space $\overline{U}$. Therefore, $F_{n_0} \cap \overline{U}$ must also contain an interior point in the whole space X, a contradiction. □

Now let $U(A(\Sigma))$ be the group of all unitary elements of $A(\Sigma)$ and denote $N(C(X), A(\Sigma))$ the subgroup of all normalizers of $C(X)$ in $U(A(\Sigma))$. We note that for a topologically free dynamical system the group $U(C(x))$ becomes a normal subgroup of $N(C(X), A(\Sigma))$ by part (3) of Theorem A. The following theorem then determines the structure of $N(C(X), A(\Sigma))$ in this dynamical system where X is an arbitrary compact Hausdorff space. The assertion (a) of the theorem was proved before by Putnam for minimal dynamical systems on the Cantor sets [**7**, Lemma 5.1 and Theorem 5.2].

Theorem 1. *Let $\Sigma = (X, \sigma)$ be a topologically free dynamical system. If $v \in N(C(X), A(\Sigma))$ we denote by τ_v the homeomorphism of X induced by Adv.*

(*a*) *The map $v \in N(C(X), A(\Sigma)) \to \tau_v^{-1} \in Homeo(X)$ induces an isomorphism between the factor group $N(C(X), A(\Sigma))/U(C(X))$ and $[Z_\sigma]$.*

(*b*) *The C^*-subalgebra $A(v)$ coincides with $A(\Sigma)$ if and only if σ belongs to $[Z_{\tau_v}]$, the topological full group with respect to the homeomorphism τ_v.*

If Σ is free, that is, $Aper(\sigma) = X$ the latter condition holds provided that $\sigma(O_{\tau_v}(x)) = O_{\tau_v}(x)$ for all $x \in X$.

Proof. Part of (a). By Lemma 1, $\tau_v(O_\sigma(x)) = O_\sigma(x)$ for every point x. Therefore we can define the function $n(x)$ on the set $Aper(\sigma)$ by $\tau_v(x) = \sigma^{n(x)}(x)$. We assert that it has a continuous extension to X. Define the sets,

$$X(\tau_v, n) = \{x \in X | \tau_v(x) = \sigma^n(x)\}.$$

We have then

$$X = \bigcup_{n\in Z} X(\tau_v, n)$$

and by Lemma 4 the space X coincides with the closure of the disjoint union of open sets $\{U_k\}$ where U_k means the nonempty interior of the set $X(\tau, n_k)$. In this situation, each set $\overline{U_k}$ is separated from the set $\overline{\bigcup_{l\neq k} U_l}$ by the continuous function $v^*(-n_k)(x)$ in such a way that

$$|v^*(-n_k)(x)| = 1 \quad \text{on } \overline{U_k},$$

whereas

$$v^*(-n_k)(x) = 0 \quad \text{on } \overline{\bigcup_{l \neq k} U_l}.$$

For, by Lemma 2 the function $v^*(-n_k)(x)$ takes such values on $U_k \cap Aper(\sigma)$ and $U_l \cap Aper(\sigma)(l \neq k)$ which are dense in $\overline{U_k}$ and $\overline{U_l}$ respectively. Thus each set $\overline{U_k}$ turns out to be open and closed. Let $p(\tau, k)$ be the characteristic function of $\overline{U_k}$. We have then for any function f in $C(X)$,

$$v^* f v p(\tau, k) = p(\tau, k)\delta^{n_k *} f \delta^{n_k}.$$

Hence,

$$v p(\tau, k)\delta^{n_k *} f = f v p(\tau, k)\delta^{n_k *},$$

and $vp(\tau, k)\delta^{n_k *}$ belongs to $C(X)$ by part (3) of Theorem A. Choose an element $\sum_{-n}^{n} f_i \delta^i$ such that

$$\left\| v - \sum_{-n}^{n} f_i \delta^i \right\| < 1$$

and suppose there exists an integer n_k with $|n_k| > n$. We have then,

$$\begin{aligned} 1 &= \|vp(\tau, k)\delta^{n_k *}\| \\ &= \left\| E\left(vp(\tau, k)\delta^{n_k *} - \sum_{-n}^{n} f_i \alpha^i(p(\tau, k))\delta^{i - n_k} \right) \right\| \\ &\leq \left\| v - \sum_{-n}^{n} f_i \delta^i \right\| \|p(\tau, k)\delta^{n_k *}\| < 1. \end{aligned}$$

This is a contradiction, and the family $\{\overline{U_k}\}$ must be a finite family, say $k = 1, 2, \ldots k(\tau)$. Hence

$$X = \bigcup_{k=1}^{k(\tau)} \overline{U_k}.$$

It is then obvious that the function $n(x)$ defined as $n(x) = n_k$ on $\overline{U_k}$ is the continuous extension of our previous function $n(x)$ on $Aper(\sigma)$.

Conversely as seen in Lemma 3 each homeomorphism τ in $[Z_\sigma]$ determines a normalizer v of $C(X)$ in $A(\Sigma)$ up to the unitary group $U(C(X))$. Therefore we have the conclusion.

(b) Suppose that $A(v) = A(\Sigma)$. Then for every point x the pure state ϕ and $\tilde{\phi}$ in the proof of Lemma 1 coincide each other, together with their representation spaces H_ϕ and H. Hence applying the same arguments there we easily see that the strict inclusion $O_\tau(x) \subset O_\sigma(x)$ never occurs in this situation, that is $O_\tau(x) = O_\sigma(x)$. Next we notice that the projection of the norm one from $A(\Sigma)$ to $C(X)$ is unique. In fact, if E' is an another projection of norm one to $C(X)$ the state $\mu_x \circ E'$ gives also a state extension of μ_x and

$$\mu_x \circ E = \mu_x \circ E' \text{ for an aperiodic point } x$$

because the state extension of μ_x to $A(\Sigma)$ is unique (cf. [**8**, Theorem 3.3.7]). Namely

$$E(a)(x) = E'(a)(x) \text{ on } Aper(\sigma) \text{ for every } a \in A(\Sigma),$$

whence $E(a) = E'(a)$. Thus the projection E may also be regarded as the canonical projection of norm one in the crossed product $A(v)$ identified with $A(\Sigma_{\tau_v})$ by remarks after Theorem B, and $E(v^n) = 0$ if $n \neq 0$. Therefore we can proceed the same arguments as in case (a) exchanging δ and v and conclude that there exists a continuous function $m(x)$ defined as $\sigma(x) = \tau^{m(x)}(x)$, namely $\sigma \in [Z_\tau]$.

For a converse, it suffices to notice that we can proceed our arguments in the same way as in the proof of Lemma 3 based on the set

$$X(\sigma, m) = \{x \in X | m(x) = m\}$$

exchanging the role of δ and v and show that δ is a finite linear combination of powers of v over $C(X)$. In fact, putting $w = \sum_k v^{m_k} p(\sigma, k)$ (finite sum) following the decomposition along with the range of $m(x)$ where $p(\sigma, k)$ is the characteristic function of the non-empty set $X(\sigma, m_k)$ we can see that w is a unitary element of $A(\Sigma)$ such that $Adw = Ad\delta$ on $C(X)$. It follows from part (3) of Theorem A that $w^*\delta$ is a unitary function u in $C(X)$ and $\delta = wu$ belongs to $A(v)$. Finally suppose that Σ_σ is free and $\sigma(O_{\tau_v}(x)) = O_{\tau_v}(x)$ for all x. Then the system Σ_{τ_v} becomes also free and we can define the sets,

$$X(\sigma, m) = \{x \in X | \sigma(x) = \tau^m(x)\}$$

without ambiguity. It follows by Lemma 2 that

$$|v^{*^m}(-1)(x)| = 1 \text{ on } X(\sigma, m) \text{ and } v^{*^m}(-1)(x) = 0 \text{ on } X(\sigma, m') \text{ if } m' \neq m.$$

Therefore, the family $\{X(\sigma, m_k)\}$ of nonempty sets becomes a family of disjoint open and closed sets whose union coincides with X. Hence the function $m(x)$ defined as m_k on $\{X(\sigma, m_k)\}$ is necessarily continuous. This completes the proof. □

In connection with the statement of part (b) of theorem 1, it would be worthwhile to notice Boyle's result [**1**, Theorem 2.6] from which we have the following corollary. Note that for a homeomorphism τ in $[Z_\sigma]$ we have already continuous function $n(x)$ for which $\tau(x) = \sigma^{n(x)}(x)$.

Corollary. *Let $\Sigma = (X, \sigma)$ be a topologically transitive dynamical system on a compact metric space X. Then the C^*-subalgebra $A(v)$ coincides with $A(\Sigma)$ if and only if τ_v is flip conjugate to σ.*

In [**1**, Theorem 2.7] Boyle has also analysed what happenes about flip conjugacy between τ and σ if we only assume the boundedness for $n(x)$ instead of continuity. When $\tau(O_\sigma(x)) = O_\sigma(x)$ for every point x we can anyway define the function $n(x)$ on the set $Aper(\sigma)$ and may extend it sometimes to a bounded function on X. The following example illustrates a gap between boundedness of $n(x)$ on $Aper(\sigma)$ and continuity of $n(x)$ on the whole space X in the present situation. Let $\Sigma = (T^2, \sigma)$ be the topologically free dynamical system on the 2-dimensional torus T^2 induced by matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let τ be the homeomorphism defined as σ on $[0, \frac{1}{2}] \times T$ and as σ^{-1} on $[\frac{1}{2}, 1] \times T$. Then

$$T^2 = X(\tau, 1) \cup X(\tau, -1)$$

where $n(x)$ may be defined as 1 on $(0, \frac{1}{2} \times T)$ and as -1 on $(\frac{1}{2}, 1) \times T$. Here we see that $O_\tau(x) = O_\sigma(x)$ for every point x and there exist no normalizer in $A(\Sigma)$ corresponding to the homeomorphism τ because of the lack of continuous extension of $n(x)$ to T^2. In fact, since T^2 is connected Theorem 1 says that the only normalizers in $A(\Sigma)$ are those elements $\{\lambda\delta^n | n \in Z, |\lambda| = 1\}$. This is also the case for the irrational rotation algebras on the torus. The above dynamical system is however not topologically transitive. We remark that two transformation group C^*-algebras $A(\Sigma_\sigma)$ and $A(\Sigma_\tau)$ are not isomorphic but there exist no isomorphism between them keeping their subalgebras of continuous functions because the maps σ and τ are not flip conjugate.

3. Isomorphism Theorem.

In the interplay between topological dynamics and theory of C^*-algebras it is one of the most basic problem to determine when two transformation group C^*-algebras are isomorphic in terms of topological dynamical systems. In connection with the Krieger's theorem for measurable dynamical systems of non-singular ergodic transformations and associated von Neuman factors, the concept of topological orbit equivalence is perhaps a good candidate to handle this problem. In fact, recent results by Giordano, Putnam and Skau [**4**] show that we have forwarded into this direction. Namely they have given complete descriptions about conditions for isomorphic relations between transformation group C^*-algebras and for those isomorphisms keeping their subalgebras of continuous functions as well when the systems are minimal dynamical systems on the Cantor set ([**4**, Theorem 2.1 and 2.4]). It seems to be however still a long way to reach the general isomorphism theorem.

Meanwhile, the above theorem leads us to the following general isomorphism theorem of transformation group C^*-algebras keeping their subalgebras of continuous functions.

Theorem 2. *Let $\Sigma_\sigma = (X, \sigma)$ and $\Sigma_\tau = (Y, \tau)$ be topologically free dynamical systems, then the algebras $A(\Sigma_\sigma)$ and $A(\Sigma_\tau)$ are isomorphic each other keeping their subalgebras $C(X)$ and $C(Y)$ if and only if there exists a homeomorphism h from X to Y such that $h(O_\sigma(x)) = O_\tau(h(x))$ for every point x in X (topological orbit equivalence) and both homeomorphisms $h^{-1} \circ \tau \circ h$ and $h \circ \sigma \circ h^{-1}$ on X and Y belong to $[Z_\sigma]$ and $[Z_\tau]$ respectively, that is, there exist continuous functions $m(x)$ and $n(x)$ on X such that*

$$h(\sigma(x)) = \tau^{m(x)}(h(x)) \text{ and } \tau(h(x)) = h(\sigma^{n(x)}(x)).$$

Proof. Suppose that Φ is an isomorphism from $A(\Sigma_\sigma)$ to $A(\Sigma_\tau)$ such that $\Phi(C(X)) = C(Y)$. Then the homeomorphism $h : X \to Y$ induced by Φ is the one such that $\Phi(f)(h(x)) = f(x)$ for every $f \in C(X)$ and $x \in X$. Put $v = \Phi^{-1}(\delta_\tau)$ for the generating unitary δ_τ of $A(\Sigma_\tau)$ and $\tilde{\tau} = h^{-1} \circ \tau \circ h$. We then easily see that v is a normalizer of $C(X)$ associated to the homeomorphism $\tilde{\tau}$ and $A(v) = A(\Sigma_\sigma)$. Hence, by part (b) of Theorem 1, $\tilde{\tau}$ belongs to $[Z_\sigma]$. Similarly we see the homeomorphism $h \circ \sigma \circ h^{-1}$ on Y belongs to $[Z_\tau]$.

Conversely if we have such a homeomorphism h from X to Y we see that $\tilde{\tau} = h^{-1} \circ \tau \circ h$ gives rise to a normalizer v such that $(vfv^*)(x) = f(\tilde{\tau}^{-1}(x))$ for $f \in C(X)$. Moreover, we have that

$$\sigma(x) = \tilde{\tau}^{m(x)}(x) \quad \text{for every } x \in X$$

hence by part (b) of Theorem 1 the C^*-subalgebra $A(v)$ coincides with $A(\Sigma_\sigma)$. On the other hand, from remark (b) followed after Theorem B the $A(v)$ is isomorphic to $A(\Sigma_\tau)$. This completes the proof. □

When Σ_σ and Σ_τ are topologically transitive dynamical systems on compact metric spaces, the statement of the above theorem can be strengthened in the following way again by Boyle's theorem cited before (cf. also Corollary of Theorem 1).

Corollary. *Let $\Sigma_\sigma = (X, \sigma)$ and $\Sigma_\tau = (Y, \tau)$ be topologically transitive dynamical systems on the compact metric spaces X and Y. Then $A(\Sigma_\sigma)$*

and $A(\Sigma_\tau)$ are isomorphic each other keeping their subalgebras $C(X)$ and $C(Y)$ if and only if σ is flip conjugate to τ.

This result has been proved in [**4**, Theorem 2.4] for minimal dynamical systems on the Cantor sets.

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