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REALIZING HOMOLOGY CLASSES UP TO COBORDISM

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Abstract

It is known that neither immersions nor maps with a fixed finite set of multisingularities are enough to realize all mod 2 homology classes in manifolds. In this paper we define the notion of realizing a homology class up to cobordism; it is shown that for realization in this weaker sense immersions are sufficient, but maps with a fixed finite set of multisingularities are still insufficient.

1. Introduction

In 1949 Steenrod [5] posed the following question: given a homology class h of a space X, does there exist a closed manifold V and a continuous map $f : V \to X$ such that $f_*[V] = h$, where $[V]$ is the fundamental class of V? Thom's famous result answers the question affirmatively if h is a $\mathbb{Z}_2$-homology class, and shows that for integral homology the answer in general is negative. It is a natural further question whether f can be chosen to be "nice" if X itself is a smooth manifold. For example, can it be always an embedding or an immersion? If not, then can f be chosen to have only mild singularities?

For embeddings Thom himself gave some necessary and sufficient conditions. From these conditions it is not hard to deduce that there are $\mathbb{Z}_2$-homology classes of codimension 2 not realizable by embeddings in some manifolds.

In [6] it was shown that for any $k > 1$ there is a manifold M (of dimension approximately $4k$) and a cohomology class $\alpha \in H^k(M; \mathbb{Z}_2)$ such that the Poincaré dual of α cannot be realized by an immersion. Moreover it was shown there that for any $k > 1$ singular maps of finite complexity (see Section 3 for the precise definition) are insufficient to realize all codimension k homology classes in manifolds.

Therefore in order to obtain positive answers it is natural to relax the notion of "realization of a homology class". The relaxed version we use will be "realization up to cobordism". For this purpose we define the cobordism group of pairs (M^n, α) where M^n is a closed smooth n-manifold and $\alpha \in H^k(M; \mathbb{Z}_2)$ for a fixed k.

Definition. Given two pairs (M^n, α) and (N^n, β) we say that they are *cobordant* if there is a pair (W^{n+1}, γ) such that W^{n+1} is a compact $(n+1)$-manifold with boundary $\partial W = M \sqcup N$ and $\gamma \in H^k(W; \mathbb{Z}_2)$ is a cohomology class such that $\gamma|_M = \alpha$ and $\gamma|_N = \beta$.

Remark. The obtained group of pairs is clearly isomorphic to $\mathfrak{N}_n(K(\mathbb{Z}_2, k))$, the nth bordism group of the Eilenberg-MacLane space $K(\mathbb{Z}_2, k)$.

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Definition. Lef $\mathcal{F}$ be a class of smooth maps (for example, embeddings, immersions, or singular maps of some given complexity). We say that a pair (M, α) is *$\mathcal{F}$-realizable* if there exist a closed manifold V and a map $f : V \to M$ such that $f \in \mathcal{F}$ and $f_*[V]$ is Poincaré dual to α.

We say that (M, α) is *$\mathcal{F}$-realizable up to cobordism* if there is an $\mathcal{F}$-realizable pair (N, β) cobordant to (M, α).

We show that this relaxation allows to give a positive answer in the case of immersions but for singular maps of finite complexity the answer remains negative. In particular, in every codimension greater than one there are homology classes which cannot be realized by embeddings even up to cobordism.

2. Realization by immersions

Theorem 1. *Any pair (M, α) is realizable by immersions up to cobordism.*

For conciseness, (co)homology coefficients $\mathbb{Z}_2$ will be omitted and K will stand for $K(\mathbb{Z}_2, k)$.

In what follows, $MO(k)$ denotes as usual the Thom space of the universal vector bundle over $BO(k)$, and for any space X we denote by ΓX the space $\Omega^\infty S^\infty X = \lim_{N\to\infty} \Omega^N S^N X$. Recall that $\Gamma MO(k)$ is the classifying space of codimension k immersions, in particular, the group of cobordism classes of codimension $k > 0$ immersions into a fixed closed manifold P (where cobordisms are codimension k immersions into $P \times [0, 1]$) is isomorphic to the group of homotopy classes $[P, \Gamma MO(k)]$.

It is well-known that $\Gamma MO(k)$ is stably equivalent to a bouquet that contains $MO(k)$ (i.e. there is a space Y such that $\Gamma MO(k) \underset{stably}{\cong} MO(k) \vee Y$). Hence $H^*(MO(k))$ embeds naturally into $H^*(\Gamma MO(k))$. In particular the Thom class $u_k \in H^k(MO(k))$ can be considered (uniquely, since Y is known to be $(2k-1)$-connected) as a cohomology class of $\Gamma MO(k)$. Denote by u the corresponding map into K, that is, $u : \Gamma MO(k) \to K$ has the property that $u^*(\iota_k) = u_k$, where $\iota_k \in H^k(K)$ is the fundamental class.

Alternatively, we may use the universal property of the functor Γ that is as follows ([3, p. 39.], [7, pp.42–43.]): for any map $f : X \to Y$ from a compactly generated Hausdorff space X to an infinite loop space Y there is a homotopically unique extension $\hat{f} : \Gamma X \to Y$ that is an infinite loop map. Applying this property to u_k yields the map u.

For any P the map $u_*^P : [P, \Gamma MO(k)] \to [P, K]$ induced by u associates to (a cobordism class of) an immersion the Poincaré dual of the homology class represented by the immersion.

This shows that Theorem 1 has the following equivalent reformulation:

Theorem 1′. *The map $u : \Gamma MO(k) \to K$ induces an epimorphism of the bordism groups in any dimension. That is, for any n*

$$u_* : \mathfrak{N}_n(\Gamma MO(k)) \to \mathfrak{N}_n(K)$$

is onto.

Proof. It is well-known ([4]) that there is an isomorphism $H_*(X; \mathbb{Z}_2) \otimes \mathfrak{N}_* \to \mathfrak{N}_*(X)$, natural in X, defined by taking a representative $[\hat{\alpha} : M_\alpha \to X] \in \mathfrak{N}_*(X)$ for all elements

α of a basis of $H_*(X)$ and mapping $\sum_j \alpha_j \otimes [N_j]$ to $\sum_j [\hat{\alpha}_j \circ pr_j : M_{\alpha_j} \times N_j \to X]$, where $pr_j : M_{\alpha_j} \times N_j \to M_{\alpha_j}$ is the projection. Hence a map induces an epimorphism of the (unoriented) bordism groups if and only if it does so in the $\mathbb{Z}_2$-homology groups.

For later use, recall that for any space X the ring $H_*(\Gamma X)$ is a polynomial ring (multiplication being the Pontryagin product) in variables x_λ, $y_{I,\lambda}$, where $\{x_\lambda\}_\lambda$ is a homogeneous basis of $H_*(X)$ and $y_{I,\lambda}$ are further variables defined using Kudo-Araki operations as $y_{I,\lambda} = Q^I x_\lambda$ (their precise description will be unimportant in our argument).

In order to show that

$$u_* : H_*(\Gamma MO(k)) \to H_*(K)$$

is onto it is enough to show that the composition $\varphi = p \circ (u_k)_*$

$$\overline{H}_*(MO(k)) \overset{(u_k)_*}{\to} \overline{H}_*(K) \overset{p}{\to} Q(H_*(K)) = \overline{H}_*(K)/\mu\left(\overline{H}_*(K) \otimes \overline{H}_*(K)\right)$$

is onto, where $\mu : H_*(K) \otimes H_*(K) \to H_*(K)$ is the multiplication map and p is the natural projection onto the quotient group of indecomposables. Indeed, assume that φ is onto and for all j choose elements in $H_j(K)$ such that they form a (linear) basis in $\overline{H}_j(K)/\mu\left(\overline{H}_j(K) \otimes \overline{H}_j(K)\right)$. It is easy to see by induction on j that the chosen elements generate $\overline{H}_*(K)$ multiplicatively and hence the subring of $H_*(\Gamma MO(k))$ generated by the preimages of these elements is mapped onto the entire $H_*(K)$ (here we use that u_* is a ring homomorphism, since u is an infinite loop map).

Hence to prove Theorem 1 we have to show that $\varphi : H_*(MO(k)) \to QH_*(K)$ is onto. This is equivalent to the dual homomorphism φ^* being injective. By [8, Proposition 3.10.], the dual of $QH_*(K)$ is $PH^*(K)$, the submodule of primitive elements of the Hopf algebra $H^*(K)$. This latter group is known to be

$$PH^*(K) = \mathbb{Z}_2 \left\langle Sq^I \iota_k : I \text{ admissible of excess } e(I) \leq k \right\rangle,$$

the vector space over $\mathbb{Z}_2$ freely generated by the $Sq^I \iota_k$ (see eg. [2, p. 23.]). The dual of $H_*(MO(k))$ is $H^*(MO(k))$ and can be identified with the ideal generated by w_k in $\mathbb{Z}_2[w_1, \ldots, w_k]$ (w_k corresponds to the Thom class u_k). The map φ^* maps ι_k to u_k and then to w_k, and commutes with the action of the Steenrod algebra, allowing to calculate the image of φ^*.

Finally, we need to show that the set $\left\{Sq^I(w_k) : I \text{ is admissible with } e(I) \leq k\right\}$ is linearly independent in the ideal $(w_k) \subset \mathbb{Z}_2[w_1, \ldots, w_k]$. This is the immediate consequence of [9, Remark 2.4.] that shows that the Steenrod algebra acts freely unstably on w_k in $H^*(BO(k))$, and this finishes the proof of Theorem 1. □

3. Non-realizability up to cobordism by singular maps of finite complexity

Recall some definitions from singularity theory that are necessary for the formulation of Theorem 2.

Definition. Fix a natural number $k \geq 1$ and consider equivalence classes of germs $\eta : (\mathbb{R}^{n-k}, 0) \to (\mathbb{R}^n, 0)$, $n \geq k$, up to left-right equivalence and stabilization, that is, we consider η to be equivalent to $\eta \times id_{\mathbb{R}^1} : (\mathbb{R}^{n-k+1}, 0) \to (\mathbb{R}^{n+1}, 0)$. An equivalence class is called a (codimension k) *local singularity* (even if its rank is maximal).

DEFINITION. A *multisingularity* is a finite multiset (set with elements equipped with multiplicities) of local singularities.

DEFINITION. Let $f : M \to N$ be a smooth map such that for any $y \in N$ the preimage $f^{-1}(y)$ is a finite set. For $y \in N$ and $f^{-1}(y) = \{x_1, \ldots, x_m\}$ let $\left[f_{x_j}\right]$ denote the local singularity class of the germ f at x_j. The multiset $\{[f_{x_1}], \ldots, [f_{x_m}]\}$ is called the *multisingularity of f at y*. A multisingularity is *locally stable* if it is the multisingularity of a stable map in the sense of [1, Section 1.4.].

DEFINITION. Let τ be a set of multisingularities. The map f is said to be a *τ-map* if its multisingularity at any point $y \in N$ belongs to τ.

Theorem 2. *Let τ be any finite set of locally stable multisingularities of codimension $k > 1$ maps and let $\mathcal{F}$ be the class of τ-maps. Then the class $\mathcal{F}$ is insufficient for realizing up to cobordism all codimension k homology classes in manifolds. That is, for any $k > 1$ there is a pair (M, α) with M a smooth manifold and $\alpha \in H^k(M)$ such that (M, α) is not $\mathcal{F}$-realizable up to cobordism.*

Proof. The proof given in [6, Theorem 1.3.] for non-realizability of homologies by τ-maps also proves the stronger statement of Theorem 2. In that proof there was a classifying space X_τ for τ-maps (analogously to $\Gamma MO(k)$ being the classifying space for immersions). X_τ has a single nonzero element in its first nontrivial (reduced) cohomology group, $H^k(X_\tau)$, which can be called the Thom class $u_\tau : X_\tau \to K$. If any pair (M, α) could be realizable up to cobordism by τ-maps, then the map u_τ would induce an epimorphism $(u_\tau)_* : \mathfrak{N}_*(X_\tau) \to \mathfrak{N}_*(K)$ between the unoriented bordism groups or, equivalently, between the homology groups (using the same argument as in the proof of Theorem 1). But [6] shows that for any sufficiently high dimension j (under the assumption that $k > 1$) we have $\dim_{\mathbb{Z}_2} H_j(X_\tau) < \dim_{\mathbb{Z}_2} H_j(K)$, hence $(u_\tau)_* : H_j(X_\tau) \to H_j(K)$ cannot be surjective. $\square$

REMARK. In particular, embeddings or immersions with self-intersection multiplicity bounded by a fixed number are insufficient for realizing all homology classes in manifolds even up to cobordism.

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