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# ON AUSLANDER-REITEN COMPONENTS AND SIMPLE MODULES FOR FINITE GROUP ALGEBRAS

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## Introduction

Let $G$ be a finite group, $k$ a field of characteristic $p>0$ and $B$ a block of the group algebra $kG$. Let $\Theta$ be a connected component (AR-component for short) of the stable Auslander-Reiten quiver of $B$. Erdmann showed that if $B$ is a wild block of $kG$, then the tree class of $\Theta$ is $A_\infty$ [6]. In this note we investigate where simple modules lie in the Auslander-Reiten quiver of $B$. Let $\Lambda$ be a symmetric algebra and $M$ a simple $\Lambda$-module. Then the Auslander-Reiten sequence $\mathcal{A}(\Omega^{-1}M)$ terminating in $\Omega^{-1}M$ is of the form $0\to\Omega M\to H_M\oplus P_M\to\Omega^{-1}M\to 0$, where $\Omega$ is the Heller operator, $P_M$ is the projective cover of $M$ and $H_M$ is the heart $\mathrm{Rad}P_M/\mathrm{Soc}P_M$ of $P_M$ (see [1, Proposition 4.11]), and sequences of this type will be called standard sequences. Therefore if the tree class of the AR-component $\Theta$ containing $M$ is $A_\infty$, then $M$ lies at the end of $\Theta$ if and only if $H_M$ is indecomposable.

In Section 1, we consider for general symmetric algebras what happens if some AR-component with tree class $A_\infty$ contains a simple module not lying at the end of its AR-component. In Section 2 we give certain conditions which imply that all simple modules in $B$ lie at the ends of AR-components.

The notation is almost standard. All the modules considered here are finite dimensional over $k$. Concerning some basic facts and terminologies used here, we refer to [2] and [5].

## 1. AR-components of symmetric algebras and simple modules

In the case of general symmetric algebras, Jost gave some conditions which imply that all simple modules contained in an AR-component with tree class $A_\infty$ lie at the end of this component [7, Theorem 3.3]. Now we consider what happens if some simple module does not lie at the end of an AR-component with tree class $A_\infty$. In this section, let $\Lambda$ be a symmetric algebra and $\Theta$ an AR-component with tree class $A_\infty$ of the stable Auslander-Reiten quiver of $\Lambda$, and suppose that $\Theta$ contains some simple $\Lambda$-module not lying at the end of $\Theta$. Under this assumption $\Theta$ is of the form $\mathbf{Z}A_\infty$ or $(\mathbf{Z}/m)A_\infty$ (so called an $m$-tube), and we may assume that $\Theta$ or $\Omega\Theta$ contains some simple module $S$ not lying at the end and that the wing $\mathcal{W}(S)$ spanned by $S$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & S & & & & & & \\
 & & & & & \beta_1 \nearrow & & \searrow \alpha_1 & & & & & \\
 & & & & \Omega^2 X_1 & & & & X_1 & & & & \\
 & & & \nearrow & & \searrow & & \nearrow & & \searrow \alpha_2 & & & \\
 & & \Omega^4 X_2 & & & & \Omega^2 X_2 & & & & X_2 & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \alpha_3 & \\
 & \cdot^{\cdot^{\cdot}} & & & & & & & & & \ddots & & \\
 \nearrow & & \searrow & & \nearrow & & & & \searrow & & \nearrow & & \searrow \alpha_{n-1} \\
 \Omega^{2(n-1)} X_{n-1} & & & & \Omega^{2(n-2)} X_{n-1} & & & & \Omega^2 X_{n-1} & & & & X_{n-1} \\
 \nearrow & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \searrow \alpha_n \\
 \Omega^{2n} X_n & & & \Omega^{2(n-1)} X_n & & & & & & \Omega^2 X_n & & & X_n
\end{array}
$$

with $\Omega^{2i}X_n$ $(0 \leq i \leq n)$ lying at the end, satisfies the condition that

$(*)$ there are no projectives in $\mathcal{A}(\Omega^{2i}X_j)$ for $0 \leq i \leq j < n$.

Indeed, if this is not the case, then the AR-sequence $\mathcal{A}(\Omega^{2i}X_j)$ terminating in $\Omega^{2i}X_j$ is standard for some $1 \leq j \leq n-1$ and some $0 \leq i \leq j$ because standard ones are only those which involve projectives. Thus, $\Omega^{2i}X_j$ is isomorphic to $\Omega^{-1}S'$ for some simple module $S'$, and $S'$ does not lie at the end. Hence we start with $S'$ instead of $S$, and therefore we finally get a wing with the above property $(*)$.

In the above situation, we shall see that the AR-sequences $\mathcal{A}(\Omega^{2i}X_n)$ terminating in $\Omega^{2i}X_n$ $(0 \leq i \leq n-1)$ are standard. Also in the case where $\Theta$ is an infinite $m$-tube, we shall see that $n+1 < m$, i.e., $X_n \not\cong \Omega^{2i}X_n$ for $0 < i \leq n$.

First we recall the following easy result (see, e.g., the argument in [3, Section 3]), which will be used repeatedly.

**Lemma 1.1.** *Let $\mathcal{A}(U) : 0 \to X \to Y \oplus Z \to U \to 0$ with $Y$ and $Z$ non-projective be an AR-sequence terminating in $U$. Assume that the irreducible map $\alpha : Y \to U$ is a monomorphism. Then the irreducible map $\alpha' : X \to Z$ is also a monomorphism and* $\mathrm{Coker}\alpha \cong \mathrm{Coker}\alpha'$. *Dually, if the irreducible map $\alpha' : X \to Z$ is an epimorphism, then the irreducible map $\alpha : Y \to U$ is also an epimorphism and* $\mathrm{Ker}\alpha \cong \mathrm{Ker}\alpha'$.

Now we give attention to the modules $X_1$, $\Omega^2 X_1$ and $\Omega^2 X_2$.

**Lemma 1.2.** *$X_1$, $\Omega^2 X_1$ and $\Omega^2 X_2$ are uniserial and their Loewy series are as follows for some simple $\Lambda$-modules $T_1$ and $T_n$ :*

$$X_1 : \begin{pmatrix} T_1 \\ S \end{pmatrix}, \quad \Omega^2 X_1 : \begin{pmatrix} S \\ T_n \end{pmatrix}, \quad \Omega^2 X_2 : \begin{pmatrix} T_1 \\ S \\ T_n \end{pmatrix}.$$

Proof. Since $S$ is simple, the irreducible map $\beta_1 : \Omega^2 X_1 \to S$ is an epimorphism and the irreducible map $\alpha_1 : S \to X_1$ is a monomorphism. By the property(*) and Lemma 1.1, it follows that $\mathcal{A}(X_n)$ and $\mathcal{A}(\Omega^{2(n-1)} X_n)$ are standard, i.e., $\Omega^{2(n-1)} X_n \cong \Omega^{-1} T_1$ and $X_n \cong \Omega^{-1} T_n$ for some simple $\Lambda$-modules $T_1$ and $T_n$. Also, Lemma 1.1 yields that $\mathrm{Coker}\alpha_1 \cong T_1$ and $\mathrm{Ker}\beta_1 \cong T_n$. □

Next we consider the modules $X_i$ $(1 \le i \le n)$.

**Lemma 1.3.** *For the modules $X_i$ and the irreducible maps $\alpha_i : X_{i-1} \to X_i$ $(1 \le i \le n)$, the following hold.*

(1) *The irreducible maps $\alpha_i$ are monomorphisms.*

(2) *$\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple $\Lambda$-module $T_i (1 \le i \le n)$.*

(3) *$T_i$ appears in the head of $X_i$ and the composition factors of $X_i$, from the head, are given by $\{T_i, T_{i-1}, \cdots, T_1, S\}$.*

(4) *The socle of $X_i$ is isomorphic to $S$.*

Proof. In the case $i = 1$, the statements follow by Lemma 1.2. Assume that the statements hold for $X_j (1 \le j \le i-1)$. Note that the AR-sequences $\mathcal{A}(X_i)(1 \le i \le n-1)$ are not standard. We consider the following mesh:

$$\begin{array}{ccccc} & & X_{i-1} & & \\ & \nearrow & & \searrow^{\alpha_i} & \\ \Omega^2 X_i & & & & X_i \\ & {}_{\alpha}\searrow & & \nearrow_{\beta} & \\ & & \Omega^2 X_{i+1} & & \end{array}$$

(1) Assume contrary that $\alpha_i$ is an epimorphism. Since the socle of $X_{i-1}$ is simple and isomorphic to $S$, the socle of $\mathrm{Ker}\alpha_i$ is isomorphic to $S$ and $S$ does not appear as a composition factor of $X_i$. Since the irreducible map $\beta : \Omega^2 X_{i+1} \to X_i$ is an epimorphism and $\mathrm{Ker}\beta \cong T_n$, $S$ does not appear as a composition factor of $\Omega^2 X_{i+1}$. Now we see that $\Omega^2 X_1 = \begin{pmatrix} S \\ T_n \end{pmatrix} \subset \Omega^2 X_i$ by induction. However, since $S$ lies in the head of $\Omega^2 X_1$, we have $\Omega^2 X_1 \subset \mathrm{Ker}\alpha$, where $\alpha$ is the irreducible map from $\Omega^2 X_i$ to $\Omega^2 X_{i+1}$, but this contradicts that $\mathrm{Ker}\alpha_i \cong \mathrm{Ker}\alpha$.

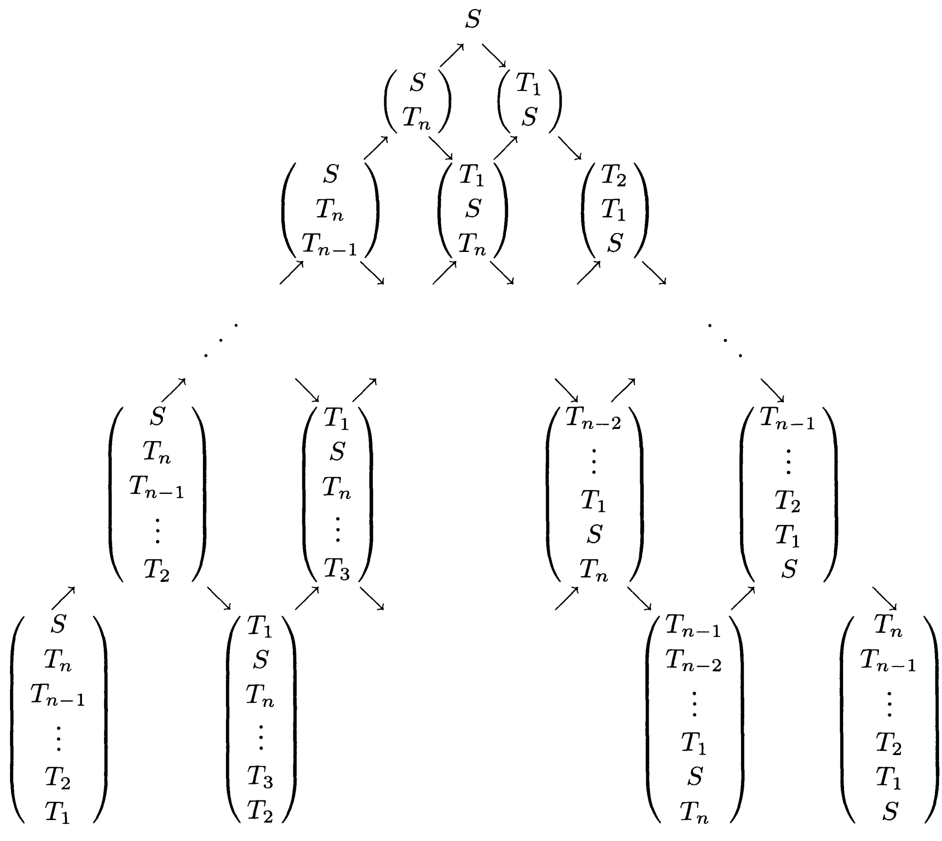
(2) Note that the statement (1) above, Lemma 1.1 and the property (*) imply that $\mathcal{A}(\Omega^{2(n-i)} X_n)$ is standard. Hence we have $\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple $\Lambda$-module $T_i$.

(3) This follows since $\mathrm{Coker}\alpha_i \cong T_i$ by (2).

(4) By the inductive hypothesis, we have $\mathrm{Soc}X_{i-1} \cong S$. Since $X_{i-1}$ is a maximal submodule of $X_i$ and $X_i$ is indecomposable, we have $\mathrm{Soc}X_{i-1} = \mathrm{Soc}X_i$.

□

**Proposition 1.4.** *Using the same notation as in Lemma* 1.3*, the wing* $\mathcal{W}(S)$ *spanned by* $S$ *is as follows.*



*In particular, all modules in* $\mathcal{W}(S)$ *are uniserial.*

Proof. We continue to use the notation in Lemma 1.3. From Lemma 1.3(2) and the property $(*)$, the irreducible maps $\Omega^{2s}X_i \to \Omega^{2s}X_{i+1}$ are monomorphisms and the irreducible maps $\Omega^{2(s+1)}X_{i+1} \to \Omega^{2s}X_i$ are epimorphisms for $1 \leq i \leq n-1$ and $0 \leq s \leq i$. Therefore $X_i$ is a homomorphic image of $\Omega^{-1}T_i$ and the head of $X_i$ is isomorphic to $T_i$. Thus $X_i$ $(1 \leq i \leq n)$ are uniserial. In particular $X_{n-1}$(=the heart of the projective cover of $T_n$) is uniserial and so is $\Omega^2 X_n (\cong \Omega T_n)$. Since

$\Omega^2 X_i$ $(1 \le i \le n)$ are submodules of $\Omega^2 X_n$, they are uniserial. Using this argument repeatedly, we see that all modules in $\mathcal{W}(S)$ are uniserial. $\square$

From Lemma 1.3 and Proposition 1.4, we have the following immediately.

**Theorem 1.5.** *Let $\Lambda$ be a symmetric algebra and $\Theta$ an AR-component of $\Lambda$ with tree class $A_\infty$. Suppose that $\Theta$ contains some simple module not lying at the end of $\Theta$. Then for some simple $\Lambda$-modules $S, T_1, \cdots, T_n$ the projective covers $P_{T_i}$ of $T_i$ $(1 \le i \le n)$ are uniserial and the Loewy series are as follows:*

$$P_{T_1}: \begin{pmatrix} T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ \\ \\ \vdots \\ T_2 \\ T_1 \end{pmatrix}, \quad P_{T_2}: \begin{pmatrix} T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ \\ \vdots \\ T_3 \\ T_2 \end{pmatrix}, \quad \cdots, \quad P_{T_i}: \begin{pmatrix} T_i \\ T_{i+1} \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \\ T_{n-1} \\ \vdots \\ T_{i+1} \\ T_i \end{pmatrix}, \quad \cdots, \quad P_{T_n}: \begin{pmatrix} T_n \\ T_{n-1} \\ \vdots \\ \\ \\ \vdots \\ T_2 \\ T_1 \\ S \\ T_n \end{pmatrix}.$$

*In particular, the Cartan matrix for $\Lambda$ looks like*
$$\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 & \\ 1 & 2 & 1 & & \vdots & \\ 1 & 1 & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 2 & 1 & \\ 1 & 1 & \cdots & 1 & & \\ & & 0 & & & * \end{pmatrix}.$$

Proof. By Lemma 1.3(2), the AR-sequences $\mathcal{A}(\Omega^{2(n-i)} X_n)$ are standard and $\Omega^{2(n-i)} X_n \cong \Omega^{-1} T_i$ for some simple $\Lambda$-modules $T_i (1 \le i \le n)$. Also by Proposition 1.4, the hearts $\Omega^{2(n-i)} X_{n-1}$ of $P_{T_i}$ are uniserial and their Loewy series, from the head, are given by $T_{i-1}, T_{i-2}, \cdots, T_1, S, T_n, T_{n-1}, \cdots, T_{i+1}$. We claim that $T_i \not\cong T_j$ if $i \ne j$. Indeed, since $S$ appears only in the $(i+1)$th head of $P_{T_i}$, we have $P_{T_i} \not\cong P_{T_j}$ if $i \ne j$. $\square$

REMARK 1.6. Under the same notation as in Proposition 1.4, suppose that $\Theta$ is an infinite $m$-tube. Then it follows that $n + 1 < m$ since $\Omega^{-1} T_i \not\cong \Omega^{-1} T_j$ if $i \ne j$.

## 2. AR-components of group algebras and simple modules

In this section, we show that under certain conditions all simple modules in a wild block $B$ of the group algebra $kG$ lie at the ends of the AR-components.

**Theorem 2.1.** *Let $B$ be a wild block of $kG$. Suppose that $G$ has a non-trivial normal $p$-subgroup and $k$ is algebraically closed. Then all simple modules in $B$ lie at the ends of the AR-components.*

Proof. Let $Q$ be a non-trivial normal $p$-subgroup of $G$. Assume contrary that some simple module in $B$ does not lie at the end. Then for some simple modules $S, T_1, \cdots, T_n$, the projective covers $P_{T_i}$ of $T_i$ $(1 \leq i \leq n)$ are uniserial and the Loewy series are as in Theorem 1.5. In particular the Cartan integers $c_{T_iT_i} = 2$.

Claim 1. $n = 1$, i.e., for some simple modules $S$ and $T$, the Loewy series of the projective cover $P_T$ of $T$ is given by $T, S, T$.

Proof of the Claim 1. From the result of Tsushima [10, Lemma 3], $T_i$ are projective as $k(G/Q)$-modules, i.e., $\mathrm{vx}(T_i) = Q$ and the trivial $kQ$-module $k_Q$ is a source of $T_i$. Now assume contrary that $n \geq 2$. Since $T_1 \cong \Omega^2 T_2$, it follows that $k_Q \cong \Omega^2 k_Q$ and $Q$ is cyclic. However, by the result of Erdmann [4, Theorem] $T_i$ belong to a block with a cyclic defect group, a contradiction.

Claim 2. We have $p = 2$ and $Q$ is the Klein four group.

Proof of the Claim 2. Since $T{\downarrow_Q}$ and $S{\downarrow_Q}$ are direct sums of copies of $k_Q$, the length of Loewy series of $P_T{\downarrow_Q}$ is at most 3. Hence $Q$ is the Klein four group by the result of Okuyama [9].

Let $H_S$ be the heart of the projective cover $P_S$ of $S$ and $\Theta$ the AR-component containing $\Omega S$. Then $H_S \cong T \oplus X$ for some indecomposable non-projective module $X$. We consider the wing spanned by $X$ :

$$\begin{array}{ccccccc} & & & X & & & \\ & & \nearrow & & \searrow & & \\ & \Omega S & \rightarrow & P_S & \rightarrow & \Omega^{-1} S & \\ \nearrow & & \searrow & & \nearrow & & \searrow \\ \Omega^2 T & & & T & & & \Omega^{-2} T. \end{array}$$

Claim 3. $\mathrm{vx}(X) \gneqq Q$.

Proof of the Claim 3. Assume contrary that $\mathrm{vx}(X) = Q$. Note that the AR-component containing $S$ is not a tube, since $k_Q | S{\downarrow_Q}$ and $k_Q$ is not a periodic module. Since $\mathrm{vx}(T) = Q$, from the result of Okuyama and Uno [8, Theorem], all the indecomposable modules in $\Theta$ have the same vertex $Q$. Since $\Theta$ is of the form $\mathbf{Z}A_\infty$, the class of the $Q$-sources of the indecomposable modules in $\Theta$ consists of infinitely many $\Omega^2$-orbits. However this would be impossible because non-periodic indecomposable $kQ$-modules are the syzygies of the trivial module $k_Q$ only (see, e.g., [2]).

Now we consider the following two cases.

CASE 1. $\mathrm{vx}(\Omega^{-1}S) \gneqq Q$. The AR-sequence $\mathcal{A}(\Omega^{-1}S)\downarrow_Q$ restricted to $Q$ splits [2, Proposition 4.12.10]. However, $\Omega^{-1}S\downarrow_Q$ (resp. $\Omega S\downarrow_Q$) is a direct sum of copies of $\Omega^{-1}k_Q$ (resp. $\Omega k_Q$) but $T\downarrow_Q$ is a direct sum of copies of $k_Q$, a contradiction.

CASE 2. $\mathrm{vx}(\Omega^{-1}S) = Q$, $\mathrm{vx}(X) \gneqq Q$. The AR-sequence $\mathcal{A}(\Omega^{-1}S)\downarrow_Q$ restricted to $Q$ is a direct sum of split sequences and $m$ copies of AR-sequence $\mathcal{A}(\Omega^{-1}k_Q)$ for some $m$. Since $S\downarrow_Q \cong (\dim S)k_Q$ and $\Omega^{-1}S\downarrow_Q \cong (\dim S)\Omega k_Q$, we have $m \leq \dim S$. On the other hand, since $\dim(\mathrm{Soc}(P_S\downarrow_Q)) \geq \dim S$ and $(\dim S)kQ|P_S\downarrow_Q$, we have $m \geq \dim S$. Therefore $m = \dim S$. This means that the AR-sequence $\mathcal{A}(\Omega^{-1}S)\downarrow_Q$ restricted to $Q$ is a direct sum of $(\dim S)$ copies of the AR-sequence $\mathcal{A}(\Omega^{-1}k_Q)$ and $X\downarrow_Q$ is a direct sum of copies of $k_Q$. Since $\mathrm{vx}(X) \gneqq Q$, the AR-sequence $\mathcal{A}(X)\downarrow_Q$ restricted to $Q$ splits. However $\Omega k_Q$ is a direct summand of $\Omega S\downarrow_Q$, which is a direct summand of the middle term of $\mathcal{A}(X)\downarrow_Q$, a contradiction. □

**Corollary 2.2.** *Let $B$ a wild block of $kG$. Suppose that $G$ is $p$-solvable and $k$ is algebraically closed. Then all simple modules in $B$ lie at the ends of the AR-components.*

Proof. Assume that some simple module does not lie at the end. Then by Theorem 1.5 and the result of Tsushima [10, Theorem], there is a finite group $H$ with normal $p$-subgroup such that $B$ and $kH$ are Morita equivalent. However by Theorem 2.1 all simple $kH$-modules lie at the ends of the AR-components, a contradiction. □

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