Nobusawa, N.
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CENTRAL EXTENSIONS OF THE SYMMETRIC CORE OF A GROUP

NOBUO NOBUSAWA

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Let (G, σ) be a pair of a group G and an automorphism σ of order 2 (an involution) of G. Define the *symmetric core* $S(G)$ of (G, σ) by

$$S(G)=\{a\,\sigma(a^{-1})\,|\,a\in G\}\ .$$

$S(G)$ is closed under the mapping (called the symmetry around s): $x\mapsto s\circ x=sx^{-1}s$ $(s, x\in S(G))$, and is a discrete symmetric space in the sense of [1]. Let (H, Z) be a central extension of G with the exact sequence $1\to Z\to H\xrightarrow{\phi}G\to 1$ where Z is contained in the center of H. Let τ be an involution of H such that it leaves every element of Z invariant and $\phi\circ\tau=\sigma\circ\phi$. In this case, we say that $(H, Z; \tau)$ (or, (H, τ)) is a *central extension of the null type of* (G, σ). In this paper, a central extension implies that of the null type.

It is easy to see that the symmetric core $S(H)$ of (H, τ) is homomorphic onto $S(G)$, and we call $S(H)$ a central extension, or, a *covering* (of the null type) of $S(G)$. The main result obtained in this note is that there exists a generic covering $S(U)$ of $S(G)$ and $S(U)$ is finite if G is finite. Here, a covering is called "generic" if every covering of $S(G)$ is its homomorphic image. We also obtain some connections between the generic covering and the (restricted) Schur multiplier of G when G is finite.

1. The symmetric core of a central extension

Groups with involutions are said to be symmetrically equivalent if they have the isomorphic symmetric cores. Let $\{t(a)\}$ $(a\in G)$ be a representative system of G in a central extension (H, τ), and let H_0 be a subgroup of H generated by $t(a)$ and $\tau(t(a))$ $(a\in G)$. Then, (H_0, τ_0) is a central extension of (G, σ) where τ_0 is the restriction of τ to H_0. (H, τ) and (H_0, τ_0) are symmetrically equivalent because they have the same symmetric core $\{t(a)\,\tau(t(a)^{-1})\,|\,a\in G\}$. We call (H_0, τ_0) a reduced central extension of (G, σ). Therefore, we may restrict to a reduced central extension as far as the symmetric equivalence is concerned.

The symmetric core $S(G)$ is identified with the homogeneous space G/I

($=$the set of left cosets of I), where $I=I(G)$ is the subgroup of σ-invariant elements of G. The identification is done through the mapping $a\sigma(a^{-1})\mapsto aI$. (See [1].) For a central extension (H, τ), $S(H)=H/I(H)$ by the above identification where $I(H)$ is the subgroup of τ-invariant elements of H. Since $I(H)$ contains $Z=$the kernel of ϕ, $H/I(H)$ is mapped bijectively by ϕ to $\phi(H)/\phi(I(H))$ $=G/I'$, where $I'=\phi(I(H))$. Clearly, $I'\subset I$.

Proposition 1. *I' is a normal subgroup of I, and I/I' is an elementary abelian 2-group.*

Proof. In the following, we let $\{t(a)\}$ be a representative system of G in H with $t(1)=1$. Then, $\tau(t(a))=z(a)\,t(\sigma(a))$ with $z(a)\in Z$. If especially $a\in I$, then $\tau(t(a))=z(a)\,t(a)$ and the mapping $\zeta\colon a\mapsto z(a)$ gives a homomorphism of I to Z. Moreover, $\tau^2=1$ implies that $z(a)^2=1$ if $a\in I$. Thus, to prove Proposition 1, it is enough to show that $\ker\zeta=I'$. First let $a\in\ker\zeta$. Then, $z(a)=1$, or $\tau(t(a))=t(a)$. $t(a)\in I(H)$. $\phi(t(a))\in\phi(I(H))=I'$. So, $a=\phi(t(a))\in I'$, and hence $\ker\zeta\subset I'$. Conversely, let $b\in I'=\phi(I(H))$. Then, $b=\phi(h)$ with $h\in I(H)$. $t(b)=zh$ with some $z\in Z$. $\tau(t(b))=\tau(zh)=z\,\tau(h)=zh=t(b)$. So, $z(b)=1$. $b\in\ker\zeta$, and hence $I'\subset\ker\zeta$.

Corollary. *If G is finite, the index of $S(G)$ in $S(H)$ is a power of 2 and divides $|Z|$ if Z is finite.*

Proposition 2. *Let $(H, Z; \tau)$ be a central extension of (G, σ). If Y is a subgroup of Z such that Y contains no element of order 2, then $(H/Y, \bar{\tau})$, where $\bar{\tau}$ is the naturally defined involution of H/Y, is a central extension of (G, σ) and is symmetrically equivalent with (H, τ).*

Proof. It is clear that $(H/Y, \bar{\tau})$ is a central extension of (G, σ). Let $\bar{\phi}$ be the natural homomorphism of H/Y to G. To show that $(H/Y, \bar{\tau})$ is symmetrically equivalent with (H, τ), it is enough to show that $\bar{\phi}(I(H/Y))=\phi(I(H))$. Let $a\in\bar{\phi}(I(H/Y))$. If we denote $\overline{t(a)}=t(a)\bmod Y\in H/Y$, then $\bar{\tau}(\overline{t(a)})=\overline{t(a)}$ as in the proof of Proposition 1. This implies that $\tau(t(a))=y\cdot t(a)$ with $y\in Y$. Since $\tau^2=1$, we have $y^2=1$. By the assumption on Y, $y=1$. Hence, $t(a)\in I(H)$. Therefore, $a=\phi(t(a))\in\phi(I(H))$. We have shown that $\bar{\phi}(I(H/Y))\subset\phi(I(H))$. $\phi(I(H))\subset\bar{\phi}(I(H/Y))$ is clear. We have proved Proposition 2.

Theorem 1. *Suppose that G is finite. Then, a central extension of (G, σ) is symmetrically equivalent with a finite central extension.*

Proof. We may consider only a reduced central extension (H, τ). When G is finite, H is finitely generated. Since Z has a finite index in H, Z is also finitely generated. (See [4].) Then, $Z=E\times J$ where E is a finite gorup and J is an infinite group by the fundamental theorem on finitely generated abelian groups. Now, let $Y=J$ in the above Corollary, and we obtain a finite central extension $(H/J, E; \bar{\tau})$ which is symmetrically equivalent with (H, τ).

The following Proposition 3 will be used later. For it, let $Z=Z_1\times\cdots\times Z_m$ (direct). We let $T_i=Z/Z_i$ and $H_i=H/T_i$. Then, $(H_i, Z_i; \tau_i)$ is a central extension with the naturally defined involution τ_i. Let ϕ_i be the natural homomorphism of H_i to G

Proposition 3. $\cap\phi_i(I(H_i))=\phi(I(H))$.

Proof. It is clear that $\phi(I(H))\subset\cap\phi_i(I(H_i))$. Conversely, let $a\in\cap\phi_i(I(H_i))$. Then, $\tau(t(a))=z\cdot t(a)$ with $z\in T_i(i=1, \cdots, m)$ as in the proof of Proposition 2. This means $z=1$, and hence $a\in\phi(I(H))$.

2. A generic central extension

Elements $[f]\in H^2(G, Z)$ correspond bijectively to isomorphism classes of central extensions (H, Z) of G. When we restrict our consideration to the central extensions $(H, Z; \tau)$ of (G, σ), we obtain a subgroup of $H^2(G, Z)$, which we call the restricted cohomology group and denote by $H^2(G, Z; \sigma)$. First, we characterize the elements of $H^2(G, Z; \sigma)$. Let $[f]\in H^2(G, Z; \tau)$, and (H, τ) the central extension of (G, σ) associated with it. Then, $t(a)\,t(b)=f(a, b)\,t(ab)$ and $\tau(t(a))=z(a)\,t(\sigma(a))$. Applying τ on the above, we obtain

$$\begin{aligned} &z(a)\,z(b)f(\sigma(a), \sigma(b)) = z(ab)f(a, b) \quad \text{and} \\ &z(a)\,z(\sigma(a)) = 1\,. \end{aligned} \tag{1}$$

We call (f, z) a solution of (1) in Z, where z implies the mapping: $a\mapsto z(a)$ of G to Z. Conversely, if a 2-cocyle f of G satisfies (1) with a mapping z, then we can show that $[f]\in H^2(G, Z; \sigma)$. For, we can define an involution of H, which is a central extension of G associated with f, by setting $\tau(t(a))=z(a)\,t(\sigma(a))$. So, a characterization of an element $[f]$ of $H^2(G, Z; \sigma)$ is that there exists a solution (f, z) of (1) in Z.

Let f be a 2-cocyle, i.e., $f\in Z^2(G, Z)$. f may not have a solution (f, z) of (1) in Z. We shall modify Z to get a modified group Z_f such that a 2-cocyle f' in Z_f obtained from f in a natural way will have a solution $(f'\ z')$ of (1) in Z_f. For it, let Ω be a free abelian group on G with the generators $\omega(a)$ $(a\in G)$. We factor it out by the relations $\omega(a)\,\omega(\sigma(a))=1$. Let Γ be the abelian group obtained from Ω by the factoring, and let $\gamma(a)$ denote the corresponding generators. We have $\gamma(a)\,\gamma(\sigma(a))=1$. Then, consider a direct product $Z\times\Gamma$. In $Z\times\Gamma$, we consider a subgroup W_f which is generated by elements $(f(a, b)f(\sigma(a), \sigma(b))^{-1}, \gamma(a)^{-1}\gamma(b)^{-1}\gamma(ab))$. Now, we define $Z_f=(Z\times\Gamma)/W_f$. Note that if f has a solution (f, z) of (1), then there is a homomorphism of Z_f onto Z. For, there is a homomorphism of Γ to Z which maps $\gamma(a)$ to $z(a)$, and hence there is a homomorphism of $Z\times\Gamma$ onto Z defined by $(u, \gamma(a))\mapsto u\cdot z(a)$ for $u\in z$. The kernel of this homomorphism contains W_f, and hence, we have a homomorphism of Z_f onto Z as required. Now we define a 2-cocycle f' of G in Z_f by $f'(a, b)=f(a, b) \bmod W_f$. Here, Z is considered as a subgroup of $Z\times\Gamma$ in a natural sense. At the same time, we define a mapping z' of G to Z_f by $z'(a)=\gamma(a) \bmod W_f$. It is easy to see that (f', z') is a solution of (1) in Z_f. Let (H_f, τ') be the central extension of (G, σ) associated with (f', z'). It follows that $H_f=(H\times\Gamma)/W_f$, and if (H, τ) is a central extension associated with (f, z) in case it exists then there is a homomorphism of (H_f, τ') onto (H, τ).

Theorem 2. *There exists a central extension (U, π) of (G, σ) such that every reduced central extension of (G, σ) is a homomorphic image of (U, π). Thus, $S(U)$ is a generic covering of $S(G)$.*

Proof. Let F be a free central extension of G with the exact sequence $1\rightarrow K\rightarrow F\rightarrow G\rightarrow 1$. (See [4].). Let $[h]\in H^2(G, K)$ be the cohomology class corresponding to the central extension F of G. Let $(H, Z; \tau)$ be the reduced central extension of (G, σ) associated with a solution (f, z) of (1) as before. Then, there exists a homomorphism of F onto H, which maps K to Z and $h(a, b)$ to $f(a, b)$. Now, we let $U=F_h$, which is constructed as H_f is constructed. So, we have a central extension (U, π) of (G, σ). It can be seen that the homomorphism of F onto H induces the homomorphism of U onto H_f, more precisely, of (U, π) onto (H_f, τ'). Since we have a homomorphism of (H_f, τ') onto (H, τ) as remarked before, we obtain a homomorphism of (U, π) onto (H, τ).

From now on, we suppose that G is finite. Since (U, π) may not be finite, we want to construct a finite central extension of (G, σ) which is symmetrically equivalent with (U, τ). It is an analogue of a representation group of G, and we closely follow the process given in [2] to construct it. Let C be the group of non-zero complex numbers. We know that $Z^2(G, C)=B^2(G, C)+M$ (direct) with a subgroup M, which is naturally isomorphic with the Schur multiplier $H^2(G, C)$. When we restrict our consideration to the restricted Schur multiplier $H^2(G, C; \sigma)$, we obtain a subgroup N of M. Define $\hat{N}=\mathrm{Hom}\,(N, C)$. Now we define a central extension R of G by $R=\sum \hat{N}r(a)$, where $\hat{N}$ is contained in the center of R and $\{r(a)\}$ is a representative system of G in R. Here, $r(a)\,r(b)=\alpha(a, b)\,r(ab)$ with $\alpha(a, b)\in\hat{N}$ such that $\alpha(a, b)\,(g)=g(a, b)$ for $g\in N$. Thus, $[\alpha]\in H^2(G, \hat{N})$ and R is the central extension of G associated with $[\alpha]$. We want to show that $[\alpha]\in H^2(G, \hat{N}; \sigma)$. We have to show the existence of a solution (α, β) of (1) in $\hat{N}$. For an element g of N, there exists a solution (g, u) of (1) in C. However, u is not unique. But, a different solution (g, u') simply gives a central extension which is isomorphic with that of (g, u). We determine u for g in a unique way as follows. Let $\{g_1, \cdots, g_n\}$ be a basis of N. For each

g_i, we select u_i such that (g_i, u_i) is a solution of (1) in C. For a general element g of N, let $g=\prod g_i^{n_i}$ and define $u=\prod u_i^{n_i}$. (g, u) is a solution of (1) in C. Note that, for a given element a of G, a mapping $g \mapsto u(a)$ is then a homomorphism of N to C, and hence it is an element of $\hat{N}$. Now, we define the mapping of G to $\hat{N}$ by $a \mapsto \beta(a)$, where $\beta(a)(g)=u(a)$. It follows that (α, β) is a solution of (1) in N. We have shown that there is a central extension (R, ρ) of (G, σ).

Proposition 4. *Let $(H, Z; \tau)$ be a reduced central extension of (G, σ), where Z is a cyclic group. Then, (R, ρ) is homomorphic onto (H, τ).*

Proof. We use the previous notation. The mapping $\nu \mapsto \nu(f)$ for $\nu \in \hat{N}$ gives a homomorphism of $\hat{N}$ to C. We imbed Z into C and consider Z as a subgroup of C. Since the order of f divides the order of Z, we can consider the above mapping as a homomorphism of $\hat{N}$ to Z. We can see that this homomorphism of $\hat{N}$ to Z maps the solution (α, β) of (1) in N to the solution (f, u') of (1) in Z with some element u'. Thus, (R, ρ) is homomorphic onto a central extension associated with (f, u'), the latter being isomorphic with (H, τ) as remarked before. This proves Proposition 4.

Theorem 3. *$S(R)$ is a generic covering of $S(G)$.*

Proof. Let ψ be the natural homomorphism of R to G. If we show that $\psi(I(R)) \subset \phi(I(H))$ for any central extension (H, τ) of (G, σ), then $\psi(I(R))= \phi(I(U))$ for the generic central extension U and hence $S(R)=S(U)$, which will prove Theorem 3. So, we prove $\psi(I(R)) \subset \phi(I(H))$. We may assume that H is finite. Then, Proposition 3 implies that it is enough to show the above in case that Z is cyclic. But, this case is clear from Proposition 4.

Corollary. The index of $S(G)$ in $S(R)$ divides the order of the restricted Schur multiplier $H^2(G, C; \sigma)$.

REMARK. When G is infinite, a generic covering of $S(G)$ may not be unique (up to isomorphism), because the existence of surjective homomorphisms of two symmetric cores from one to the other in both ways does not mean the isomorphism. However, if G is finite, a generic covering is uniquely determined up to isomorphism.

EXAMPLE. Let a be an element of order 2 of G, and σ the inner automorphism of G induced by a. Then, the symmetric core of (G, σ) is $S(G)= \{x^{-1} a^{-1} xa \mid x \in G\} = C[a]\, a$, where $C[a]$ is the conjugacy class of a in G. Suppose that G is finite, and let R be a representation group of G. If there exists a representative $t(a)$ of a in R such that $t(a)$ has the order 2, let τ be the inner automorphism of R induced by $t(a)$. Then, the symmetric core of (R, τ) is $S(R)=C[t(a)]\, t(a)$. It is easy to see that $S(R)$ is the generic covering of

$S(G)$. The conjugacy class $C[t(a)]$ is homomorphic to the conjugacy class $C[a]$. Generally, any covering of $S(G)$ is a homomorphic image of $S(R)$ and hence it is of form $C[b]\,b$, where $C[b]$ is an extension of the conjugacy class $C[a]$. We can conclude that every (central) extension of $C[a]$ is an image of the conjugacy class $C[t(a)]$ of R. This result was obtained in [3].

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Department of Mathematics
University of Hawaii
Honolulu, HI. 96822
U.S.A.