Fujii, M.
Osaka J. Math.
27 (1990), 539-553

HYPERBOLIC 3-MANIFOLDS WITH TOTALLY GEODESIC BOUNDARY

Michihiko FUJII

(Received August 30, 1989)

0. Introduction

In this paper, we study compact oriented hyperbolic 3-manifolds each of which has a totally geodesic boundary. By a hyperbolic manifold, we mean a Riemannian manifold with constant sectional curvature -1. A totally geodesic boundary of such a 3-manifold becomes a hyperbolic surface.

Let g be an integer greater than or equal to 2 and let $\boldsymbol{M}_g$ be the Riemann moduli space consisting of all isometry classes of closed hyperbolic surfaces of genus g. Let S be the subset of $\boldsymbol{M}_g$ consisting of those hyperbolic surfaces which are boundaries of compact hyperbolic 3-manifolds with totally geodesic boundaries. It is well known that S should be a countably infinite subset of $\boldsymbol{M}_g$. But there are very few information about the characterization of S. The only one which the author knows is the following claim which W.P. Thurston gave in a lecture at the University of Warwick in July in 1984 (the author learned it from Professor Sadayoshi Kojima).

The countably infinite subset S should be dense in $\boldsymbol{M}_g$.

Under the above circumstances, it would be desirable to give explicit constructions of compact hyperbolic 3-manifolds with totally geodesic boundaries as many as possible.

By the way, we can apply Thurston's uniformization theorem to determine whether a given compact 3-manifold admits a complete hyperbolic structure with totally geodesic boundary or not. Unfortunately however the verification of the relevant conditions is not easy in most cases. Moreover even if the answer is positive, this theorem only ensures the existence of a hyperbolic structure. So if we would like to construct such 3-manifolds explicitly we have to take another device. In this paper we use a hyperbolic truncated tetrahedron as such a device.

As a consequence of the above considerations, we prove the following result in §3. Namely for each genus $g \geq 2$ we explicitly construct infinitely many mutually non-isometric compact oriented hyperbolic 3-manifolds each of which has a totally geodesic boundary of genus g. We shall construct such hyperbolic 3-manifolds, by considering a complete hyperbolic 3-manifold with one cusp and one totally geodesic boundary of genus g (which is a g-fold branched covering of the Whitehead link complement in S^3) and taking the double of such a 3-manifold and then performing the hyperbolic Dehn surgeries at both cusps of this double with the same coefficients (see Theorem 3.3). Now it seems to be plausible that these 3-manifolds can be obtained by gluing the faces of hyperbolic truncated tetrahedra according to the *same* combinatorial pattern (see Conjecture 3.4). If this argument would be successfully completed, we would have a good chance to prove the following. Namely the moduli of the boundary closed surfaces of genus g of the above mentioned hyperbolic 3-manifolds move in $\boldsymbol{M}_g$ (see Remark below Conjecture 3.4). We may have obtained a concrete infinite subset of S.

The author would like to express his sincere gratitude to Professor Shigeyuki Morita and Professor Sadayoshi Kojima for their constant encouragement and many useful suggestions.

1. Preliminaries

We start with describing a main constituent element of hyperbolic 3-manifolds with boundaries, called *a hyperbolic truncated tetrahedron with one ideal vertex* which is a polyhedron in the hyperbolic 3-space $\boldsymbol{H}^3$ bounded by three right-angled pentagons with one ideal vertex, one right-angled hexagon and three triangles (see Fig. 1.3).

Before giving the precise definition in Proposition 1.3, we prepare some formulae in hyperbolic geometry (cf. Beardon [1]).

Proposition 1.1. *For polygons in the hyperbolic plane* $\boldsymbol{H}^2$, *as indicated in Fig.* 1.1, *the following equations hold*:

$$\cosh C = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta} \tag{a}$$

$$\cos\gamma = \frac{\cosh A\cosh B - \cosh C}{\sinh A\sinh B} \tag{a}$$

$$\cosh C = \frac{\cosh\alpha\cosh\beta + \cosh\gamma}{\sinh\alpha\sinh\beta} \tag{b}$$

$$\cosh C = \frac{\cosh\alpha\cosh\beta + 1}{\sinh\alpha\sinh\beta} \tag{c}$$

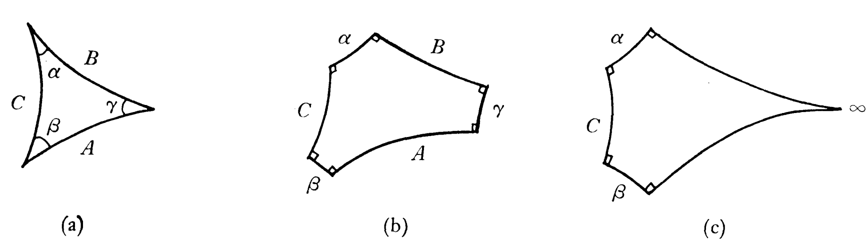


Fig. 1.1. Various values indicated above give the lengths of the corresponding geodesics or the angles.

In the rest of this paper, for any geodesic arc in $\boldsymbol{H}^3$, let us always take the hyperboliccosine value of the actual value of its length.

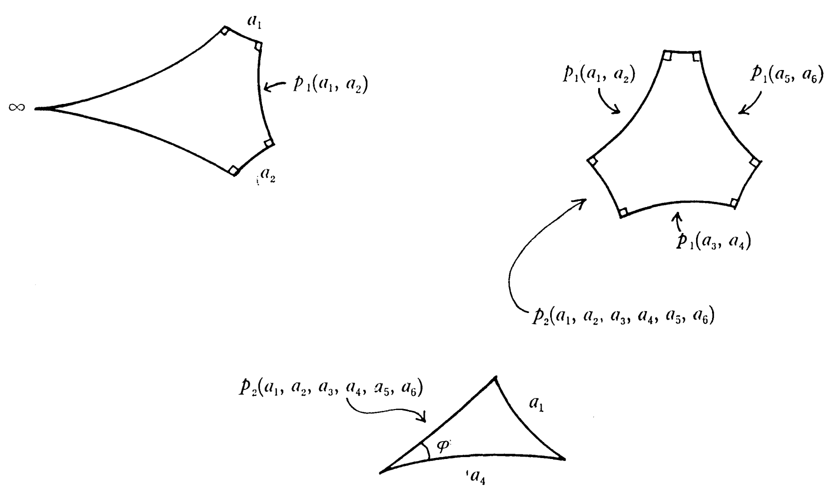
DEFINITION 1.2. *We define the following three functions p_1, p_2, p_3 for later use*:

$$p_1(a_1, a_2) := \frac{a_1 a_2 + 1}{\sqrt{a_1^2-1}\sqrt{a_2^2-1}}$$

$$p_2(a_1, a_2, a_3, a_4, a_5, a_6) := \frac{p_1(a_1, a_2)p_1(a_3, a_4)+p_1(a_5, a_6)}{\sqrt{p_1(a_1, a_2)^2-1}\sqrt{p_1(a_3, a_4)^2-1}}$$

$$p_3(a_1, a_2, a_3, a_4, a_5, a_6) := \frac{p_2(a_1, a_2, a_3, a_4, a_5, a_6)a_4-a_1}{\sqrt{p_2(a_1, a_2, a_3, a_4, a_5, a_6)^2-1}\sqrt{a_4^2-1}}$$

where $a_i \in \boldsymbol{R}$, $>1\,(i=1, \cdots, 6)$. The above functions are defined so that in the pictures of polyhedra in $\boldsymbol{H}^2$ in Fig. 1.2, *various values indicated there give the lengths of the corresponding geodesic arcs or the angle.*



$\cos\varphi = p_3(a_1, a_2, a_3, a_4, a_5, a_6)$

Fig. 1.2.

Proposition 1.3. *Let $a_i\,(i=1, \cdots, 6)$ be six real numbers each greater than* 1. *Suppose they satisfy the following equation*:

$$p_3(a_3, a_4, a_1, a_2, a_5, a_6) = p_3(a_6, a_5, a_2, a_1, a_3, a_4)\,. \tag{*}$$

Then we can construct a hyperbolic truncated tetrahedron in $\boldsymbol{H}^3$ with one ideal vertex as shown in Fig. 1.3.

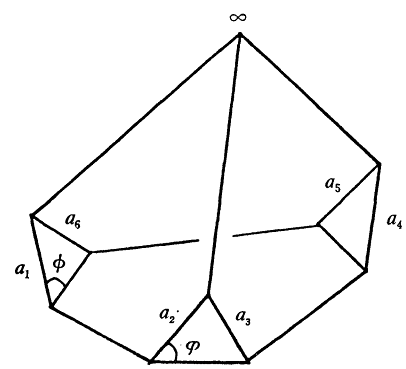


Fig. 1.3. The boundary consists of three right-angled pentagons with one ideal vertex, one right-angled hexagon and three triangles. These boundary faces are all totally geodesic. So the six dihedral angles between pentagon and triangle are $\pi/2$, and the three dihedral angles between hexagon and triangle are $\pi/2$. Parameters $a_1, \ldots, a_6$ indicated above give the lengths of the corresponding geodesic arcs and ψ and ϕ give the corresponding angles. They satisfy the following:

$$\cos \varphi = p_3(a_3, a_4, a_1, a_2, a_5, a_6)$$
$$\cos \phi = p_3(a_6, a_5, a_2, a_1, a_3, a_4)$$

Thus the equation (*) means $\varphi = \phi$.

Proof. First of all, construct a right-angled hexagon in $\boldsymbol{H}^2 \subset \boldsymbol{H}^3$ whose alternating edge lengths are

$$p_2(a_1, a_2, a_3, a_4, a_5, a_6),\ p_2(a_3, a_4, a_5, a_6, a_1, a_2),\ p_2(a_5, a_6, a_1, a_2, a_3, a_4)\,,$$

as illustrated in Fig. 1.4.

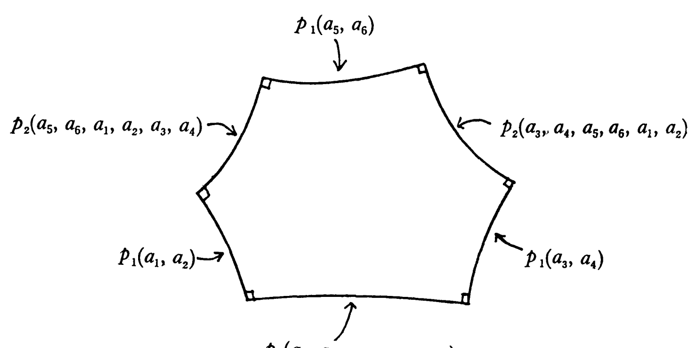


Fig. 1.4.

Then put two triangles whose edge lengths are

$$p_2(a_1, a_2, a_3, a_4, a_5, a_6),\ a_2,\ a_3$$

and

$$p_2(a_5, a_6, a_1, a_2, a_3, a_4),\ a_1,\ a_6$$

respectively on the edges of the hexagon as shown in Fig. 1.5.

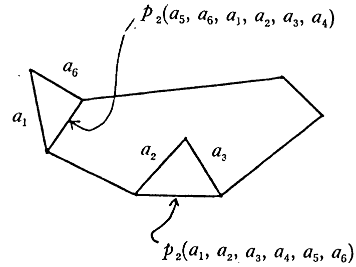


Fig. 1.5

Next we make both dihedral angles equal to $\pi/2$. If the equation (*) is satisfied, then between these two triangles there is a totally geodesic right-angled pentagon with one ideal vertex (see Fig. 1.6 (a)).

Then take another totally geodesic right-angled pentagon with one ideal

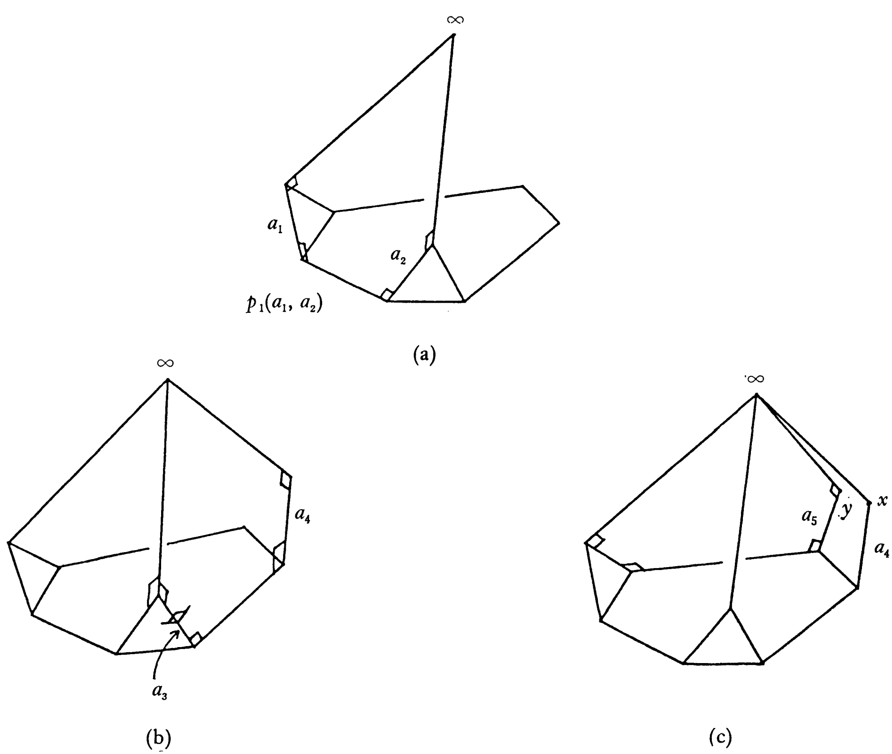


Fig. 1.6

vertex, and put it on the two edges as shown in Fig. 1.6 (b), where we make the dihedral angle between the pentagon and the triangle equal to $\pi/2$. In this case, our two pentagons have a common edge coming from ∞.

In the same way, we take one more pentagon as shown in Fig. 1.6 (c). (Since the hexagon has special edge lengths, we can take these three pentagons. See Definition 1.2 carefully.)

Now the two edges a_4 and a_5 are on the same geodesic ball (i.e. a hypersurface), so their endpoints x, y coincides each other. (Think the unique perpendicular geodesic from ∞ to this geodesic ball.)

In this way, we have constructed the required truncated tetrahedron.

Q.E.D. ■

2. A g-fold branched covering of the Whitehead link complement in S^3

In this section, for each number $g \geq 2$, we construct combinatorially a manifold which is a g-fold branched covering of the Whitehead link complement in S^3 branched along some codimension 2 submanifold by gluing all faces of $4g$ tetrahedra with vertices deleted.

First consider the following labelled four tetrahedra with vertices deleted and glue the faces according to the combinatorial gluing diagram in Fig. 2.1. Then we obtained a manifold which is homeomorphic to the Whitehead link complement in S^3 (see Thurston [3]).

Now for each number $g \geq 2$, we construct a 3-manifold which is a g-fold

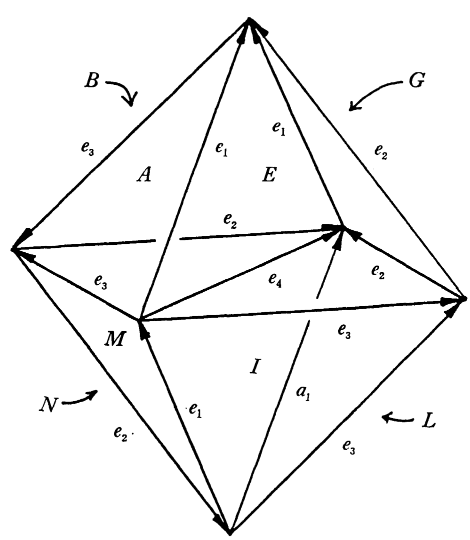


Fig. 2.1. Glue B to M, G to N, E to L and A to I so that arrows are matched.

covering of the Whitehead link complement in S^3 branched along the edge e_1. In order to give the cell decomposition of this 3-manifold, we prepare $4g$ tetrahedra with vertices deleted.

Let i be an integer with $1 \leq i \leq g$. If i is odd, then consider the following labelled four tetrahedra with vertices deleted and glue the faces according to the combinatorial gluing diagram in Fig. 2.2.

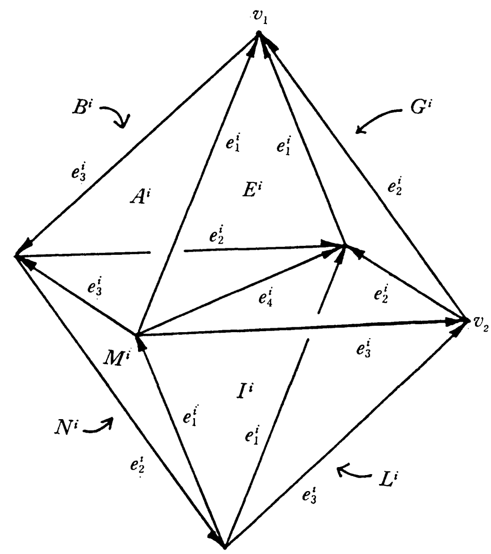


Fig. 2.2. Glue B^i to M^i, G^i to N^i and E^i to L^i so that arrows are matched ($i=1, 3, 5, \cdots$).

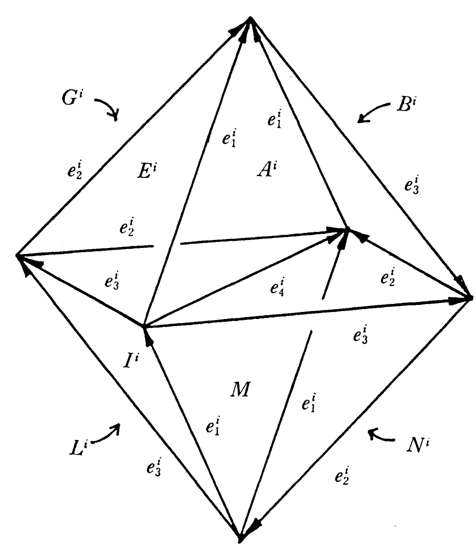


Fig. 2.3. Glue B^i to M^i, G^i to N^i and E^i to L^i so that arrows are matched ($i=2, 4, 6, \cdots$).

If i is even, then consider another combinatorial gluing diagram which is the mirror image of the above one (see Fig. 2.3).

Now we construct the desired manifold by gluing the faces of the above tetrahedra. If g is even, then glue A^1 to A^2, I^2 to I^3, A^3 to A^4, I^4 to I^5, $\cdots$, A^{g-1} to A^g and I^g to I^1 respectively. If g is odd, then glue A^1 to A^2, I^2 to I^3, A^3 to A^4, I^4 to I^5, $\cdots$, A^{g-2} to A^{g-1}, I^{g-1} to I^g and A^g to I^1 respectively.

Let N_g^3 be the resulting 3-manifold. This manifold N_g^3 is a g-fold branched covering of the Whitehead link complement in S^3 branched along edges $e_1^1, \cdots, e_1^g$ (for any g, all $e_1^i\,(i=1, \cdots, g)$ are identified). The manifold N_g^3 has two ends. Let $\mathcal{E}_i$ be the end which contains the boundary ∂_i of the neighborhood of the removed vertex $v_i\,(i=1, 2)$. Let M_g^3 be the 3-manifold which is obtained by truncating the 3-manifold N_g^3 along ∂_1. This 3-manifold M_g^3 has one boundary ∂_1 and one end $\mathcal{E}_2$. The edges of tetrahedra are classified into two kinds: one is the edge which starts from $\mathcal{E}_1$ and returns back to $\mathcal{E}_1$ and the other is the edge which connects $\mathcal{E}_1$ and $\mathcal{E}_2$. There are $g+1$ edges of the first kind

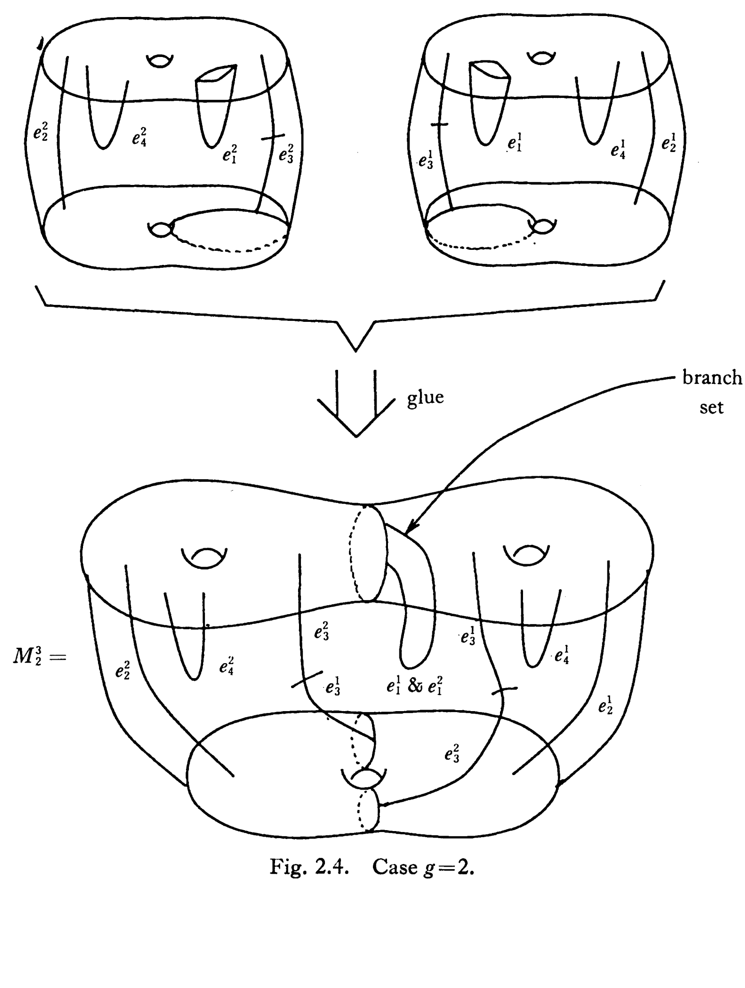


Fig. 2.4. Case $g=2$.

and $2g$ edges of the second kind. Now let us consider the triangulations of ∂_1 and ∂_2 induced by these tetrahedra. On ∂_1, there are $12g$ 2-simplices, $12g\times 3/2$ 1-simplices and $2(g+1)+2g$ 0-simplices. On ∂_2, there are $4g$ 2-simplices, $4g\times 3/2$ 1-simplices and $2g$ 0-simplices. Thus, the genus of ∂_1 is g, and the genus of ∂_2 is 1 (see Fig. 2.4 and Fig. 2.5).

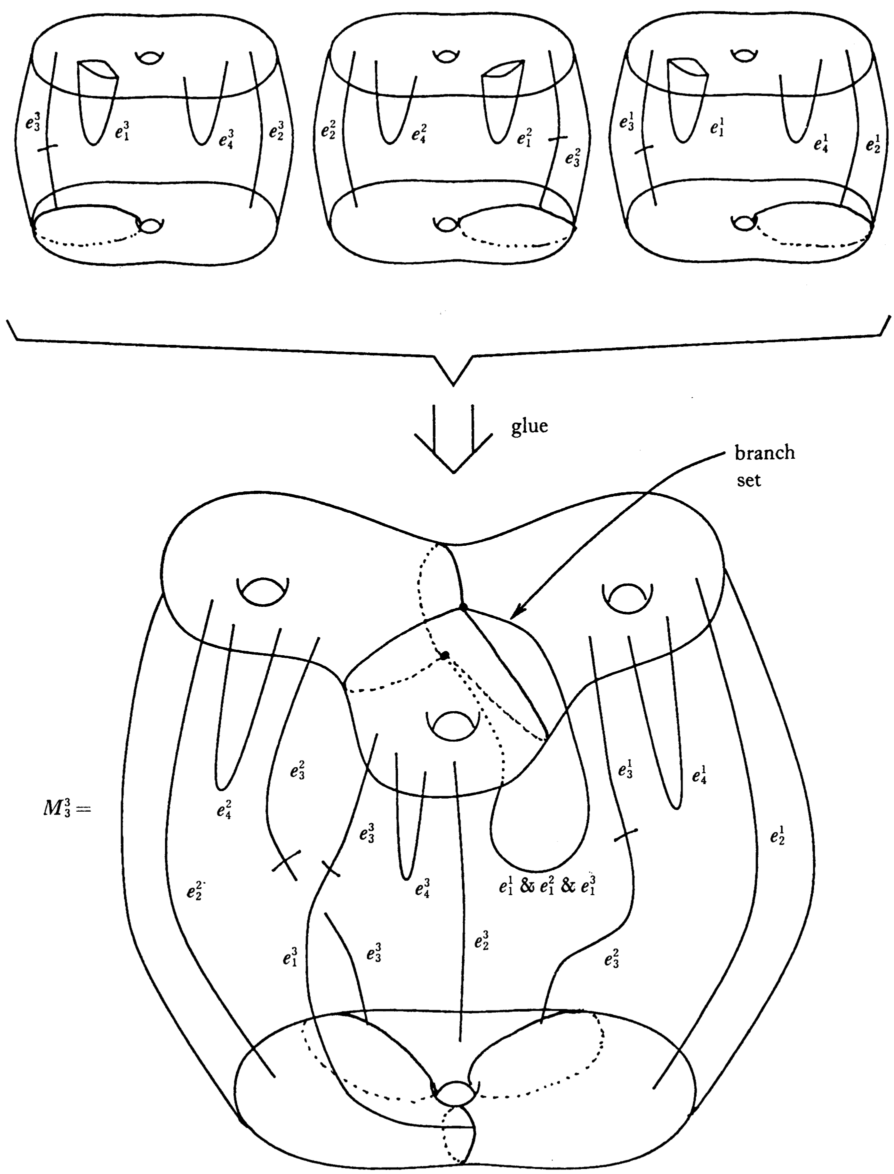


Fig. 2.5. Case $g=3$.

3. Infinitely many mutually non-isometric hyperbolic 3-manifolds with totally geodesic boundary of genus g

In this section we construct infinitely many mutually non-isometric hyperbolic 3-manifolds with totally geodesic boundary of genus g by using the manifold M_g^3 described above.

First we give a complete hyperbolic structure to the manifold M_g^3 by realizing the tetrahedra in §2 as the hyperbolic truncated tetrahedra which we give just below.

Let g be any integer greater than or equal to 2 and we fix it in the following arguments. By Proposition 1.3, there is a truncated tetrahedron with one ideal vertex as shown in Fig. 3.1. Each value indicated in the tpic ure represents the length of the corresponding geodesic or the angle (the values of angles are calculated by Proposition 1.1).

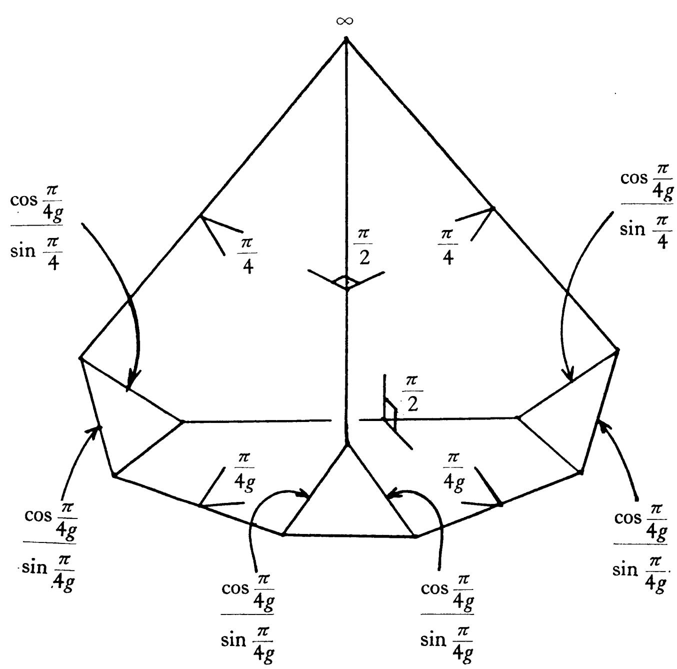


Fig. 3.1.

Let i be an integer with $1 \leq i \leq g$.

If i is odd, then take four copies of this truncated tetrahedron and glue the faces as in Fig. 3.2.

If i is even, then take four copies of this truncated tetrahedron and glue the faces as in Fig. 3.3.

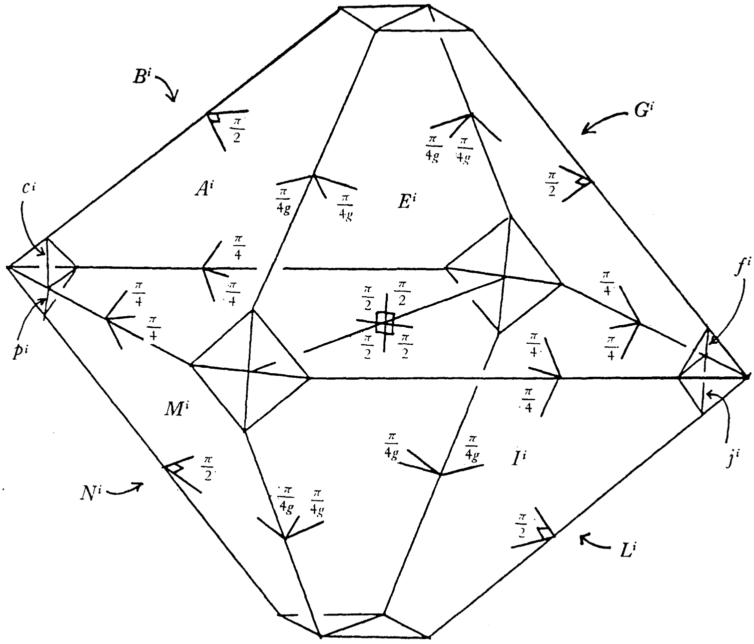


Fig. 3.2. Glue B^i to M^i, G^i to N^i and E^i to L^i. Triangles c^i, p^i, f^i and j^i are Euclidean ones each of which is realized as the intersection of the link of the corresponding ideal vertex of the truncated tetrahedron with a horosphere centered at this ideal vertex.

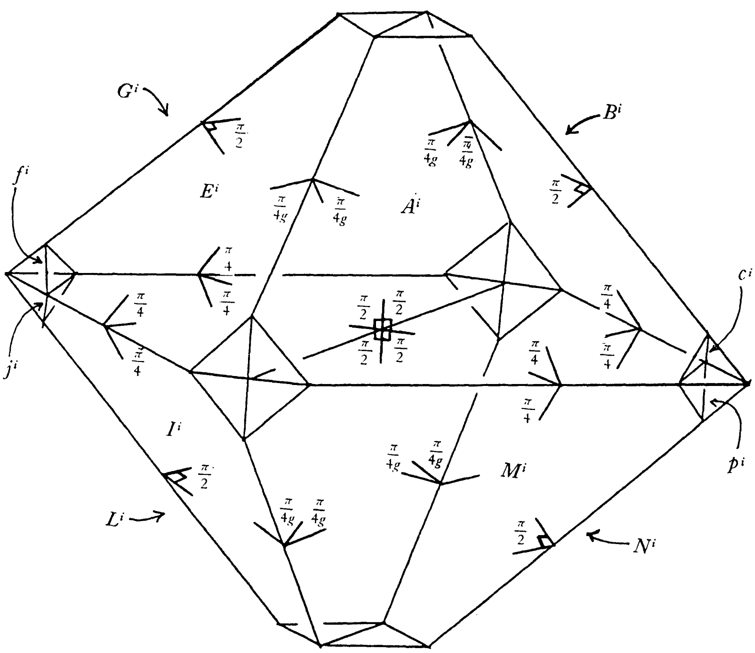


Fig. 3.3. Glue B^i to M^i, G^i to N^i and E^i to L^i. Triangles c^i, p^i, f^i and j^i are Euclidean ones each of which is realized as the intersection of the link of the corresponding ideal vertex of the truncated tetrahedron with a horosphere centered at this ideal vertex.

The realization of the tetrahedra in §2 as these hyperbolic truncated tetrahedra determines a hyperbolic structure on $(M_g^3\setminus\{\text{edges}\})$, because the corresponding two faces are always isometric to each other (see Remark below).

Remark. *Let θ be a real number satisfying the following:*

$$0<\theta<\pi/4.$$

Let a and b be the lengths of edges of the triangles in Fig. 3.4. *Then $a=b$ by Proposition* 1.1, *because*

$$\begin{aligned} a &= \frac{\cos\theta\cos(\pi/2)+\cos\theta}{\sin\theta\sin(\pi/2)} \\ &= \frac{\cos\theta}{\sin\theta} \\ &= \frac{\cos\theta\cos(\pi/4)+\cos(\pi/2)}{\sin\theta\sin(\pi/4)} \\ &= b\,. \end{aligned}$$

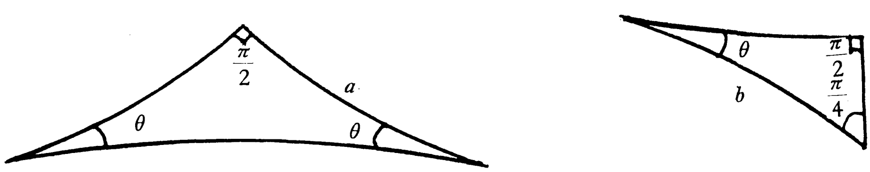


Fig. 3.4.

Proposition 3.1. *The above construction gives a complete hyperbolic structure to the resulting* 3-*manifold M_g^3.*

To prove this proposition we need the following:

Proposition 3.2 (Thurston [3]). *Let M^3 be a non-singular hyperbolic* 3-*manifold, possibly with boundary ∂M^3, obtained by gluing together the faces of polyhedra in $\boldsymbol{H}^3$ with some vertices at infinity. Then M^3 is complete if and only if the similarity structure on each link of an ideal vertex is actually a Euclidean structure.*

Proof of Proposition 3.1. Around each identified edge in M_g^3, the dihedral angles add up to 2π. Hence M_g^3 has a non-singular hyperbolic structure.

Now we look at the boundary of the toral end of M_g^3. If g is even, it is triangulated as in Fig. 3.5.

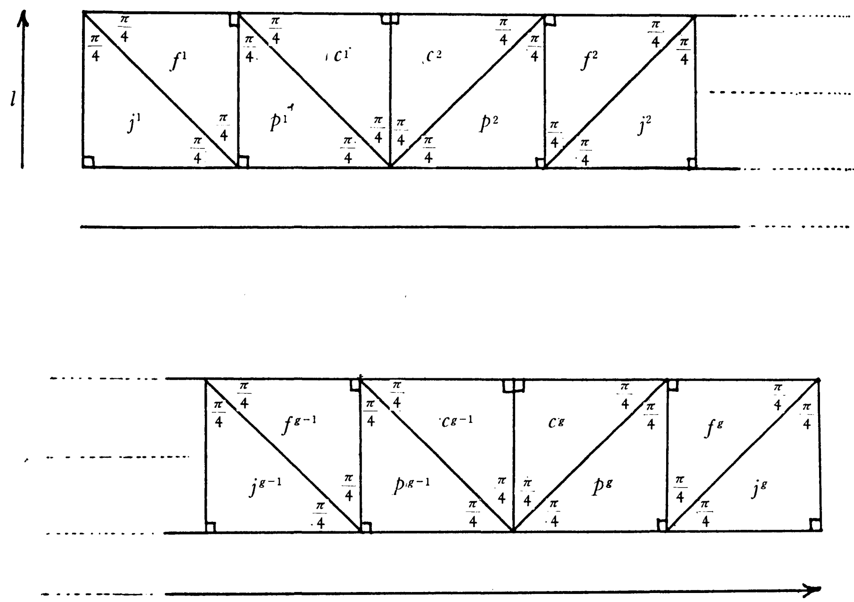


Fig. 3.5.

If g is odd, its triangulation is illustrated as in Fig. 3.6.

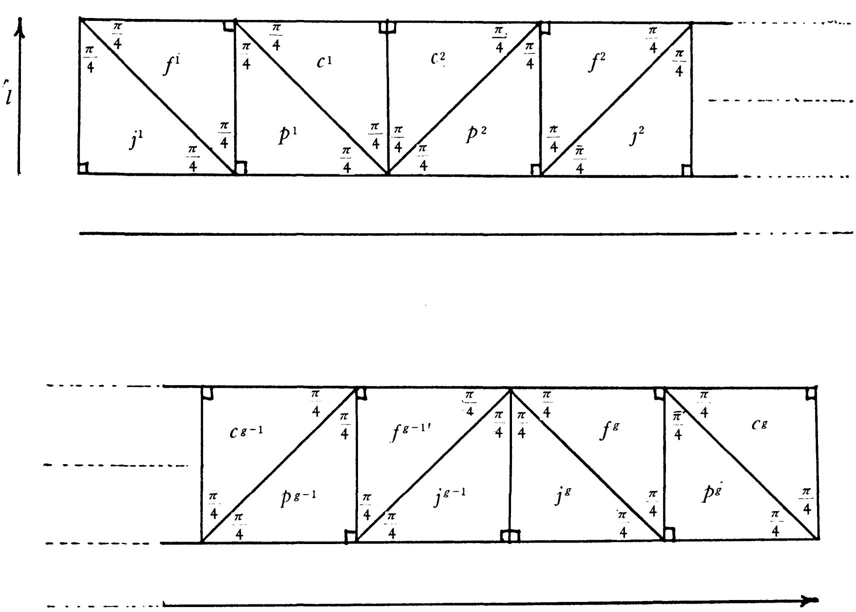


Fig. 3.6.

From the above pictures it is easy to observe that each of them is a torus with a Euclidean structure. Therefore by Proposition 3.2, M_g^3 has a complete hyperbolic structure. ■

Observe the pictures in the proof of Proposition 3.1 again. Choose generators m, l for $\pi_1(L(v))$ as indicated in these pictures, where $L(v)$ is the link of the ideal vertex v of M_g^3.

Now we can show the following:

Theorem 3.3. *Let g be an integer greater than or equal to 2. Let M_g^3 be the g-fold branched covering of the Whitehead link complement in S^3 which has a complete hyperbolic structure r^0 with one cusp and one totally geodesic boundary closed surface of genus g as was constructed in Proposition 3.1, and let $M_{g(\mu,\lambda)}^3$ be the topological 3-manifold obtained by performing the Dehn surgery of type (μ, λ) on the framed toral end of M_g^3. If (μ, λ) is a coprime pair of integers and $|\mu|+|\lambda|$ is sufficiently large, then $M_{g(\mu,\lambda)}^3$ has a hyperbolic structure with totally geodesic boundary. Thus we have explicitly constructed infinitely many mutually non-isometric compact oriented hyperbolic 3-manifolds each of which has a totally geodesic boundary closed surface of genus g.*

Proof. Take the double $D(M_g^3)$ of M_g^3. $D(M_g^3)$ has a complete hyperbolic structure with 2 toral ends (see Proposition 3.1). Let $D(M_g^3)_{(\mu_1,\lambda_1),(\mu_2,\lambda_2)}$ be the topological 3-manifold obtained by performing the Dehn surgeries of types (μ_1, λ_1), (μ_2, λ_2) on the two framed toral ends of $D(M_g^3)$ respectively.

By Theorem 5.8.2 in chapter 5.8 of Thurston [3], if (μ_i, λ_i) is a coprime pair of integers and $|\mu_i|+|\lambda_i|$ is sufficiently large ($i=1, 2$), then $D(M_g^3)_{(\mu_1,\lambda_1),(\mu_2,\lambda_2)}$ has a complete hyperbolic structure with finite volume. Now consider only the case $(\mu_1, \lambda_1)=(\mu_2, \lambda_2)$. Then there is an involution τ of $D(M_g^3)_{(\mu_1,\lambda_1),(\mu_1,\lambda_1)}$ which interchanges two copies of M_g^3 as well as two ends each other and leaves a closed surface of genus g invariant. By the Mostow's rigidity theorem, τ is homotopic to an isometry T which is still an involution;

$$T\colon D(M_g^3)_{(\mu_1,\lambda_1),(\mu_1,\lambda_1)} \xrightarrow{\cong} D(M_g^3)_{(\mu_1,\lambda_1),(\mu_1,\lambda_1)}\ ;\ \text{isometric involution}$$
$$\text{s.t.,}\quad T\simeq\tau;\ \text{homotopic.}$$

Consider now a surface which is invariant by T. It is homotopic to the above closed surface of genus g and must be totally geodesic. Consider the half of $D(M_g^3)_{(\mu_1,\lambda_1),(\mu_1,\lambda_1)}$ cutted along this surface (it is topologically $M_{g(\mu_1,\lambda_1)}^3$). It is a complete hyperbolic 3-manifold with totally geodesic boundary closed surface of genus g.

If (μ_1, λ_1) is near ∞ in $\boldsymbol{R}^2\cup\{\infty\}$, the shortest closed geodesic in $M_{g(\mu_1,\lambda_1)}^3$ is one of the two closed geodesics that we add to make $D(M_g^3)$ complete and there are infinitely many adjoning geodesics which have different lengths (cf. Neumann-Zagier [2]). Thus we have obtained infinitely many mutually non-isometric hyperbolic 3-manifolds each of which satisfies the condition in our theorem. Q.E.D. ■

Still stronger, we make the following:

Conjecture 3.4. *With the same hypotheses as in Theorem* 3.3, $M^3_{g(\mu,\lambda)}\backslash$ {the adjoining closed geodesic} *have the same combinatorial decomposition by means of truncated tetrahedra as that of* M^3_g *which was constructed in Proposition* 3.1.

REMARK. *The last part of the above theorem was proved by considering the lengths of the adjoining closed geodesics. However, we conjecture that the following stronger statement must hold. Namely there should exist infinitely many boundary closed surfaces of genus g which have different moduli. To show this, it is enough to prove Conjecture* 3.4 *and then that the natural map from the parameter space of geometric triangulation of the boundary surface of genus g, which has real dimension* 2, *to its Teichmüller space is* C^∞ *and has a full rank at the point where* M^3_g *is complete. If we can prove these, then there are infinitely many boundary surfaces of genus g which represent different points in an arbitrarily small neighborhood of one particular point in the Teichmüller space. Hence we can conclude that there are infinitely many boundary surfaces of genus g which have different moduli, because the modular group acts on the Teichmüller space properly discontinuously.*

References

[1] A.F. Beardon: The Geometry of Discrete Groups (GTM 91), Springer-Verlag, New York, 1983.

[2] W.D. Neumann and D. Zagier: *Volumes of hyperbolic three-manifolds,* Topology **24** (1985), 307–332.

[3] W.P. Thurston: The Geometry and Topology of 3-Manifolds, to be published by Princeton University Press, 1978/79.

Department of Mathematics
Tokyo Institute of Technology
Ohokayama, Meguro-ku, Tokyo
152, Japan