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THE MODULES INDUCED FROM A NORMAL SUBGROUP AND THE AUSLANDER-REITEN QUIVER

Dedicated to Professor Manabu Harada on his 60th birthday

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1. Introduction

Let G be a finite group and k a field of characteristic $p>0$. Let N be a normal subgroup of G. Let Θ be a connected component of the stable Auslander-Reiten quiver $\Gamma_s(kG)$ of the group algebra kG. We assume that there exist indecomposable N-projective kG-modules in Θ throughout this paper.

Choose an indecomposable N-projective kG-module L_0 in Θ. Let S_0 be an indecomposable kN-module such that $L_0 \mid S_0\uparrow^G$, and Ξ be a connected component of $\Gamma_s(kN)$ containing S_0. Set $T(\Xi):=\{g\in G \mid \Xi^g=\Xi\}=\{g\in G \mid S_0^g\in\Xi\}$, the inertia group of Ξ in G. Suppose that $S_0\uparrow^{T(\Xi)}=U_0\oplus U_1\oplus\cdots\oplus U_n$, where U_i is indecomposable and $L_0\simeq U_0\uparrow^G$. (Note that $T(\Xi)\supset T(S_0)=\{g\in G \mid S_0^g\simeq S_0\}$. Hence each $U_i\uparrow^G$ is indecomposable by [7], VII, 9.6 Theorem.) Let Λ be the connected component of $\Gamma_S(kT(\Xi))$ containing U_0. Now the purpose of this paper is to show the following theorem.

Theorem. *With the same notation and assumption as above, let U be an indecomposable $kT(\Xi)$-module in Λ. Then;*

(1) *The induced module $U\uparrow^G$ is indecomposable,*

(2) *The inducing from $T(\Xi)$ to G gives a graph isomorphism from Λ onto Θ which preserves edge-multiplicity and direction.*

The notation is almost standard. All the modules considered here are finite dimensional over k. We write $W \mid W'$ for kG-modules W and W' if W is isomorphic to a direct summand of W'. For an indecomposable non-projective kG-module M, we write $\mathcal{A}(M)$ to denote the Auslander-Reiten sequence (AR-sequence) $0\rightarrow\Omega^2M\rightarrow m(M)\rightarrow M\rightarrow 0$ terminating at M, and also we write $m(M)$ to denote the middle term of $\mathcal{A}(M)$. Here Ω denotes the Heller operator. A sequence $L_0-L_1-\cdots-L_t$ of indecomposable modules $L_i(0\leq i\leq t)$ is said to be a *walk* if there exists an irreducible map either from L_i to L_{i+1} or from L_{i+1} to L_i for $0\leq i\leq t-1$. Concerning some basic facts and terminologies used here, we refer to [1], [4] and [5].

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2. Preliminaries

Here we recall some basic results on AR-sequences of the group algebra kG.

Lemma 2.1 ([1], Proposition 2.17.10). *Let M be an indecomposable non-projective kG-module and H be a subgroup of G. Then the AR-sequence $\mathcal{A}(M)$ splits on restriction to H if and only if M is not H-projective.*

Lemma 2.2 ([3], Lemma 1.5 and [6], Theorem 7.5). *Let H be a subgroup of G. Let L and U be indecomposable non-projective modules for G and H respectively. Assume that U is a direct summand of $(U\uparrow^G)\downarrow_H$ with multiplicity one, and that L is an indecomposable direct summand of $U\uparrow^G$ such that $U \mid L\downarrow_H$. Then $\mathcal{A}(U)\uparrow^G \simeq \mathcal{A}(L)\oplus\mathcal{E}$, where $\mathcal{E}$ is a split sequence.*

Let $(\ ,\ )$ denote the inner product on the Green ring $a(kG)$ induced by $\dim_k \mathrm{Hom}_{kG}(\ ,\ )$ [2]. For an exact sequence of kG-modules $\mathcal{S}$: $0\to \mathrm{A}\to \mathrm{B}\to \mathrm{C}\to 0$, let $[\mathcal{S}]\in a(kG)$ be the element $\mathrm{B}-\mathrm{A}-\mathrm{C}$.

Lemma 2.3. *Let H be a normal subgroup of G. Let M be an indecomposable H-projective (but non-projective) kG-module and S be an indecomposable kH-module such that $M \mid S\uparrow^G$. Then $[\mathcal{A}(M)\downarrow_H]=n(\sum_{g\in X}[\mathcal{A}(S^g)])$, where X is a right transversal of $T(S)$ in G and n is the multiplicity of M as a summand of $S\uparrow^G$.*

Proof. By [2], Theorem 3.4, it suffices to show that $(V, [\mathcal{A}(M)\downarrow_H]-n(\sum_{g\in X}[\mathcal{A}(S^g)]))=0$ for any indecomposable kH-module V. Using the Frobenius reciprocity, we have

$$\begin{aligned}
&(V, [\mathcal{A}(M)\downarrow_H]-n(\textstyle\sum_{g\in X}[\mathcal{A}(S^g)]))\\
&\quad = (V, [\mathcal{A}(M)\downarrow_H])-(V, n(\textstyle\sum_{g\in X}[\mathcal{A}(S^g)]))\\
&\quad = (V\uparrow^G, [\mathcal{A}(M)])-n(V, (\textstyle\sum_{g\in X}[\mathcal{A}(S^g)])) .
\end{aligned}$$

Now $M \mid V\uparrow^G$ if and only if V is isomorphic to S^g for some $g\in G$ since $M\downarrow_H \mid \bigoplus_{g\in G/H} S^g$ and $(V\uparrow^G)\downarrow_H \simeq \bigoplus_{g\in G/H} V^g$ by the Mackey decomposition. If $V\simeq S^g$ for some $g\in G$, then M is a direct summand of $V\uparrow^G$ with multiplicity n. Hence we get $(V, [\mathcal{A}(M)\downarrow_H]-n(\sum_{g\in X}[\mathcal{A}(S^g)]))=0$ as desired.

3. Indecomposable modules

In this section, we shall give a proof of the main theorem. Returning to the situation of the Introduction, we assume that N is a normal subgroup of

G. Let $T=T(\Xi)$ be the inertia group of Ξ in G.

Lemma 3.1. *Let L be an indecomposable kG-module in Θ. Then every direct summand of $L\downarrow_N$ lies in $\cup_{g\in G}\Xi^g$. In particular, some summand lies in Ξ.*

Proof. Let $L_0-L_1-\cdots-L_t=L$ be a walk in Θ. We prove the assertion by induction on t.

If $t=0$, then $L_0\downarrow_N|(S_0\uparrow^G)\downarrow_N\simeq\oplus_{g\in G/N}S_0^g$ and each S_0^g lies in Ξ^g. Hence the assertion follows for $t=0$.

Suppose the assertion holds for L_{t-1}. We distinguish the following two cases.

Case 1. L_{t-1} is N-projective. Let S_{t-1} be an indecomposable kN-module such that $L_{t-1}|S_{t-1}\uparrow^G$. Since every direct summand of $L_{t-1}\downarrow_N$ lies in $\cup_{g\in G}\Xi^g$, we may assume that S_{t-1} lies in Ξ. From Lemma 2.3, we have $[\mathcal{A}(L_{t-1})\downarrow_N]=n(\sum_{g\in X}[\mathcal{A}(S_{t-1}{}^g)])$, where X is a right transversal of $T(S_{t-1})$ in G and n is the multiplicity of L_{t-1} as a summand of $S_{t-1}\uparrow^G$. This implies that $m(L_{t-1})\downarrow_N|\oplus_{g\in X}m(S_{t-1}{}^g)\oplus(L_{t-1}\oplus\Omega^2L_{t-1})\downarrow_N$ and every direct summand of $m(L_{t-1})\downarrow_N$ lies in $\cup_{g\in G}\Xi^g$ (Recall that the Auslander-Reiten translation τ is Ω^2 here). Since $L_t|m(L_{t-1})$ or $L_t|m(\Omega^{-2}L_{t-1})$, we have $L_t\downarrow_N|(m(L_{t-1})\oplus m(\Omega^{-2}L_{t-1}))\downarrow_N$. Therefore every direct summand of $L_t\downarrow_N$ lies in $\cup_{g\in G}\Xi^g$.

Case 2. L_{t-1} is not N-projective. Then AR-sequences $\mathcal{A}(L_{t-1})$ and $\mathcal{A}(\Omega^{-2}L_{t-1})$ split on restriction to N by Lemma 2.1. Hence we have $m(L_{t-1})\downarrow_N\simeq(L_{t-1}\oplus\Omega^2L_{t-1})\downarrow_N$ and $m(\Omega^{-2}L_{t-1})\downarrow_N\simeq(\Omega^{-2}L_{t-1}\oplus L_{t-1})\downarrow_N$. Since $L_t|m(L_{t-1})$ or $L_t|m(\Omega^{-2}L_{t-1})$, we have $L_t\downarrow_N|(\Omega^2L_{t-1}\oplus L_{t-1}\oplus\Omega^{-2}L_{t-1})\downarrow_N$ and so every direct summand of $L_t\downarrow_N$ lies in $\cup_{g\in G}\Xi^g$.

The following is immediate from Lemma 3.1.

Corollary 3.2. *Let U be an indecomposable kT-module in Λ. Then every direct summand of $U\downarrow_N$ lies in Ξ.*

Lemma 3.3. *Let U be an indecomposable kT-module in Λ. Let $(U\uparrow^G)\downarrow_T\simeq U\oplus Z$. Then $Z\downarrow_N$ has no indecomposable direct summand which lies in Ξ. In particular U is a direct summand of $(U\uparrow^G)\downarrow_T$ with multiplicity one.*

Proof. By the Mackey decomposition, we have

$$Z\simeq\bigoplus_{\substack{g\in T\backslash G/T\\ g\notin T}}(U^g\downarrow_{T^g\cap T})\uparrow^T$$

and

$$Z\downarrow_N\simeq\bigoplus_{\substack{g\in T\backslash G/T\\ g\notin T}}\Big(\bigoplus_{h\in(T^g\cap T)\backslash T/N}U^{gh}\downarrow_N\Big).$$

Now each indecomposable direct summand of $U\downarrow_N$ lies in Ξ by Corollary 3.2.

For $g \notin T = T(\Xi)$ and $h \in T$, $(U\downarrow_N)^{gh}$ does not have an indecomposable direct summand which lies in Ξ, and thus $Z\downarrow_N$ does not, either. This implies that Z has no indecomposable direct summand which lies in Λ by Corollary 3.2.

Lemma 3.4. *Let U and U' be indecomposable kT-modules in Λ. Then $U\uparrow^G \simeq U'\uparrow^G$ if and only if $U \simeq U'$.*

Proof. If $U \simeq U'$, then $U\uparrow^G \simeq U'\uparrow^G$ clearly. To show the converse, assume by way of contradiction that $U\uparrow^G \simeq U'\uparrow^G$ but $U \not\simeq U'$. Then $U' \mid (U'\uparrow^G)\downarrow_T \simeq (U\uparrow^G)\downarrow_T$ and hence we have $U \oplus U' \mid (U\uparrow^G)\downarrow_T$. Lemma 3.3 implies that $U'\downarrow_N$ has no direct summand contained in Ξ, which contradicts Corollary 3.2.

We are now ready to prove the theorem stated in the Introduction.

Proof of Theorem. (1) Let $U_0 - U_1 - \cdots - U_t = U$ be a walk in Λ. If $t=0$, i.e., $U \simeq U_0$, then $U_0\uparrow^G \simeq L_0$ as we have seen in the Introduction. Suppose then that $U_{t-1}\uparrow^G$ is indecomposable. We shall derive a contradiction assuming that $U_t\uparrow^G$ is decomposable. Let $U_t\uparrow^G = L \oplus M$ and $(U_t\uparrow^G)\downarrow_T = U_t \oplus Z_t$. We may assume that $U_t \mid L\downarrow_T$. Hence $M\downarrow_T \mid Z_t$, and Lemma 3.3 implies that any direct summand of $M\downarrow_N$ does not lie in Ξ. On the other hand, by Lemmas 2.2 and 3.3, we have $\mathcal{A}(U_{t-1})\uparrow^G \simeq \mathcal{A}(U_{t-1}\uparrow^G)$ and $\mathcal{A}(\Omega^{-2} U_{t-1})\uparrow^G \simeq \mathcal{A}(\Omega^{-2} U_{t-1}\uparrow^G)$ since $U_{t-1}\uparrow^G$ is indecomposable. Since $U_t \mid m(U_{t-1})$ or $U_t \mid m(\Omega^{-2} U_{t-1})$, $U_t\uparrow^G$ is a direct summand of $(m(U_{t-1}) \oplus m(\Omega^{-2} U_{t-1}))\uparrow^G \simeq m(U_{t-1}\uparrow^G) \oplus m(\Omega^{-2} U_{t-1}\uparrow^G)$. This means that every indecomposable direct summand of $U_t\uparrow^G$ lies in Θ. In particular, each direct summand of M lies in Θ, and hence Lemma 3.1 implies that $M\downarrow_N$ has an indecomposable direct summand contained in Ξ, which is a desired contradiction.

(2) From Lemmas 2.2, 3.3 and (1), we have $\mathcal{A}(U)\uparrow^G = \mathcal{A}(U\uparrow^G)$ for an indecomposable kT-module U in Λ. This and an inductive argument yield that the inducing from T to G gives an epimorphism from Λ onto Θ. Also, it must be a graph epimorphism. On the other hand, Lemma 3.4 implies that it is a monomorphism.

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