Komatsu, H.
Osaka J. Math.
22 (1985), 811-814

A COMMUTATIVITY THEOREM FOR RINGS. II

HIROAKI KOMATSU

(Received October 22, 1984)

Throughout the present paper, R will represent a ring with center C, and D the commutator ideal of R. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$; R is called s-unital if R is both left and right s-unital. Given a positive integer n, we say that R has the property $Q(n)$ if for any $x, y \in R$, $n[x, y]=0$ implies $[x, y]=0$ (see [1]).

Our present objective is to generalize [2, Theorem] for left s-unital rings as follows:

Theorem. *Let $n>0$, r, s and t be non-negative integers and let $f(X, Y)=\sum_{i=1}^{r}\sum_{j=2}^{s} f_{ij}(X, Y)$ be a polynomial in two noncommuting indeterminates X, Y with integer coefficients such that each f_{ij} is a homogeneous polynomial with degree i in X and degree j in Y and the sum of the coefficients of f_{ij} equals zero. Suppose a left s-unital ring R satisfies the polynomial identity*

$$X^t[X^n, Y]-f(X, Y) = 0\,. \tag{1}$$

If either $n=1$ or $r=1$ and R has the property $Q(n)$, then R is commutative.

We shall use freely the following well known result stated without proof.

Lemma. *Let x, y be elements of a ring with 1, and let k be a positive integer. If $x^k y=0=(x+1)^k y$ then $y=0$.*

Proof of Theorem. Let y be an arbitrary element of R, and choose an element e of R such that $ey=y$. Then (1) gives $y-ye^n=f(e, y)\in yR$. We have thus seen that R is right s-unital, and hence s-unital. Therefore, in view of [1, Proposition 1], it suffices to prove the theorem for R with 1.

Observe that D is a nil ideal of R, by a theorem of Kezlan-Bell (see, e.g., [1, Proposition 2]), since $x=e_{11}$ and $y=e_{12}$ fail to satisfy (1).

I) We consider first the case $n=1$. Let a, b be elements of R. By Lemma, it is easy to see that if $x^t a[x, b]=0$ for all $x\in R$ then $a[x, b]=0$. Noting this fact, we can apply the argument employed in the proof of [2, Theorem] to see the commutativity of R.

II) Next, suppose that $n>1$, $r=1$ and R has the property $Q(n)$. We claim that $D\subseteq C$. In fact, if $a\in D\backslash C$ then there exists a positive integer p such that $a^p\notin C$ and $a^k\in C$ for all $k>p$. For any $y\in R$, by repeated use of (1), we have $n(1+a^p)^t[a^p, y]=(1+a^p)^t[(1+a^p)^n, y]=f(1+a^p, y)=f(1, y)+f(a^p, y)=f(a^p, y)=a^{pt}[a^{pn}, y]=0$. Since $1+a^p$ is a unit in R, we have

$$n[a^p, y]=0. \tag{2}$$

Hence, $[a^p, y]=0$ by $Q(n)$, a contradiction. We have thus seen that $D\subseteq C$. We write $f_{1j}(X, Y)=\sum_{k=0}^{j} \alpha_{jk} Y^k X Y^{j-k}$. Since $\sum_{k=0}^{j} \alpha_{jk}=0$ by assumption, we have $f_{1j}(x, y)=\sum_{k=0}^{j-1} \alpha_{jk}(y^k x y^{j-k}-y^j x)=\sum_{k=0}^{j-1} \alpha_{jk} y^k[x, y^{j-k}]=\sum_{k=0}^{j-1} (j-k)\, \alpha_{jk} y^{j-1}[x, y]$ for any $x, y\in R$. Therefore, we can write $f(X, Y)=g(Y)\,[X, Y]$ with some polynomial g with integer coefficients, and (1) becomes

$$nX^{t'}[X, Y]-g(Y)\,[X, Y]=0, \text{ where } t'=n+t-1>0. \tag{3}$$

For any positive integers k, l, we denote by $h_{kl}(X, Y)$ the polynomial $(k+1)\,(n^{kl}-g(Y)^{kl})\,[X, Y]$. By repeated use of (3), for any x, $y\in R$ we have $(k+1)n^{kl} x^{t'+k}[x, y]=(k+1)\, n^{kl-1}\, x^k[x, y]\, g(y)=n^{kl-1}[x^{k+1}, y]\, g(y)=n^{kl}\, x^{(k+1)t'}[x^{k+1}, y]=(k+1)\, n^{kl}\, x^{(k+1)t'+k}[x, y]$. Then, $(k+1)\, n^{kl}\, x^{t'+k}[x, y]=(k+1)\, n^{kl}\, x^{t'+k}[x, y]\, x^{kt'}=(k+1)\, n^{kl}\, x^{t'+k}[x, y]\, x^{klt'}=(k+1)\, x^{t'+k}\, g(y)^{kl}[x, y]$. Therefore, $(k+1)\, x^{t'+k}(n^{kl}-g(y)^{kl})[x, y]=0$, and hence $h_{kl}(x, y)=0$ (Lemma). In particular, $n^2[x, (1-x^{2t'})\, y]=n^2(1-x^{2t'})\,[x, y]=(n^2-g(y)^2)\,[x, y]=h_{21}(x, y)-h_{12}(x, y)=0$, and therefore $(1-x^{2t'})[x, y]=0$. Exchanging x and y, we have $[x, y]=y^{2t'}\,[x, y]$, which comes under the case I). This completes the proof.

As an application of our theorem, we shall prove the following which includes [3, Theorem], [4, Theorem] and [5, Theorems 1 and 2].

Corollary 1. *Let $n>0$, m, t and s be fixed non-negative integers such that $(n, t, m, s)\neq(1, 0, 1, 0)$. Suppose a left s-unital ring R satisfies the polynomial identity*

$$X^t[X^n, Y]-[X, Y^m]Y^s=0. \tag{4}$$

(a) *If R has the property $Q(n)$ then R is commutative.*
(b) *If n and m are relatively prime then R is commutative.*

Proof. Let x, y be arbitrary elements of R, and choose an element e of R such that $ex=x$ and $ey=y$. If $(m, s)\neq(1, 0)$ then (4) gives $y=ye^n+ey^{m+s}-y^m e y^s\in yR$. On the other hand, if $(m, s)=(1, 0)$ then $(n, t)\neq(1, 0)$ and (4) gives $x=xe-x^{n+t}e+x^{n+t}\in xR$. We have thus seen that R is s-unital. Therefore, by [1, Proposition 1], we may assume that R has 1.

If $m=0$ (in the case of (a)), the assertion is clear by Theorem. Next, we consider the case $n=1$. If $m>0$ and $(m, s)\neq(1, 0)$ then $m+s>1$, and hence the assertion is clear, again by Theorem. Also, if $(m, s)=(1, 0)$ then, exchanging the roles of X and Y, we get the assertion. Similarly, we can prove the assertion for $m=1$. Therefore, we may assume henceforth that $n>1$ and $m>1$. For the case (a), the assertion is immediate by Theorem. So, we consider the case (b). Let $a\in D$, and $y\in R$. If a is not in C then there exists a positive integer p such that $a^p\notin C$ and $a^k\in C$ for all $k>p$ and $n[a^p, y]=0$ by (2); similarly we can prove $m[a^p, y]=0$. Hence, $[a^p, y]=0$. This contradiction shows that $D\subseteq C$, and (4) becomes

$$(5) \qquad nX^{n+t-1}[X, Y]=mY^{m+s-1}[X, Y].$$

If $n[x, y]=0$ $(x, y\in R)$ then (5) gives $my^{m+s-1}[x, y]=nx^{n+t-1}[x, y]=0=nx^{n+t-1}[x, y+1]=m(y+1)^{m+s-1}[x, y]$, whence $m[x, y]=0$ follows by Lemma, and hence $[x, y]=0$. This prove that R has the property $Q(n)$. Hence, R is commutative by Theorem, completing the proof.

Corollary 2. *Let $n>0$ and m be fixed non-negative integers. Suppose a left s-unital ring R satisfies the polynomial identity $[XY, X^n+Y^m]=0$. If either R has the property $Q(n)$ or n and m are relatively prime, then R is commutative.*

Proof. Actually, R satisfies the polynomial identity $X[X^n, Y]-[X, Y^m]Y=0$. Hence R is commutative by Corollary 1.

REMARK. In case $n>0$ and $m=0$, Corollary 1 need not be true for right s-unital rings (see [3, Remark]).

The author wishes to express his indebtedness and gratitude to Prof. H. Tominaga for his helpful suggestions and valuable comments.

References

[1] Y. Hirano, Y. Kobayashi and H. Tominaga: *Some polynomial identities and commutativity of s-unital rings,* Math. J. Okayama Univ. **24** (1982), 7–13.

[2] T.P. Kezlan: *On identities which are equivalent with commutativity,* Math. Japon. **29** (1984), 135–139.

[3] H. Komatsu: *A commutativity theorem for rings,* Math. J. Okayama Univ. **26** (1984), 109–111.

[4] E. Psomopoulos: *A commutativity theorem for rings involving a subset of the ring,* Glasnik Mat. **18** (1983), 231–236.

[5] E. Psomopoulos: *Commutativity theorems for rings and groups with constraints on commutators,* to appear in Internat. J. Math. Math. Sci.

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558
Japan