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ON THE SPECTRUM REPRESENTING ALGEBRAIC K-THEORY FOR A FINITE FIELD

Dedicated to Professor Nobuo Shimada on his sixtieth birthday

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Let r be an odd prime power. Let $\boldsymbol{F}_r$ denote the field with r elements. According to [11] and others, there exists a (-1)-connected Ω-spectrum $\boldsymbol{KF}_r$ whose 0-th space is $\boldsymbol{Z}\times BGL\boldsymbol{F}_r^+$, where $BGL\boldsymbol{F}_r^+$ is the plus construction of the classifying space of $GL\boldsymbol{F}_r$. $\boldsymbol{KF}_r$ is a ring spectrum with a unit.

Let p be an odd prime. The object of this paper is the localization of $\boldsymbol{KF}_r$ at p, $\boldsymbol{KF}_{r(p)}$, for the case that r gives a generator of the group of units $(\boldsymbol{Z}/p^2)^\times$. Then the associated generalized cohomology theory $KF_r^*(\ ;\boldsymbol{Z}_{(p)})$ appears as a secondary cohomology theory determined by a certain stable operation in connected complex K-theory localized at p. From this interpretation we deduce some results about the multiplicative structure on $\boldsymbol{KF}_{r(p)}$, which are basic to the study of the ring structure of $KF_{r*}(CP^\infty;\boldsymbol{Z}_{(p)})$ etc. In particular we can characterize the product on $\boldsymbol{KF}_{r(p)}$ by a certain property.

For simplicity we write $\boldsymbol{A}$ for $\boldsymbol{KF}_{r(p)}$ (see [8]). We shall work in the homotopy category of CW-spectra (see [3, III]).

The paper is organized as follows. In § 0 we collect several results on $\boldsymbol{A}$. In § 1 we compute $H^*(\boldsymbol{A};\boldsymbol{Z}/p)$. In § 2 we compute $H_*(\boldsymbol{A};\boldsymbol{Z}/p)$. In § 3 we consider the left coaction of $\mathcal{A}_*$ on $H_*(\boldsymbol{A};\boldsymbol{Z}/p)$ and discuss the $\mathcal{B}$-module structure of $H^*(\boldsymbol{A};\boldsymbol{Z}/p)$, where $\mathcal{B}=\Lambda(Q_0,Q_1)\subset\mathcal{A}$. In § 4 we prove our main results, which are Theorems 4.3 and 4.5.

0. The spectrum A

Let p be a fixed odd prime. Let $\boldsymbol{bu}_{(p)}$ be the Ω-spectrum representing connected complex K-theory localized at p. This is a ring spectrum with a unit and $\pi_*(\boldsymbol{bu}_{(p)})=\boldsymbol{Z}_{(p)}[u]$ where $|u|=2$. It is known that

$$\boldsymbol{bu}_{(p)}=\bigvee_{j=1}^{p-1}\Sigma^{2(j-1)}\boldsymbol{G}$$

for a spectrum $\boldsymbol{G}$ [6]. This is a ring spectrum with a unit and $\pi_*(\boldsymbol{G})=\boldsymbol{Z}_{(p)}[v]$ where $|v|=2(p-1)$. According to [4], if $\kappa\colon \boldsymbol{G}\to\boldsymbol{bu}_{(p)}$ is the injection, then the diagram

(0.1)
$$\begin{tikzcd}
\Sigma^{2(p-1)}G \arrow[r, "v"] \arrow[d, "\Sigma^{2(p-1)}\kappa"'] & G \arrow[dd, "\kappa"] \\
\Sigma^{2(p-1)}\boldsymbol{bu}_{(p)} \arrow[d, "u^{p-2}"'] & \\
\Sigma^2\boldsymbol{bu}_{(p)} \arrow[r, "u"] & \boldsymbol{bu}_{(p)}
\end{tikzcd}$$

commutes, where (by abuse of notation) u, v denote the composites $S^2\wedge\boldsymbol{bu}_{(p)} \xrightarrow{u\wedge 1} \boldsymbol{bu}_{(p)}\wedge\boldsymbol{bu}_{(p)}\to\boldsymbol{bu}_{(p)}$ and $S^{2(p-1)}\wedge G\xrightarrow{v\wedge 1} G\wedge G\to G$ respectively. Furthermore, for each r prime to p, there exists a map of ring spectra ψ^r: $G\to G$ which makes the diagram

(0.2)
$$\begin{tikzcd}
G \arrow[r, "\psi^r"] \arrow[d, "\kappa"'] & G \arrow[d, "\kappa"] \\
\boldsymbol{bu}_{(p)} \arrow[r, "\psi^r"] & \boldsymbol{bu}_{(p)}
\end{tikzcd}$$

commute, where the lower ψ^r is derived from the Adams operation in complex K-theory.

Consider the fibre sequence

$$\Sigma^2\boldsymbol{bu}_{(p)} \xrightarrow{u} \boldsymbol{bu}_{(p)} \xrightarrow{\rho} \boldsymbol{HZ}_{(p)}$$

(where $\boldsymbol{HZ}_{(p)}$ denotes the Eilenberg-MacLane spectrum for $\boldsymbol{Z}_{(p)}$). This leads to an exact sequence

$$0\to[\boldsymbol{bu}_{(p)}, \Sigma^2\boldsymbol{bu}_{(p)}] \xrightarrow{u_*} [\boldsymbol{bu}_{(p)}, \boldsymbol{bu}_{(p)}] \xrightarrow{\rho_*} [\boldsymbol{bu}_{(p)}, \boldsymbol{HZ}_{(p)}]$$

where we have used the fact that $H^{-1}(\boldsymbol{bu}_{(p)}; \boldsymbol{Z}_{(p)})=0$. Consider the element $\psi^r-1\in[\boldsymbol{bu}_{(p)}, \boldsymbol{bu}_{(p)}]$. Since $\rho_*(\psi^r-1)=0$, there is a unique $\theta\in[\boldsymbol{bu}_{(p)}, \Sigma^2\boldsymbol{bu}_{(p)}]$ such that $u_*(\theta)=\psi^r-1$. Denote by $\boldsymbol{A}$ the fibre spectrum of θ; that is,

$$\boldsymbol{A} \xrightarrow{\eta} \boldsymbol{bu}_{(p)} \xrightarrow{\theta} \Sigma^2\boldsymbol{bu}_{(p)} \tag{0.3}$$

is a fibre sequence.

From now on we deal with a case such that r is a generator of $(\boldsymbol{Z}/p^2)^\times$. Then $\boldsymbol{A}$ does not depend on the choice of r. In fact, since $(\psi^r-1)_*(u^s)=(r^s-1)u^s$ in $\pi_*(\boldsymbol{bu}_{(p)})$, [1, Lemma (2.12)] yields

$$\pi_i(\boldsymbol{A})=\begin{cases}\boldsymbol{Z}_{(p)} & \textit{if}\quad i=0\\ \boldsymbol{Z}/p^{1+\nu_p(t)} & \textit{if}\quad i=2t(p-1)-1\ \ (t>0)\\ 0 & \textit{otherwise}\end{cases} \tag{0.4}$$

where $\nu_p(t)$ is the power of p in t.

Consider the fibre sequence

$$\Sigma^{2(p-1)}G \xrightarrow{v} G \longrightarrow HZ_{(p)}\,.$$

By a similar argument we have a unique lift $\theta' \in [G, \Sigma^{2(p-1)}G]$ of $\psi^r - 1 \in [G, G]$. Let A' denote the fibre of θ'. Then from (0.1) and (0.2) it follows that there is a commutative diagram of fibre sequences

$$\begin{tikzcd}
A' \arrow[r, "\eta'"] \arrow[dd, dotted, "\kappa'"'] & G \arrow[r, "\theta'"] \arrow[dd, "\kappa"] & \Sigma^{2(p-1)}G \arrow[d, "\Sigma^{2(p-1)}\kappa"] \\
 & & \Sigma^{2(p-1)}bu_{(p)} \arrow[d, "u^{p-2}"] \\
A \arrow[r, "\eta"] & bu_{(p)} \arrow[r, "\theta"] & \Sigma^2 bu_{(p)}
\end{tikzcd}$$

It is easily verified that the induced map $\kappa'\colon A' \to A$ is an equivalence. So we may identify them.

Choose r to be an odd prime power so that it satisfies our hypothesis. In view of [12, VIII] it seems that there exists a map of ring spectra $Br\colon KF_{r(p)} \to bu_{(p)}$ and its lift $KF_{r(p)} \to A$ in (0.3) becomes an equivalence. We identify them and then η can be regarded as a map of ring spectra (cf. [15, p. 252]). Since κ is a (split injective) map of ring spectra, so is η'. In § 4 we give a different approach to this fact.

It is not an accident that $\pi_*(A)$ is isomorphic to $\operatorname{Im} J_{(p)}$ which is a direct summand of $\pi_*(S^0)_{(p)}$. In fact, Tornehave [19] showed that

(0.5) *The unit* $\hat{\imath}\colon S^0 \to A$ *realizes the projection of* $\pi_*(S^0)_{(p)}$ *onto* $\operatorname{Im} J_{(p)}$.

Hereafter for brevity we write

$$\Sigma^{2p-3}G \xrightarrow{\Delta} A \xrightarrow{\eta} G \xrightarrow{\theta} \Sigma^{2(p-1)}G\,. \tag{0.6}$$

We will use only this fibre sequence in later sections.

1. The mod p cohomology of A

Let $\mathcal{A}$ be the mod p Steenrod algebra. As an $\mathcal{A}$-module,

$$H^*(G; Z/p) \cong \mathcal{A}/\mathcal{A}(Q_0, Q_1) \tag{1.1}$$

where $Q_0 = \delta$, $Q_1 = \mathcal{P}^1\delta - \delta\mathcal{P}^1$ and $\mathcal{A}(\ )$ denotes the left ideal in $\mathcal{A}$ generated by the set in parentheses. Apply the functor $H^*(\ ; Z/p)$ to (0.6). Then we have

Lemma 1.1. *If f is the generator of $H^0(G; Z/p)$, then $\theta^*(\sigma^{2(p-1)}f) = c\cdot\mathcal{P}^1 f$ for some non-zero $c \in Z/p$ (where σ^i denotes the increase of degrees by i).*

Proof. By (1.1), $H^{2(p-1)}(\boldsymbol{G}; \boldsymbol{Z}/p)=\boldsymbol{Z}/p\{\mathcal{P}^1 f\}$. Hence we may set $\theta^*(\sigma^{2(p-1)}f)=c\cdot\mathcal{P}^1 f$ for some $c\in\boldsymbol{Z}/p$. It is sufficient to show that c is non-zero. Suppose $c=0$. Then it follows that $\tilde{H}^*(\boldsymbol{A}; \boldsymbol{Z}/p)=\boldsymbol{Z}/p\{\eta^*(\mathcal{P}^1 f)\}$ in degrees less than $2p(p-1)-1$. On the other hand, by (0.5) or [17], $\hat{\iota}_*: \pi_i(\boldsymbol{S}^0)_{(p)}\to\pi_i(\boldsymbol{A})$ is an isomorphism for $i<|\beta_1|=2p(p-1)-2$ (where $\beta_1\in\pi_*(\boldsymbol{S}^0)_{(p)}$ is the first element which does not belong to $\operatorname{Im} J_{(p)}$). By the Whitehead theorem, $\tilde{H}_*(\boldsymbol{A}; \boldsymbol{Z}/p)=0$ in degrees less than $2p(p-1)-2$. This is a contradiction.

REMARK. As in [5] one can prove this lemma by calculating the Adams spectral sequence for $\pi_*(\boldsymbol{A})$ and using (0.4). See also [10, p. 421].

For $a\in\mathcal{A}$ let $L(a)\colon \Sigma^{|a|}\mathcal{A}\to\mathcal{A}$ and $R(a)\colon \Sigma^{|a|}\mathcal{A}\to\mathcal{A}$ be defined by $L(a)(\sigma^{|a|}b)=ab$ and $R(a)(\sigma^{|a|}b)=ba$ respectively.

Corollary 1.2. *The following square commutes:*

$$\begin{array}{ccc} \Sigma^{2(p-1)}\mathcal{A} & \xrightarrow{c\cdot R(\mathcal{P}^1)} & \mathcal{A} \\ \downarrow & & \downarrow \\ \Sigma^{2(p-1)}\mathcal{A}/\mathcal{A}(Q_0, Q_1) & \xrightarrow{\theta^*} & \mathcal{A}/\mathcal{A}(Q_0, Q_1)\,. \end{array}$$

From this corollary we see that

$$\operatorname{Coker}(\theta^*\colon \Sigma^{2(p-1)}\mathcal{A}/\mathcal{A}(Q_0, Q_1)\to\mathcal{A}/\mathcal{A}(Q_0, Q_1))\cong\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)\,.$$

We also have an isomorphism

(1.2) $\operatorname{Ker}(\theta^*\colon \Sigma^{2(p-1)}\mathcal{A}/\mathcal{A}(Q_0, Q_1)\to\mathcal{A}/\mathcal{A}(Q_0, Q_1))\cong\Sigma^{2p(p-1)}\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)$

the inverse of which is induced by $R(\mathcal{P}^{p-1})$. (Although it is easy for a specialist to prove this fact directly, we do it by a different method in § 2.) Combining these, we get a short exact sequence of $\mathcal{A}$-modules

$$0\to\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)\xrightarrow{\hat{\eta}^*} H^*(\boldsymbol{A}; \boldsymbol{Z}/p)\xrightarrow{\hat{\Delta}^*}\Sigma^q\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)\to 0$$

where $q=2p(p-1)-1$. Put $g=\hat{\eta}^*(1)\in H^0(\boldsymbol{A}; \boldsymbol{Z}/p)$ and let $\sigma^q h\in H^q(\boldsymbol{A}; \boldsymbol{Z}/p)$ be the element such that $\hat{\Delta}^*(\sigma^q h)=\sigma^q 1$. Since $\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)^{q+1}=\boldsymbol{Z}/p\{\mathcal{P}^p\}$ and $\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)^{q+2(p-1)}=0$, we may set

$$H^*(\boldsymbol{A}; \boldsymbol{Z}/p)=\mathcal{A}\{g\}\oplus\Sigma^q\mathcal{A}\{h\}/\mathcal{A}(Q_0 g\oplus 0, \mathcal{P}^1 g\oplus 0, d\cdot\mathcal{P}^p g\oplus\sigma^q Q_0 h, 0\oplus\sigma^q\mathcal{P}^1 h)$$

for some $d\in\boldsymbol{Z}/p$. Here $d\neq 0$. For if $d=0$, then by looking at the cell structure of $\boldsymbol{A}$, we find that there is a CW-spectrum $(\boldsymbol{S}^0\cup e^{2p(p-1)})_{(p)}$ in which $\mathcal{P}^p$ is non-zero. This contradicts the tiiviality of the mod p Hopf invariant [16].

Theorem 1.3. *As a left $\mathcal{A}$-module $H^*(\boldsymbol{A}; \boldsymbol{Z}/p)$ is generated by g and $\sigma^q h$ subject to the relations*

$$Q_0(g) = 0\,,\ \mathcal{P}^1(g) = 0\,,\ \mathcal{P}^p(g) = Q_0(\sigma^q h) \quad and \quad \mathcal{P}^1(\sigma^q h) = 0\,.$$

Proof. Change $\sigma^q h$ for $d\cdot\sigma^q h$ if necessary.

2. The mod p homology of A

Most of this section is an odd prime version of [14].

Let $\mathcal{A}_*$ be the dual of $\mathcal{A}$. It is the tensor product of an exterior algebra and a polynomial algebra:

$$\mathcal{A}_* = \Lambda(\tau_0, \tau_1, \cdots)\otimes \boldsymbol{Z}/p[\xi_1, \xi_2, \cdots]$$

where $|\tau_n|=2p^n-1$ and $|\xi_n|=2p^n-2$. $\mathcal{A}_*$ is a left and right $\mathcal{A}$-module; respective actions are given by

$$\langle a(\alpha), b\rangle = \langle \alpha, ba\rangle \quad \text{and} \quad \langle (\alpha)a, b\rangle = \langle \alpha, ab\rangle$$

for all $a, b\in\mathcal{A}$ and $\alpha\in\mathcal{A}_*$. By abuse of notation, for $a\in\mathcal{A}$ let $L(a)\colon \mathcal{A}_*\to\Sigma^{|a|}\mathcal{A}_*$ and $R(a)\colon \mathcal{A}_*\to\Sigma^{|a|}\mathcal{A}_*$ be defined by $L(a)(\alpha)=\sigma^{|a|}a(\alpha)$ and $R(a)(\alpha)=\sigma^{|a|}(\alpha)a$ respectively; note that $R(a)\colon \Sigma^{|a|}\mathcal{A}\to\mathcal{A}$ and $L(a)\colon \mathcal{A}_*\to\Sigma^{|a|}\mathcal{A}_*$ are dual. Define $\mathcal{P}(\ )$, $(\ )\mathcal{P}\colon \mathcal{A}_*\to\mathcal{A}_*$ by $\mathcal{P}(\alpha)=\sum_{i\geq 0}\mathcal{P}^i(\alpha)$ and $(\alpha)\mathcal{P}=\sum_{i\geq 0}(\alpha)\mathcal{P}^i$ respectively. They are ring homomorphisms, since Cartan formulas $\mathcal{P}^n(\alpha\beta)=\sum_{i+j=n}\mathcal{P}^i(\alpha)\mathcal{P}^j(\beta)$ and $(\alpha\beta)\mathcal{P}^n=\sum_{i+j=n}(\alpha)\mathcal{P}^i(\beta)\mathcal{P}^j$ hold.

Proposition 2.1. *The following formulas hold:*

(i)
$$\begin{aligned}
&\mathcal{P}(\tau_n) = \tau_n\\
&\mathcal{P}(\xi_n) = \xi_n+\xi_{n-1}^p \qquad (\text{i.e., } \mathcal{P}^1(\xi_n) = \xi_{n-1}^p)\\
&\delta(\tau_n) = \xi_n\\
&\delta(\xi_n) = 0\,.
\end{aligned}$$

(ii)
$$\begin{aligned}
&(\tau_n)\mathcal{P} = \tau_n+\tau_{n-1} \qquad (\text{i.e., } (\tau_n)\mathcal{P}^{p^{n-1}} = \tau_{n-1})\\
&(\xi_n)\mathcal{P} = \xi_n+\xi_{n-1} \qquad (\text{i.e., } (\xi_n)\mathcal{P}^{p^{n-1}} = \xi_{n-1})\\
&(\tau_n)\delta = \begin{cases} 0 & \text{if } n>0\\ 1 & \text{if } n=0\end{cases}\\
&(\xi_n)\delta = 0\,.
\end{aligned}$$

Proof. Recall the definitions of τ_n and ξ_n.

By abuse of notation, let χ denote the conjugation in $\mathcal{A}$ or $\mathcal{A}_*$; note that $\chi\colon \mathcal{A}\to\mathcal{A}$ and $\chi\colon \mathcal{A}_*\to\mathcal{A}_*$ are dual.

Proposition 2.2. *For each $a\in\mathcal{A}$ with $\chi a=-a$, the following squares commute:*

$$\text{(i)}\quad \begin{array}{ccc} \mathcal{A}_* & \xrightarrow{L(a)} & \mathcal{A}_* \\ \chi\downarrow & & \downarrow\chi \\ \mathcal{A}_* & \xrightarrow{-R(a)} & \mathcal{A}_*\,. \end{array} \qquad \text{(ii)}\quad \begin{array}{ccc} \mathcal{A}_* & \xrightarrow{R(a)} & \mathcal{A}_* \\ \chi\downarrow & & \downarrow\chi \\ \mathcal{A}_* & \xrightarrow{-L(a)} & \mathcal{A}_*\,. \end{array}$$

The proof is immediate.

REMARK. This proposition can be applied to the cases $a=Q_0$, $\mathcal{P}^1$ and $\mathcal{P}^p$ (see [13, §7]).

By Theorem 1.3 there is an exact sequence of $\mathcal{A}$-modules

$$\Sigma\mathcal{A}\oplus\Sigma^{2(p-1)}\mathcal{A}\oplus\Sigma^{q+1}\mathcal{A}\oplus\Sigma^{q+2(p-1)}\mathcal{A} \xrightarrow{\substack{R(Q_0\oplus 0)\oplus R(\mathcal{P}^1\oplus 0)\oplus \\ R(-\mathcal{P}^p\oplus\sigma^q Q_0)\oplus R(0\oplus\sigma^q\mathcal{P}^1)}} \mathcal{A}\oplus\Sigma^q\mathcal{A} \xrightarrow{\varepsilon} H^*(\boldsymbol{A}: \boldsymbol{Z}/p) \to 0\,.$$

Dualizing this gives

$$\Sigma\mathcal{A}_*\oplus\Sigma^{2(p-1)}\mathcal{A}_*\oplus\Sigma^{q+1}\mathcal{A}_*\oplus\Sigma^{q+2(p-1)}\mathcal{A}_* \xleftarrow{\substack{L(Q_0)\oplus L(\mathcal{P}^1)\oplus \\ (-L(\mathcal{P}^p)+L(\sigma^q Q_0))\oplus L(\sigma^q\mathcal{P}^1)}} \mathcal{A}_*\oplus\Sigma^q\mathcal{A}_* \xleftarrow{\varepsilon_*} H_*(\boldsymbol{A}; \boldsymbol{Z}/p) \leftarrow 0\,.$$

Using Proposition 2.2 (i) we get an exact sequence

$$\Sigma\mathcal{A}_*\oplus\Sigma^{2(p-1)}\mathcal{A}_*\oplus\Sigma^{q+1}\mathcal{A}_*\oplus\Sigma^{q+2(p-1)}\mathcal{A}_* \xleftarrow{\substack{R(Q_0)\oplus R(\mathcal{P}^1)\oplus \\ (-R(\mathcal{P}^p)+R(\sigma^q Q_0))\oplus R(\sigma^q\mathcal{P}^1)}} \mathcal{A}_*\oplus\Sigma^q\mathcal{A}_* \leftarrow A_*(\boldsymbol{HZ}/p) \leftarrow 0$$

(where $A_*(\ \ )$ denotes the generalized homology theory associated with $\boldsymbol{A}$). In order to describe $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$, we calculate the kernel of $R(Q_0)\oplus R(\mathcal{P}^1)\oplus (-R(\mathcal{P}^p)+R(\sigma^q Q_0))\oplus R(\sigma^q\mathcal{P}^1)$ and apply $\chi\oplus\chi$ to it.

Using Proposition 2.1, we easily see that

$$\mathrm{Ker}\,(R(Q_0)\colon \mathcal{A}_*\to\Sigma\mathcal{A}_*) = \Lambda(\tau_1, \tau_2, \cdots)\otimes\boldsymbol{Z}/p[\xi_1, \xi_2, \cdots]$$

and

$$\mathrm{Ker}\,(R(\mathcal{P}^1)\colon \mathcal{A}_* \to \Sigma^{2(p-1)}\mathcal{A}_*) = \boldsymbol{Z}/p\{1, \tau_0, \tau_0\xi_1-\tau_1, \tau_0\tau_1\}\otimes\Lambda(\tau_2, \tau_3, \cdots)\otimes\boldsymbol{Z}/p[\xi_1^p, \xi_2, \xi_3, \cdots]\,.$$

Therefore

$$\mathrm{Ker}\,(R(Q_0)\oplus R(\mathcal{P}^1)\colon \mathcal{A}_* \to \Sigma\mathcal{A}_*\oplus\Sigma^{2(p-1)}\mathcal{A}_*) = \Lambda(\tau_2, \tau_3, \cdots)\otimes\boldsymbol{Z}/p[\xi_1^p, \xi_2, \xi_3, \cdots]\,.$$

We write B for this kernel.

Lemma 2.3. *For any non-zero $\alpha \in B$ there exists a unique $\alpha' \in \operatorname{Ker} R(\mathcal{P}^1)$ such that $(\alpha')Q_0 = (\alpha)\mathcal{P}^p$ (where if $\alpha \in \operatorname{Ker} R(\mathcal{P}^p)$, we take $\alpha' = 0$).*

Proof. Direct calculations using Proposition 2.1.

Henceforth for each non-zero $\alpha \in B$ we use α' to denote such an element. Define two subsets of $\mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*$ as

$$\tilde{B} = \{\alpha \oplus \sigma^q \alpha' \mid \alpha \in B\} \quad \text{and} \quad \sigma^q B = \{0 \oplus \sigma^q \alpha \mid \alpha \in B\}.$$

Then it is evident that $A_*(\boldsymbol{HZ}/p) \cong \tilde{B} + \sigma^q B$. Thus we obtain

Theorem 2.4. *As a $\boldsymbol{Z}/p$-module,*

$$H_*(\boldsymbol{A}; \boldsymbol{Z}/p) \cong (\chi \oplus \chi)(\tilde{B}) + (\chi \oplus \chi)(\sigma^q B).$$

Proof of (1.2). Starting from (1.1), we go a similar way to the above and get

$$H_*(\boldsymbol{G}; \boldsymbol{Z}/p) \cong \Lambda(\alpha_2, \alpha_3, \cdots) \otimes \boldsymbol{Z}/p[\beta_1, \beta_2, \cdots]$$

where $\alpha_n = \chi\tau_n$ and $\beta_n = \chi\xi_n$. By the dual of Corollary 1.2, θ_* can be identified with $c \cdot L(\mathcal{P}^1)$. Using Propositions 2.1 and 2.2 (ii), we see that

$$\theta_*(\alpha_2^{\varepsilon_2}\alpha_3^{\varepsilon_3} \cdots \beta_1^{r_1}\beta_2^{r_2}\beta_3^{r_3}\cdots) = \begin{cases} -cr_1 \cdot \sigma^{2(p-1)}\alpha_2^{\varepsilon_2}\alpha_3^{\varepsilon_3}\cdots\beta_1^{r_1-1}\beta_2^{r_2}\beta_3^{r_3}\cdots & \text{if } r_1 > 0 \\ 0 & \text{if } r_1 = 0 \end{cases}$$

where $\varepsilon_i = 0, 1$ and $r_i \geq 0$. This shows that

$$\begin{aligned} &\operatorname{Coker}(\theta_*: H_*(\boldsymbol{G}; \boldsymbol{Z}/p) \to H_*(\Sigma^{2(p-1)}\boldsymbol{G}; \boldsymbol{Z}/p)) \cong \\ &\qquad \Sigma^{2(p-1)}(\Lambda(\alpha_2, \alpha_3, \cdots) \otimes \boldsymbol{Z}/p[\beta_1^p, \beta_2, \beta_3, \cdots])\{\beta_1^{p-1}\}. \end{aligned}$$

Since the dual of $\mathcal{A}/\mathcal{A}(Q_0, \mathcal{P}^1)$ is just

$$\chi B = \Lambda(\alpha_2, \alpha_3, \cdots) \otimes \boldsymbol{Z}/p[\beta_1^p, \beta_2, \beta_3, \cdots],$$

the result follows by dualization.

3. The $\mathcal{A}_*$-coaction on $H_*(A; Z/p)$

Let $\phi: H_*(\boldsymbol{A}; \boldsymbol{Z}/p) \to \mathcal{A}_* \otimes H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ be the dual of the usual $\mathcal{A}$-action map $\mathcal{A} \otimes H^*(\boldsymbol{A}; \boldsymbol{Z}/p) \to H^*(\boldsymbol{A}; \boldsymbol{Z}/p)$. It gives $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ the structure of an $\mathcal{A}_*$-comodule. We study this coaction.

Since $\varepsilon_*: H_*(\boldsymbol{A}; \boldsymbol{Z}/p) \to \mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*$ is an injective homomorphism of $\mathcal{A}_*$-comodules, it suffices to determine the $\mathcal{A}_*$-comodule structure of $\mathcal{A}_* \oplus \Sigma^q \mathcal{A}_*$. Let $\phi_*: \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ be the coproduct on $\mathcal{A}_*$. It also gives an $\mathcal{A}_*$-comodule structure on $\mathcal{A}_*$. Recall the following properties of ϕ_*: for $\alpha, \beta \in \mathcal{A}_*$,

$$\phi_*(\alpha\beta) = \phi_*(\alpha)\phi_*(\beta);$$
$$\phi_*\chi = (\chi\otimes\chi)T\phi_* \quad \text{where} \quad T(\alpha\otimes\beta) = (-1)^{|\alpha||\beta|}\beta\otimes\alpha;$$
$$\phi_*(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{p^i}\otimes\xi_i \quad \text{and} \quad \phi_*(\tau_n) = \tau_n\otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{p^i}\otimes\tau_i\,.$$

The composite

$$\Sigma^q\mathcal{A}_* \xrightarrow{\Sigma^q\phi_*} \Sigma^q(\mathcal{A}_*\otimes\mathcal{A}_*) \xrightarrow{\cong} \mathcal{A}_*\otimes\Sigma^q\mathcal{A}_*\,,$$

which we denote by $\sigma^q\phi_*$, gives an $\mathcal{A}_*$-comodule structure on $\Sigma^q\mathcal{A}_*$. Moreover the composite

$$\mathcal{A}_*\oplus\Sigma^q\mathcal{A}_* \xrightarrow{\phi_*\oplus\Sigma^q\phi_*} (\mathcal{A}_*\otimes\mathcal{A}_*)\oplus(\mathcal{A}_*\otimes\Sigma^q\mathcal{A}_*) \xrightarrow{\cong} \mathcal{A}_*\otimes(\mathcal{A}_*\oplus\Sigma^q\mathcal{A}_*),$$

which may be written as $\phi_*+\sigma^q\phi_*$, gives an $\mathcal{A}_*$-comodule structure on $\mathcal{A}_*\oplus\Sigma^q\mathcal{A}_*$. Combining these and Theorem 2.4, one can evaluate $\phi(x)$ for every $x\in H_*(\boldsymbol{A};\boldsymbol{Z}/p)$.

It is convenient to introduce the following (artificial) multiplication on $H_*(\boldsymbol{A};\boldsymbol{Z}/p)$. For non-zero α, $\beta\in B$ define

(1) $(\chi\alpha\oplus\sigma^q\chi\alpha')\circ(\chi\beta\oplus\sigma^q\chi\beta') = \chi(\alpha\beta)\oplus\sigma^q\chi(\alpha'\beta+\alpha\beta')$

(2) $(\chi\alpha\oplus\sigma^q\chi\alpha')\circ(0\oplus\sigma^q\chi\beta) = 0\oplus\sigma^q\chi(\alpha\beta)$

(3) $(0\oplus\sigma^q\chi\alpha)\circ(\chi\beta\oplus\sigma^q\chi\beta') = 0\oplus\sigma^q\chi(\alpha\beta)$

(4) $(0\oplus\sigma^q\chi\alpha)\circ(0\oplus\sigma^q\chi\beta) = 0\,.$

This is well defined. To check this assertion we first observe that if $\alpha\in B$ then $(\alpha)Q_0=0$ and $(\alpha)\mathcal{P}^i=0$ for $0<i<p$. Therefore, if α, $\beta\in B$ we have $\alpha\beta\in B$, $(\alpha'\beta+\alpha\beta')\mathcal{P}^1=0$ and

$$\begin{aligned}(\alpha'\beta+\alpha\beta')Q_0 &= (\alpha')Q_0\cdot\beta+\alpha\cdot(\beta')Q_0\\ &= (\alpha)\mathcal{P}^p\cdot\beta+\alpha\cdot(\beta)\mathcal{P}^p\\ &= (\alpha\beta)\mathcal{P}^p\,.\end{aligned}$$

This implies that (1) is well defined. The other cases are obvious.

We now show that the formula

$$\phi(x\circ y) = \phi(x)\circ\phi(y)$$

holds for all x, $y\in H_*(\boldsymbol{A};\boldsymbol{Z}/p)$. For example, if $x=\chi\alpha\oplus\sigma^q\chi\alpha'$ and $y=\chi\beta\oplus\sigma^q\chi\beta'$, then we have

$$\begin{aligned}\phi(x\circ y) &= \phi(\chi(\alpha\beta)\oplus\sigma^q\chi(\alpha'\beta+\alpha\beta'))\\ &= \phi_*(\chi\alpha\cdot\chi\beta)+\sigma^q\phi_*(\chi\alpha'\cdot\chi\beta+\chi\alpha\cdot\chi\beta')\\ &= \phi_*(\chi\alpha)\cdot\phi_*(\chi\beta)+\sigma^q(\phi_*(\chi\alpha')\phi_*(\chi\beta)+\phi_*(\chi\alpha)\phi_*(\chi\beta'))\end{aligned}$$

$$
\begin{aligned}
&= (\phi_*(\chi\alpha)+\sigma^q\phi_*(\chi\alpha'))\circ(\phi_*(\chi\beta)+\sigma^q\phi_*(\chi\beta')) \\
&= \phi(x)\circ\phi(y) .
\end{aligned}
$$

The other cases are obvious.

REMARK. As seen in [9], $\boldsymbol{KF}_r$ has a natural product. So it induces a multiplication on $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$. We cannot confirm whether $\circ$ coincides with this one; however, we believe so (cf. Theorem 4.3).

By virtue of Lemma 2.3 we may put

$$\widetilde{\chi\alpha} = \chi\alpha\oplus\sigma^q\chi\alpha' \quad \text{and} \quad \sigma^q\chi\alpha = 0\oplus\sigma^q\chi\alpha$$

for each non-zero $\alpha\in B$. With this notation the multiplication $\circ$ is given by

(1) $\tilde{x}\circ\tilde{y} = \tilde{x}\tilde{y}$ (2) $\tilde{x}\circ\sigma^q y = \sigma^q xy$ (3) $\sigma^q x\circ\tilde{y} = \sigma^q xy$
(4) $\sigma^q x\circ\sigma^q y = 0$

for all $x, y\in\chi B$. Notice that as an algebra $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ is generated by the elements $\sigma^q 1$, $\tilde{\beta}_1^p$, $\tilde{\beta}_n$, $\tilde{\alpha}_n$ with $n\geq 2$.

Theorem 3.1. *The $\mathcal{A}_*$-coaction on $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ is given by*

$$
\begin{aligned}
&\phi(\sigma^q 1) = 1\otimes\sigma^q 1 \\
&\phi(\tilde{\beta}_1^p) = \chi\xi_1^p\otimes\tilde{1}+\chi\tau_0\otimes\sigma^q 1+1\otimes\tilde{\beta}_1^p \\
&\phi(\tilde{\beta}_2) = \chi\xi_2\otimes\tilde{1}+\chi(\tau_0\xi_1-\tau_1)\otimes\sigma^q 1+\chi\xi_1\otimes\tilde{\beta}_1^p+1\otimes\tilde{\beta}_2 \\
&\phi(\tilde{\alpha}_2) = \chi\tau_2\otimes\tilde{1}+\chi(\tau_0\tau_1)\otimes\sigma^q 1+\chi\tau_1\otimes\tilde{\beta}_1^p+\chi\tau_0\otimes\tilde{\beta}_2+1\otimes\tilde{\alpha}_2 \\
&\phi(\tilde{\beta}_n) = \sum_{i=0}^{n}\chi\xi_{n-i}\otimes\tilde{\beta}_i^{p^{n-i}} \qquad \textit{for} \quad n\geq 3 \\
&\phi(\tilde{\alpha}_n) = \sum_{i=0}^{n}\chi\tau_{n-i}\otimes\tilde{\beta}_i^{p^{n-i}}+1\otimes\tilde{\alpha}_n \qquad \textit{for} \quad n\geq 3 .
\end{aligned}
$$

Let $\mathcal{B}$ be the exterior subalgebra of $\mathcal{A}$ generated by Q_0 and Q_1. In the next section we need to know the $\mathcal{B}$-module structure of $H^*(\boldsymbol{A}; \boldsymbol{Z}/p)$. But it can be read off from Theorem 3.1. We give its details.

Define a left action of $\mathcal{A}$ on $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ by

$$\langle f, a(x)\rangle = (-1)^{|a||x|}\langle(\chi a)(f), x\rangle$$

for all $a\in\mathcal{A}$, $x\in H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ and $f\in H^*(\boldsymbol{A}; \boldsymbol{Z}/p)$ (cf. [2, p. 76]).

Corollary 3.2. *For $i=0$ or 1, Q_i acts on $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ as a derivation (with respect to $\circ$). So the $\mathcal{B}$-action on $H_*(\boldsymbol{A}; \boldsymbol{Z}/p)$ is given by*

$$
\begin{aligned}
&Q_0(\tilde{\beta}_1^p) = \sigma^q 1 , \quad Q_0(\tilde{\alpha}_n) = \tilde{\beta}_n \qquad \textit{for} \quad n\geq 2 , \\
&Q_1(\tilde{\beta}_2) = -\sigma^q 1 , \quad Q_1(\tilde{\alpha}_n) = \tilde{\beta}_{n-1}^p \qquad \textit{for} \quad n\geq 2 .
\end{aligned}
$$

We define a weight function $w\colon H_*(\boldsymbol{A};\boldsymbol{Z}/p)\to\boldsymbol{Z}$ by

$$w(\tilde{1})=0\,,\quad w(\tilde{\beta}_1^p)=w(\sigma^q 1)=p\,,$$
$$w(\tilde{\alpha}_n)=w(\tilde{\beta}_n)=p^{n-1}\qquad\text{for}\quad n\geq 2$$

together with the rules

$$w(x+y)=\max\{w(x),\,w(y)\}\quad\text{and}$$
$$w(x\circ y)=w(x)+w(y)$$

for all $x,\,y\in H_*(\boldsymbol{A};\boldsymbol{Z}/p)$. By Corollary 3.2 the $\mathcal{B}$-action preserves weight. For $j\geq 0$ let N_j denote the submodule of $H_*(\boldsymbol{A};\boldsymbol{Z}/p)$ spanned by elements of weight jp. Then $H_*(\boldsymbol{A};\boldsymbol{Z}/p)\cong\bigoplus_{j\geq 0}N_j$ as $\mathcal{B}$-modules. It suffices to examine the $\mathcal{B}$-module structure of N_j. For this purpose the Q_i-homology

$$H_*(\quad;Q_i)=\operatorname{Ker}Q_i/\operatorname{Im}Q_i$$

is useful.

Lemma 3.3. *For $j\geq 0$ we have*

(i)
$$H_*(N_j;Q_0)=\begin{cases}\boldsymbol{Z}/p\{\tilde{1}\} & \text{if}\quad j=0\\ \boldsymbol{Z}/p\{\sigma^q(\beta_1^p)^{np-1},\ (\tilde{\beta}_1^p)^{np}\} & \text{if}\quad j=np\ (n\geq 1)\\ 0 & \text{otherwise}\end{cases}$$

(ii)
$$H_*(N_j;Q_1)=\begin{cases}\boldsymbol{Z}/p\{\sigma^q\beta_2^{p-1}\cdots\beta_{k+2}^{p-1}\beta_{k+3}^{n_k-1}\beta_{k+4}^{n_{k+1}}\cdots\beta_{l+3}^{n_l}\,, & \\ \qquad\tilde{\beta}_{k+3}^{n_k}\tilde{\beta}_{k+4}^{n_{k+1}}\cdots\tilde{\beta}_{l+3}^{n_l}\} & \text{if}\quad j=np\quad(n\geq 0)\\ 0 & \text{otherwise}\end{cases}$$

where $k=\nu_p(n)$ and $n=n_kp^k+n_{k+1}p^{k+1}+\cdots+n_lp^l$ is the p-adic expansion of n.

Proof. From § 1 we have a short exact sequence of $\mathcal{B}$-modules

$$0\to\Sigma^q\chi B\to H_*(\boldsymbol{A};\boldsymbol{Z}/p)\to\chi B\to 0\,.$$

This yields a long exact sequence

$$\cdots\to H_m(H_*(\boldsymbol{A};\boldsymbol{Z}/p);Q_i)\to H_m(\chi B;Q_i)\xrightarrow{\partial}H_{m-1}(\Sigma^q\chi B;Q_i)\to\cdots$$

Since the $\mathcal{B}$-action on χB is given by

$$Q_0(\alpha_n)=\beta_n\quad\text{and}\quad Q_1(\alpha_n)=\beta_{n-1}^p\qquad\text{for}\quad n\geq 2\,,$$

it follows that

$$H_*(\chi B;Q_0)=\boldsymbol{Z}/p[\beta_1^p]\quad\text{and}$$
$$H_*(\chi B;Q_1)=\bigotimes_{n\geq 2}\boldsymbol{Z}/p[\beta_n]/(\beta_n^p)\,.$$

An inspection of weight shows that to calculate $H_*(N_j; Q_i)$ it suffices to determine the behavior of

$$\begin{array}{l} H_{2j(p-1)}(\chi B; Q_0) = \boldsymbol{Z}/p\{(\beta_1^p)^j\} \\ \quad \Big\downarrow \partial \\ H_{2j(p-1)-1}(\Sigma^q \chi B; Q_0) = \boldsymbol{Z}/p\{\sigma^q(\beta_1^p)^{j-1}\} \end{array}$$

and

$$\begin{array}{l} H_{2\nu_p((jp^2)!)(p-1)}(\chi B; Q_1) = \boldsymbol{Z}/p\{\beta_2^{j_0}\beta_3^{j_1}\cdots\beta_{s+2}^{j_s}\} \\ \quad \Big\downarrow \partial \\ H_{2\nu_p((jp^2)!)(p-1)-(2p-1)}(\Sigma^q \chi B; Q_1) = \boldsymbol{Z}/p\{\sigma^q\beta_2^{j_0-1}\beta_3^{j_1}\cdots\beta_{s+2}^{j_s}\} \end{array}$$

where $j=j_0+j_1p+\cdots+j_sp^s$ is the p-adic expansion of j. By the definition of ∂ and Corollary 3.2, we find that

$$\partial((\beta_1^p)^j) = j\cdot\sigma^q(\beta_1^p)^{j-1} \quad \text{and}$$
$$\partial(\beta_2^{j_0}\beta_3^{j_1}\cdots\beta_{s+2}^{j_s}) = \begin{cases} -j_0\cdot\sigma^q\beta_2^{j_0-1}\beta_3^{j_1}\cdots\beta_{s+2}^{j_s} & \text{if } j_0>0 \\ 0 & \text{if } j_0=0\,. \end{cases}$$

This gives the result.

It is easy to carry these results to those for the usual $\mathcal{B}$-action (cf. [7, II]). Hereafter we talk about the usual action.

According to [3,III], there is a classification of finite dimensional $\mathcal{B}$-modules, which we use implicitly. We fix some notation. Let I be defined by the exact sequence of $\mathcal{B}$-modules

$$0 \to I \to \mathcal{B} \to \boldsymbol{Z}/p \to 0\,.$$

Put $I^n = I \underset{\mathcal{B}}{\otimes} \cdots \underset{\mathcal{B}}{\otimes} I$ (n-factors). Note that $H_*(I^n; Q_0) = \boldsymbol{Z}/p\{\overset{n}{\underset{\mathcal{B}}{\otimes}} Q_0\}$ and $H_*(I^n; Q_1) = \boldsymbol{Z}/p\{\overset{n}{\underset{\mathcal{B}}{\otimes}} Q_1\}$ where $|\overset{n}{\underset{\mathcal{B}}{\otimes}} Q_0| = n$ and $|\overset{n}{\underset{\mathcal{B}}{\otimes}} Q_1| = n+2n(p-1)$.

The above discussion can be summarized as follows.

Theorem 3.4. *As a $\mathcal{B}$-module, ignoring free summands,*

$$H^*(\boldsymbol{A}; \boldsymbol{Z}/p) \cong \boldsymbol{Z}/p \oplus \bigoplus_{n\ge 1} (\Sigma^{a(n)} I^{b(n)} \oplus \Sigma^{c(n)} I^{d(n)})$$

where

$$\begin{aligned} a(n)+b(n) &= 2np^2(p-1)-1\,, \\ b(n) &= \nu_p((np^3)!)-np^2-\nu_p(n)-2\,, \\ c(n)+d(n) &= 2np^2(p-1)\,, \\ d(n) &= \nu_p((np^3)!)-np^2\,. \end{aligned}$$

4. The multiplicative structure on A

The first half of this section is heavily influenced by [18].

Let μ: $G\wedge G\to G$ be the product on G. Consider the external product

$$\times : G^*(G)\otimes G^*(G) \xrightarrow{\wedge} (G\wedge G)^*(G\wedge G) \xrightarrow{'\mu_*} G^*(G\wedge G) .$$

Lemma 4.1. *The element $\theta\in G^{2(p-1)}(G)$ satisfies*

$$\theta\mu = (\Sigma^{2(p-1)}\mu)(\theta\wedge 1_G+1_G\wedge\theta+v\theta\wedge\theta) .$$

Proof. Put $1=1_G\in G^0(G)$. By the definition of θ, we have

$$\psi^r{}_*(1\times 1) = 1\times 1+v_*\theta_*(1\times 1)=1\times 1+v_*(\theta\mu) .$$

On the other hand, since ψ^r is multiplicative and $\times$ is bilinear, we have

$$\begin{aligned}\psi^r{}_*(1\times 1) &= \psi^r{}_*\mu_*(1\wedge 1) = \mu_*(\psi^r\wedge\psi^r)_*(1\wedge 1)\\ &= \psi^r{}_*(1)\times\psi^r{}_*(1)\\ &= (1+v_*\theta_*(1))\times(1+v_*\theta_*(1))\\ &= 1\times 1+v_*(\theta_*(1)\times 1+1\times\theta_*(1)+v\theta_*(1)\times\theta_*(1))\\ &=1\times 1+v_*((\Sigma^{2(p-1)}\mu)(\theta\wedge 1+1\wedge\theta+v\theta\wedge\theta)) .\end{aligned}$$

Since v_*: $G^{2(p-1)}(G\wedge G)\to G^0(G\wedge G)$ is injective, the result follows.

Lemma 4.2. *We have*

(i) $[A, \Sigma^{2p-3}G]=0$.

(ii) $[A\wedge A, \Sigma^{2p-3}G]=0$.

Proof. Consider the Adams spectral sequence $\{E_r^{s,t}, d_r\}$ converging to $G^*(X)$, where $X=A$ or $A\wedge A$. It has the form

$$E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{B}}^{s,t}(Z/p, H^*(X; Z/p)) \Rightarrow G^{t-s}(X) .$$

(For this details see [3, III].) In view of Theorem 3.4 (where a similar result for $A\wedge A$ follows from this and the Künneth theorem), all we need to do is the calculation of $\mathrm{Ext}_{\mathcal{B}}^{*,*}(Z/p, M)$ for $M=\Sigma^m\mathcal{B}$, $\Sigma^m Z/p$, $\Sigma^m I^n$ and their direct sums. As is well known, for all $\mathcal{B}$-modules M and N,

$$\mathrm{Ext}_{\mathcal{B}}^{s,t}(Z/p, M\oplus N)\cong \mathrm{Ext}_{\mathcal{B}}^{s,t}(Z/p, M)\oplus\mathrm{Ext}_{\mathcal{B}}^{s,t}(Z/p, N)$$
$$\mathrm{Ext}_{\mathcal{B}}^{s,t}(Z/p, \Sigma^m N)\cong \mathrm{Ext}_{\mathcal{B}}^{s,m+t}(Z/p, N)$$

and

$$\mathrm{Ext}_{\mathcal{B}}^{*,*}(Z/p, \mathcal{B})\cong Z/p\{z\} \quad\text{where}\quad |z|=(0, -2p)$$
$$\mathrm{Ext}_{\mathcal{B}}^{*,*}(Z/p, Z/p)\cong Z/p[q_0, q_1] \quad\text{where}\quad |q_i|=(1, 2p^i-1)$$
$$\mathrm{Ext}_{\mathcal{B}}^{s,t}(Z/p, I^n)\cong \mathrm{Ext}_{\mathcal{B}}^{s-n,t}(Z/p, Z/p) .$$

Using these data, one can describe the figure of $E_2^{*,*}$; in particular, we have $E_2^{s,t}=0$ if $t-s=2p-3$. This implies the result.

Theorem 4.3. *A is a ring spectrum and $\eta\colon A\to G$ is a map of ring spectra. The product on A satisfying such property is unique.*

Proof. Consider the exact sequence

$$0\to[A\wedge A,\ A]\xrightarrow{\eta_*}[A\wedge A,\ G]\xrightarrow{\theta_*}[A\wedge A,\ \Sigma^{2(p-1)}G]$$

where we have used Lemma 4.2 (ii). By Lemma 4.1 we have

$$\begin{aligned}\theta_*(\mu(\eta\wedge\eta)) &= \theta\mu(\eta\wedge\eta)\\ &= (\Sigma^{2(p-1)}\mu)(\theta\wedge 1_G+1_G\wedge\theta+v\theta\wedge\theta)(\eta\wedge\eta)\end{aligned}$$

which is clearly equal to zero, since $\theta\eta=0$. Hence there exists a unique $\hat{\mu}\in[A\wedge A,\ A]$ such that $\eta\hat{\mu}=\mu(\eta\wedge\eta)$.

Let $\iota\colon S^0\to G$ be the unit on G. Then there is a unique $\hat{\iota}\in[S^0,\ A]$ such that $\eta\hat{\iota}=\iota$ (see (0.5)). Consider the exact sequence

$$0\to[S^0\wedge A,\ A]\xrightarrow{\eta_*}[S^0\wedge A,\ G]$$

where we have used Lemma 4.2 (i). Then we have

$$\begin{aligned}\eta_*(\hat{\mu}(\hat{\iota}\wedge 1_A)) &= \eta\hat{\mu}(\hat{\iota}\wedge 1_A) = \mu(\eta\wedge\eta)(\hat{\iota}\wedge 1_A)\\ &= \mu(\iota\wedge\eta) = \mu(\iota\wedge 1_G)(1_{S^0}\wedge\eta)\\ &= \eta = \eta_*(1_A)\,.\end{aligned}$$

This proves that $\hat{\mu}(\hat{\iota}\wedge 1_A)=1_A$. Another equation $\hat{\mu}(1_A\wedge\hat{\iota})=1_A$ is obtained similarly.

Lemma 4.4. *Under the above notation we have*

(i) $\hat{\mu}(1_A\wedge\Delta)=\Delta(\Sigma^{2p-3}\mu)(\Sigma^{2p-3}\eta\wedge 1_G)$.

(ii) $\hat{\mu}(\Delta\wedge 1_A)=\Delta(\Sigma^{2p-3}\mu)(\Sigma^{2p-3}1_G\wedge\eta)$.

Proof. Because an argument is quite parallel, we show (ii) only. By smashing (0.6) to the right with A, we have a diagram

$$\begin{tikzcd}
S^{2p-3}\wedge G\wedge A \arrow[r, "\Delta\wedge 1_A"] \arrow[d, "\Sigma^{2p-3}1_G\wedge\eta"'] & A\wedge A \arrow[r, "\eta\wedge 1_A"] \arrow[dd, "\hat{\mu}"] & G\wedge A \arrow[r, "\theta\wedge 1_A"] \arrow[d, "1_G\wedge\eta"] & S^{2p-2}\wedge G\wedge A \arrow[d, "\Sigma^{2p-2}1_G\wedge\eta"] \\
S^{2p-3}\wedge G\wedge G \arrow[d, "\Sigma^{2p-3}\mu"'] & \text{①} \quad\quad \text{②} & G\wedge G \arrow[d, "\mu"] & \text{③} \quad S^{2p-2}\wedge G\wedge G \arrow[d, "\Sigma^{2p-2}\mu"] \\
S^{2p-3}\wedge G \arrow[r, "\Delta"] & A \arrow[r, "\eta"] & G \arrow[r, "\theta"] & S^{2p-2}\wedge G
\end{tikzcd}$$

in which rows are fibre sequences. Part ② commutes by Theorem 4.3. To prove the commutativity of part ①, it suffices to show that of part ③. But by Lemma 4.1 we have

$$\begin{aligned}\theta\mu(1_G\wedge\eta) &= (\Sigma^{2p-2}\mu)(\theta\wedge 1_G+1_G\wedge\theta+v\theta\wedge\theta)(1_G\wedge\eta)\\ &= (\Sigma^{2p-2}\mu)(\theta\wedge\eta)\\ &= (\Sigma^{2p-2}\mu)(\Sigma^{2p-2}1_G\wedge\eta)(\theta\wedge 1_A)\,.\end{aligned}$$

Let us consider $\tilde{A}_*(CP^\infty)$. Since G-theory is complex oriented, $\tilde{G}_n(CP^\infty)=0$ if n is odd. From (0.6) we have an exact sequence

$$0\to\tilde{A}_{2n}(CP^\infty)\xrightarrow{\ \eta\ }\tilde{G}_{2n}(CP^\infty)\xrightarrow{\ \theta\ }\tilde{G}_{2n-2(p-1)}(CP^\infty)\xrightarrow{\ \Delta\ }\tilde{A}_{2n-1}(CP^\infty)\to 0$$

for all $n\geq 0$ (where of course $\eta=(\eta\wedge 1_{CP^\infty})_*$ etc.). Thus we may use the following notation:

$$\begin{aligned}\eta(\tilde{x}) &= x \qquad \text{for}\quad x\in\operatorname{Ker}\theta;\\ \Delta(x) &= \bar{x} \qquad \text{for}\quad x\in\tilde{G}_*(CP^\infty)\,.\end{aligned}$$

The multiplication m: $CP^\infty\times CP^\infty\to CP^\infty$ induces a product $\cdot$ on $\tilde{G}_*(CP^\infty)$ and a product $*$ on $\tilde{A}_*(CP^\infty)$.

Theorem 4.5. *The following formulas hold.*

(i) $\tilde{x}*\tilde{y}=\widetilde{x\cdot y}$.

(ii) $\tilde{x}*\bar{y}=\overline{x\cdot y}$.

(iii) $\bar{x}*\tilde{y}=\overline{x\cdot y}$.

(iv) $\bar{x}*\bar{y}=0$.

Proof. Since η is multiplicative by Theorem 4.3, (i) follows.

Similarly, using $\eta\Delta=0$, we have

$$\eta(\bar{x}*\bar{y}) = \eta(\bar{x})*\eta(\bar{y}) = \eta\Delta(x)*\eta\Delta(y) = 0$$

which proves (iv).

For (ii), by definition and Lemma 4.4 (i), we have

$$\begin{aligned}\tilde{x}*\bar{y} &= m_*(\tilde{x}\times\bar{y}) = m_*\hat{\mu}(\tilde{x}\wedge\bar{y})\\ &= m_*\hat{\mu}(\tilde{x}\wedge\Delta(\Sigma^{2p-3}y))\\ &= m_*\hat{\mu}(1_A\wedge\Delta)(\Sigma^{2p-3}\tilde{x}\wedge y)\\ &= m_*\Delta(\Sigma^{2p-3}\mu)(\Sigma^{2p-3}\eta\wedge 1_G)(\Sigma^{2p-3}\tilde{x}\wedge y)\\ &= m_*\Delta(\Sigma^{2p-3}\mu)(\Sigma^{2p-3}x\wedge y)\\ &= m_*\Delta(\Sigma^{2p-3}x\times y)\\ &= \Delta(\Sigma^{2p-3}m_*(x\times y))\end{aligned}$$

$$= \Delta(\Sigma^{2p-3}x\cdot y)$$
$$=\overline{x\cdot y}\,.$$

Similarly (iii) follows from Lemma 4.4 (ii).

REMARK. The argument of this section assures us that the same formulas as above hold with respect to the fibre sequences (0.3) and

$$\boldsymbol{AZ}/p \to \boldsymbol{buZ}/p \to \Sigma^2\boldsymbol{buZ}/p$$

where $\boldsymbol{XZ}/p$ represents the mod p X-theory (cf. [15, p. 254]).

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