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THE VANISHING OF COHOMOLOGY ASSOCIATED TO DISCRETE SUBGROUPS OF COMPLEX SIMPLE LIE GROUPS*

HANS-CHRISTOPH IM HOF AND ERNST A. RUH

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1. Introduction

Let G denote a connected complex simple Lie group and K a maximal compact subgroup of G. The quotient $M=G/K$ is a riemannian symmetric space of non-compact type. Let Γ denote a discrete subgroup of G with compact quotient $\Gamma\backslash G$, and let ρ denote an irreducible non-trivial complex representation of G in a finite dimensional complex vector space F. In this paper we prove that for such representations a certain quadratic form defined by Matsushima and Murakami [3] is positive definite, and hence $H^*(\Gamma, M, \rho)$ vanishes.

The motivation for this paper is a result of Min-Oo and Ruh [4] on comparison theorems for non-compact symmetric spaces, where an estimate from below for the first eigenvalue of the Laplace operator on 2-forms with values in a bundle associated to the adjoint representation is essential. This estimate is an immediate consequence of the positivity of the above quadratic form. The vanishing of $H^*(\Gamma, M, \rho)$, without the information on the first eigenvalue, is a special case of [1, Ch. VII, Th. 6. 7].

2. The result

Let $\mathfrak{g}$ denote the Lie algebra of left-invariant vector fields of the simple Lie group G, ρ: $\mathfrak{g}\to\mathfrak{gl}(F)$ the representation induced by ρ: $G\to GL(F)$, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ with $\mathfrak{k}$ the Lie algebra of a maximal compact subgroup K. We identify the Lie algebra $\mathfrak{g}$ with the corresponding vector fields on $\Gamma\backslash G$.

Let $A(\Gamma, M, \rho)$ ($A_0(\Gamma, M, \rho)$ in the notation of Matsushima and Murakami [3]) denote the vector space of F-valued differential forms on $\Gamma\backslash G$ which are horizontal and ad$K$-equivariant, i.e., $\eta\in A(\Gamma, M, \rho)$ satisfies $i_X\eta=0$ and $\theta_X\eta=-\rho(X)\eta$ for all $X\in\mathfrak{k}$, where i_X is interior multiplication and θ_X is the Lie

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derivative. A q-form $\eta\in A(\Gamma, M, \rho)$ is determined by its values $\eta_{i_1,\cdots,i_q}=\eta(Y_{i_1},\cdots, Y_{i_q})$ on q-tuples of basis vectors of $\mathfrak{p}$. According to [3, (6.7)], the Laplace operator

$$\Delta: A(\Gamma, M, \rho)\rightarrow A(\Gamma, M, \rho)$$

is a sum of a differential operator Δ_D and an operator Δ_ρ associated to the representation ρ. Restricted to q-forms these operators have the following coordinate expressions.

$$(\Delta_D\eta)(Y_{i_1},\cdots, Y_{i_q})=-\sum_{k=1}^{n} Y_k^2\eta_{i_1,\cdots,i_q}+\sum_{k=1}^{n}\sum_{u=1}^{q}(-1)^u\,[Y_{i_u}, Y_k]\eta_{ki_1,\cdots,\hat{i}_u,\cdots,i_q}\,,$$

$$(\Delta_\rho\eta)(Y_{i_1},\cdots, Y_{i_q})=\sum_{k=1}^{n}\rho(Y_k)^2\eta_{i_1,\cdots,i_q}-\sum_{k=1}^{n}\sum_{u=1}^{q}(-1)^u\rho([Y_{i_u}, Y_k])\eta_{ki_1,\cdots\hat{i}_u,\cdots,i_q}\,,$$

where $\{Y_i; i=1,\cdots, n=\dim M\}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form φ of $\mathfrak{g}$ restricted to $\mathfrak{p}$. As in [3], the definition of Δ requires a choice of an admissible hermitean inner product on F. The inner product $\langle\,,\rangle_F$ is called admissible if for all $u, v\in F$ the following conditions hold:

$$\begin{aligned}\langle\rho(X)u, v\rangle_F &= -\langle u, \rho(X)v\rangle_F &&\text{for } X\in\mathfrak{k}\,,\\ (\rho\langle Y)u, v\rangle_F &= \langle u, \rho(Y)v\rangle_F &&\text{for } Y\in\mathfrak{p}\,.\end{aligned}$$

Matsushima and Murakami [3] prove that admissible hermitean inner products always exist.

The following result is well known.

Proposition 1. *The vector space $H^*(\Gamma, M, \rho)$ is canonically isomorphic to the vector space $\{\eta\in A(\Gamma, M, \rho);\ \Delta\eta=0\}$ of harmonic forms.*

The restriction of the Killing form φ to $\mathfrak{p}$ together with the scalar product $\langle\,,\rangle_F$ on F induce a hermitean scalar product $(\ ,\ )$ on $A(\Gamma, M, \rho)$, obtained by integrating the pointwise defined scalar product

$$\langle\eta, \omega\rangle=\sum_{i_1<\cdots<i_q}\langle\eta_{i_1,\cdots,i_q}, \omega_{i_1,\cdots,i_q}\rangle_F\,.$$

Here $\eta_{i_1,\cdots,i_q}$ and $\omega_{i_1,\cdots,i_q}$ are the coordinates of q-forms with respect to an orthonormal basis in $\mathfrak{p}$, and $\langle\eta, \omega\rangle$ is defined to be zero if η and ω are of different degrees.

The following result is proved in [3].

Proposition 2. *The quadratic forms $\eta\mapsto(\Delta_D\eta, \eta)$ and $\eta\mapsto(\Delta_\rho\eta, \eta)$ are positive semi-definite.*

A differential form $\eta\in A(\Gamma, M, \rho)$ is a section of the trivial vector bundle on

$\Gamma\backslash G$ with fibre $\mathrm{Hom}(\Lambda\mathfrak{p}, F)$, the homomorphisms from the exterior algebra over $\mathfrak{p}$ to F. The operator Δ_ρ does not involve derivatives and thus can be viewed as a linear map

$$\Delta_\rho\colon \mathrm{Hom}(\Lambda\mathfrak{p}, F) \to \mathrm{Hom}(\Lambda\mathfrak{p}, F).$$

Our main result concerns the positivity of the quadratic form $\eta\mapsto\langle\Delta_\rho\eta, \eta\rangle$ on $\mathrm{Hom}(\Lambda\mathfrak{p}, F)$, which by Proposition 2 implies the vanishing of the cohomology vector space $H^*(\Gamma, M, \rho)$.

Theorem. *Let ρ denote an irreducible non-trivial complex representation of a complex simple Lie algebra $\mathfrak{g}$ on a finite dimensional complex vector space F. Then the quadratic form $\eta\mapsto\langle\Delta_\rho\eta, \eta\rangle$ on $\mathrm{Hom}(\Lambda\mathfrak{p}, F)$ is positive definite, and therefore $H^*(\Gamma, M, \rho)=(0)$.*

The basic ideas of the proof are similar to those of Raghunathan [6]. Our restriction to complex Lie groups allows us to prove the optimal result. In addition, Assertions III and IV of [6], which lead to difficulties, can be avoided.

3. The proof

The restriction to complex Lie algebras $\mathfrak{g}$ allows us to identify $\mathrm{Hom}_{\mathbf{R}}(\Lambda\mathfrak{p}, F)$ with $\mathrm{Hom}_{\mathbf{C}}(\Lambda\mathfrak{g}, F)$. In the following we suppress the subscripts $\mathbf{R}$ and $\mathbf{C}$. Since $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ and $\mathfrak{p}=i\mathfrak{k}$, multiplication with i is a $\mathbf{R}$-vector space isomorphism $J\colon\mathfrak{k}\to\mathfrak{p}$. Let $\Lambda J\colon\Lambda\mathfrak{k}\to\Lambda\mathfrak{p}$ denote the induced isomorphism and define

$$\mathrm{Hom}(\Lambda\mathfrak{p}, F) \xrightarrow{\simeq} \mathrm{Hom}(\Lambda\mathfrak{k}, F) \xrightarrow{\simeq} \mathrm{Hom}(\Lambda\mathfrak{g}, F),$$

where the first isomorphism is composition with ΛJ, and the image ξ of $\xi'\in\mathrm{Hom}(\Lambda\mathfrak{k}, F)$ under the second isomorphism is defined by $\xi(X\otimes\lambda)=\lambda\xi'(X)$, for $X\in\Lambda\mathfrak{k}$ and $\lambda\in\mathbf{C}$.

From now on we identify $\mathrm{Hom}(\Lambda\mathfrak{g}, F)$ with $F\otimes\Lambda\mathfrak{g}^*$ and view Δ_ρ as an element in the endomorphism ring of $F\otimes\Lambda\mathfrak{g}^*$. Let c denote the Casimir element with respect to the Killing form φ in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The representation ρ extends to $U(\mathfrak{g})$. In the following lemma σ denotes the dual of the representation $\Lambda\mathrm{ad}$ induced by the adjoint representation of $\mathfrak{g}$.

Lemma 1. $2\Delta_\rho = 3(\rho\otimes 1)(c)+(1\otimes\sigma)(c)-(\rho\otimes\sigma)(c)$

This lemma proves in particular that Δ_ρ is a selfadjoint endomorphism with respect to the scalar product $\langle\,,\rangle$ introduced earlier.

Proof. Let $\{X_k;\ k=1, \cdots, n\}$ be an orthonormal basis of $\mathfrak{k}$ with respect to $-\varphi$ restricted to $\mathfrak{k}$, and $\{Y_k\}=\{iX_k\}$ the corresponding basis in $\mathfrak{p}$. The image

$\xi\in\mathrm{Hom}(\Lambda\mathfrak{g}, F)$ of $\eta\in\mathrm{Hom}(\Lambda\mathfrak{p}, F)$ under the isomorphism defined above evaluated on $(X_{i_1}, \cdots, X_{i_q})$ is

$$\xi(X_{i_1}, \cdots, X_{i_q}) = \eta(iX_{i_1}, \cdots, iX_{i_q}) = \eta(Y_{i_1}, \cdots, Y_{i_q}) .$$

With this identification of $\mathrm{Hom}_{\mathbf{C}}(\Lambda\mathfrak{g}, F)$ and $\mathrm{Hom}_{\mathbf{R}}(\Lambda\mathfrak{p}, F)$, Δ_ρ operates on ξ as follows:

$$\begin{aligned}(\Delta_\rho\xi)(X_{i_1}, \cdots, X_{i_q}) &= \sum_{k=1}^{n} \rho(iX_k)^2\xi(X_{i_1}, \cdots, X_{i_q}) \\ &\quad - \sum_{k=1}^{n}\sum_{u=1}^{q}(-1)^u\rho([iX_{i_u}, iX_k])\xi(X_k, X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_q}) \\ &= (S\xi)(X_{i_1}, \cdots, X_{i_q})+(T\xi)(X_{i_1}, \cdots, X_{i_q}) .\end{aligned}$$

In view of the identification $\mathrm{Hom}(\Lambda\mathfrak{g}, F)=F\otimes\Lambda\mathfrak{g}^*$, the first summand is given in terms of the Casimir element c as

$$S = (\rho\otimes 1)(c) ,$$

since $\{X_k\}$ and $\{-X_k\}$ are dual bases with respect to φ and therefore $c=-\sum X_k^2$. To deal with the second summand, we abbreviate $E=\Lambda\mathfrak{g}^*$ and specialize to $\xi=f\otimes e$ with $f\in F$ and $e\in E$. The immediate goal is to prove that in this case

$$T(f\otimes e) = \sum_{k=1}^{n} \rho(X_k)f\otimes\sigma(X_k)e .$$

Let c_{ij}^k denote the structure constants of $\mathfrak{k}$ (and $\mathfrak{g}$) with respect to the basis $\{X_k\}$; thus $\sum_{k=1}^{n} c_{ij}^k X_k=[X_i, X_j]$, and

$$\rho([iX_{i_u}, iX_k]) = -\rho([X_{i_u}, X_k]) = -\sum_{s=1}^{n} c_{i_u k}^s\rho(X_s) = -\sum_{s=1}^{n} c_{s i_u}^k\rho(X_s) ,$$

where the last equality holds because c_{ij}^k, in terms of an orthonormal basis with respect to $-\varphi$, is skew symmetric in each pair of indices. We have

$$(T\xi)(X_{i_1}, \cdots, X_{i_q}) = \sum_{s=1}^{n}\sum_{u=1}^{q}(-1)^u\rho(X_s)\xi([X_s, X_{i_u}], X_{i_1}, \cdots, \hat{X}_{i_u}, \cdots, X_{i_q}) .$$

Abbreviating $X=X_{i_1}\wedge\cdots X_{i_q}$ we obtain $(T\xi)(X)=\sum_{k=1}^{n} \rho(X_k)\xi(-\Lambda\mathrm{ad}(X_k)X)$, and for $\xi=f\otimes e$ and σ the dual representation of $\Lambda\mathrm{ad}$ we obtain

$$T(f\otimes e) = \sum_{k=1}^{n} \rho(X_k)f\otimes\sigma(X_k)e .$$

To conclude the proof we compute as in [6]

$$2\rho(X_k)\otimes\sigma(X_k) = (\rho\otimes\sigma)(X_k)^2-\rho(X_k)^2\otimes id_E-id_F\otimes\sigma(X_k) ,$$

and obtain

$$2T=(\rho\otimes 1)(c)+(1\otimes\sigma)(c)-(\rho\otimes\sigma)(c)\,.$$

To prove the Theorem we will show that all the eigenvalues of Δ_ρ are positive. The basic observation (see Lemma 2 below) is that for any irreducible representation ρ, the endomorphism $\rho(c)$ is a scalar operator whose eigenvalue is given in terms of the highest weight of ρ. This fact will be applied individually to the irreducible components of $\rho\otimes 1$, $1\otimes\sigma$, and $\rho\otimes\sigma$.

First we introduce some notation. As above we fix a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$ and $\mathfrak{p}=i\mathfrak{k}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ compatible with the given Cartan decomposition. Then $\mathfrak{h}=\mathfrak{h}_\mathfrak{k}\oplus\mathfrak{h}_\mathfrak{p}$, where $\mathfrak{h}_\mathfrak{k}=\mathfrak{h}\cap\mathfrak{k}$ and $\mathfrak{h}_\mathfrak{p}=\mathfrak{h}\cap\mathfrak{p}=i\mathfrak{h}_\mathfrak{k}$. Let Δ denote the root system of the pair $(\mathfrak{g},\mathfrak{h})$. To each $\alpha\in\Delta$ we associate $H_\alpha\in\mathfrak{h}$ such that $\alpha(H)=\langle H_\alpha, H\rangle$ for all $H\in\mathfrak{h}$, where the Killing form is denoted by $\langle\,,\rangle$ from now on. Then $\mathfrak{h}_\mathfrak{p}$ coincides with the real vector space spanned by $\{H_\alpha;\ \alpha\in\Delta\}$, so Δ may be viewed as a subset of $\mathfrak{h}_\mathfrak{p}^*$, the real dual of $\mathfrak{h}_\mathfrak{p}$. The Killing form $\langle\,,\rangle$ is real and positive definite on $\mathfrak{h}_\mathfrak{p}$, hence it induces a scalar product $\langle\,,\rangle$ on $\mathfrak{h}_\mathfrak{p}^*$. By fixing a basis of Δ we once and for all determine a set Δ^+ of positive roots. We define $\delta=\sum_{\alpha\in\Delta^+}\alpha$.

Lemma 2. *Let $\rho\colon\mathfrak{g}\to\mathfrak{gl}(F)$ be any irreducible representation of $\mathfrak{g}$ with highest weight λ, then $\rho(c)=\langle\lambda,\lambda+\delta\rangle\cdot id_F$.*

For a proof see Raghunathan [5, Lemma 4], or Bourbaki [2, Ch. 8, §6, n° 4].

Lemma 2 immediately applies to our given representation ρ and thus enables us to compute the contribution of $3(\rho\otimes 1)(c)$ to the eigenvalues of $2\Delta_\rho$. The second term $(1\otimes\sigma)(c)$ involves the representation $\sigma=\Lambda\mathrm{ad}^*$ of $\mathfrak{g}$ on $E=\Lambda\mathfrak{g}^*$. This representation is no longer irreducible, so Lemma 2 applies to each component of σ separately. Thus the knowledge of the highest weights of the irreducible components of σ is required.

Lemma 3. *Let μ be the highest weight of an irreducible component of σ. Then μ is of the form $\mu=\sum_{\alpha\in\Delta}m_\alpha\alpha$, with $m_\alpha\in\{0,1\}$.*

Proof. The weight space decomposition of $\mathrm{ad}\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ with respect to the Cartan subalgebra $\mathfrak{h}$ equals the root space decomposition of the pair $(\mathfrak{g},\mathfrak{h})$, i.e.,

$$\mathfrak{g}=\mathfrak{h}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha\,.$$

The dual representation $\mathrm{ad}^*\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}^*)$ leads to the analogous decomposition

$$\mathfrak{g}^*=\mathfrak{h}^*\oplus\sum_{\alpha\in\Delta}(\mathfrak{g}^*)_\alpha\,,\qquad\text{with}\qquad(\mathfrak{g}^*)_\alpha=(\mathfrak{g}_{-\alpha})^*\,.$$

Now let $n=\dim\mathfrak{g}^*$, $r=\dim\mathfrak{h}^*$, and observe $\dim(\mathfrak{g}^*)_\alpha=1$. Then $E=\Lambda\mathfrak{g}^*=\Lambda(\mathfrak{h}^*\oplus\sum_{\alpha\in\Delta}(\mathfrak{g}^*)_\alpha)$ is isomorphic to a sum of subspaces of the form

$$\Lambda^h(\mathfrak{h}^*)\otimes(\mathfrak{g}^*)_{\alpha_1}\otimes\cdots(\mathfrak{g}^*)_{\alpha_k},$$

where $0\leqslant h\leqslant r$, $0\leqslant k\leqslant n-1$, $\alpha_i\in\Delta$, and $\alpha_i\neq\alpha_j$ for $i\neq j$.

Such a subspace is invariant under the action of $\mathfrak{h}$, it has weight $\alpha_1+\cdots+\alpha_k$. This implies in particular that the highest weights of the irreducible components occuring in σ are of the form $\alpha_1+\cdots+\alpha_k$, $\alpha_i\in\Delta$, $\alpha_i\neq\alpha_j$ for $i\neq j$.

Let now $E=\sum E_\mu$ be the decomposition of E into its irreducible components E_μ indexed by their respective highest weights. Lemma 2 enables us to compute the eigenvalues of $(1\otimes\sigma)(c)$ on $F\otimes E_\mu$. The third term, $(\rho\otimes\sigma)(c)$, in Lemma 1 involves the representation $\rho\otimes\sigma$ of $\mathfrak{g}$ on $F\otimes E$. This space certainly decomposes into the sum $\sum F\otimes E_\mu$, but each of the components $F\otimes E_\mu$ may further decompose into a sum $F\otimes E_\mu=\sum V_\mu^\nu$, where the subspaces V_μ^ν, irreducible under $\rho\otimes\sigma$, are indexed bei their respective heighest weights ν.

For each μ there is exactly one component $V_\mu^{\lambda+\mu}$ of $F\otimes E_\mu$ with highest weight $\lambda+\mu$. All other components V_μ^ν have highest weights $\nu<\lambda+\mu$. The following lemma allows us to restrict our attention to the spaces $V_\mu^{\lambda+\mu}$.

Lemma 4. *Let ρ_1, ρ_2 be two irreducible representations of $\mathfrak{g}$ with respective highest weights λ_1, λ_2. Then $\lambda_1>\lambda_2$ implies*

$$\langle\lambda_1, \lambda_1+\delta\rangle>\langle\lambda_2, \lambda_2+\delta\rangle$$

Proof. Let $\beta=\lambda_1-\lambda_2$ and assume $\beta>0$. Then

$$\langle\lambda_1, \lambda_1+\delta\rangle-\langle\lambda_2, \lambda_2+\delta\rangle=$$
$$2\langle\lambda_2, \beta\rangle+\langle\beta, \beta\rangle+\langle\beta, \delta\rangle>0,$$

since λ_2 and δ are dominant.

According to Lemma 4, the maximal eigenvalue of $(\rho\otimes\sigma)(c)$ restricted to $F\otimes E_\mu$ is attained on the space $V_\mu^{\lambda+\mu}$. Since $(\rho\otimes 1)(c)$ and $(1\otimes\sigma)(c)$ are positive scalar operators on the whole space $F\otimes E_\mu$, and $(\rho\otimes\sigma)(c)$ occurs with a minus sign in $2\Delta_\rho$, the *minimal* eigenvalue of $2\Delta_\rho$ restricted to $F\otimes E_\mu$ is attained on the space $V_\mu^{\lambda+\mu}$. This minimal eigenvalue involves only λ and μ, according to Lemma 2. Our claim is now reduced to the

Assertion. *Let μ be any of the highest weights occuring in the decomposition $E=\sum E_\mu$. Then the eigenvalue of $2\Delta_\rho$ is positive on $V_\mu^{\lambda+\mu}$.*

Proof. On $V_\mu^{\lambda+\mu}$ we have

$$2\Delta_\rho=\{3\langle\lambda, \lambda+\delta\rangle+\langle\mu, \mu+\delta\rangle-\langle\lambda+\mu, \lambda+\mu+\delta\rangle\}\cdot\mathrm{id}.$$

By a straightforward computation this reduces to

$$\Delta_\rho = \{\langle\lambda, \lambda\rangle + \langle\lambda, \delta-\mu\rangle\}\cdot \mathrm{id}\,.$$

The term $\langle\lambda, \lambda\rangle$ is obviously positive, since λ is the highest weight of a non-trivial representation. Now $\delta = \sum_{\alpha\in\Delta^+} \alpha$, and according to Lemma 3, $\mu = \sum_{\alpha\in\Delta} m_\alpha \alpha$ with $m_\alpha \in \{0, 1\}$, hence $\delta-\mu = \sum_{\alpha\in\Delta^+} n_\alpha \alpha$ with $n_\alpha \in \{0, 1, 2\}$. Therefore $\langle\lambda, \delta-\mu\rangle = \sum_{\alpha\in\Delta^+} n_\alpha \langle\lambda, \alpha\rangle \geqslant 0$, since λ is dominant.

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Hans-Christoph Im Hof and Ernst A. Ruh

Mathematisches Institut der Universität Bonn
Wegelerstrasse 10
D-5300 Bonn, Germany