

BOUNDEDNESS OF SOME PSEUDO-DIFFERENTIAL OPERATORS

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1. Introduction. In a recent paper [1] H. O. Cordes develops a new method to deal with pseudo-differential operators. He shows, among others, that if a symbol $a(x, \xi)$ defined on $R^n \times R^n$ has bounded derivatives $D_x^\beta D_\xi^\alpha a$ for $|\alpha|, |\beta| \leq [n/2] + 1$, then the associated pseudo-differential operator $A = a(X, D)$ is L^2 -bounded. (A similar result was previously given by Calderón and Vaillancourt [2].) This result has been partially generalized by A. G. Childs [3], who shows by the same method that a uniform *multiple* Hölder-continuity of a with an exponent larger than $1/2$ is sufficient for the boundedness of A .

The main purpose of the present paper is to show that the same method can be used to prove that A is L^2 -bounded if $|D_x^\beta D_\xi^\alpha a(x, \xi)| \leq M(1 + |\xi|)^{(|\beta| - |\alpha|)\rho}$ for $|\alpha| \leq [n/2] + 1$ and $|\beta| \leq [n/2] + 2$, where $0 < \rho < 1$ (see Theorem 5.3 below). A similar result is contained, as a special case, in Calderón and Vaillancourt [4] and H. Kumano-go [5, 6], except that the numbers of the required derivatives are different. But it may be of some interest to give a new proof, which requires relatively little amount of computation.

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2. A formal identity. We find it convenient to start with formulating the basic idea of Cordes in a slightly different form.

Given a tempered distribution a on $R^n \times R^n$, the pseudo-differential operator $A = a(X, D)$ may be defined by

$$(2.1) \quad \langle Au, v \rangle = \langle a, w \rangle, \quad w(x, \xi) = (2\pi)^{-n/2} e^{i\xi x} \hat{u}(\xi) v(x),$$

where $u, v \in \mathcal{S}(R^n)$ (the Schwartz space) so that $w \in \mathcal{S}(R^n \times R^n)$; here $\hat{}$ denotes the Fourier transform, and \langle , \rangle the pairing between \mathcal{S}' and \mathcal{S} . A is a continuous operator on $\mathcal{S}(R^n)$ to $\mathcal{S}'(R^n)$. As usual, (2.1) may be written symbolically as

$$(2.2) \quad Au(x) = a(X, D)u(x) = (2\pi)^{-n/2} \int_{R^n} e^{i\xi x} a(x, \xi) \hat{u}(\xi) d\xi.$$

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In what follows we are interested only in symbols a which are locally integrable functions on $R^n \times R^n$.

If in particular $a(x, \xi) = x_j$, we have $A = X_j$, the operator of multiplication with x_j . If $a(x, \xi) = \xi_j$, we have $A = D_j = -i\partial/\partial x_j$. As usual we write $X = (X_1, \dots, X_n)$ and $D = (D_1, \dots, D_n)$. Also we use the customary multi-index notation such as $X^\alpha = a(X)$ where $a(x) = x^\alpha = \prod_j x_j^{\alpha_j}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Note that X_j and D_j map $\mathcal{S}(R^n)$ into itself and can be extended, in an obvious way, to operators on $\mathcal{S}'(R^n)$ to itself. The same is true of the operators $e^{i\xi X}$ and e^{ixD} , where ξ and x are in R^n and $\xi X = \sum_j \xi_j X_j$, etc.; these are given by

$$e^{i\xi X}u(x') = e^{i\xi x'}u(x'), \quad e^{ixD}u(x') = u(x' + x), \quad x' \in R^n.$$

A basic tool in this paper is given by the following lemma.

Lemma 2.1. *Let $b \in L^\infty(R^n \times R^n)$ and $g \in L^1(R^n \times R^n)$. Then*

$$(2.3) \quad (b * g)(X, D) = \iint_{R^n \times R^n} b(x, \xi) e^{i\xi X} e^{-ixD} g(X, D) e^{ixD} e^{-i\xi X} dx d\xi,$$

where $*$ denotes convolution (so that $b * g \in L^\infty(R^n \times R^n)$).

Because of the properties of the operators $e^{i\xi X}$ and e^{ixD} mentioned above, the integrand in (2.3) is an operator on $\mathcal{S}(R^n)$ to $\mathcal{S}'(R^n)$. Thus (2.3) makes sense if the integral is taken in the weak sense. We may omit the straightforward verification of (2.3).

It may be noted that (2.3) is valid under more general conditions, such as $b \in L^p$ and $g \in L^q$, where $1 \leq p, q \leq \infty$ and $p^{-1} + q^{-1} \geq 1$, so that $b * g \in L^r$, $r^{-1} = p^{-1} + q^{-1} - 1$.

3. An operator calculus. Let $H = L^2(R^n)$. We denote by $\mathbf{B}(H)$ the set of all bounded linear operators on H , with the operator norm $\|\cdot\|$. $\mathbf{B}_1(H)$ denotes the trace class of compact operators on H , with the associated trace norm $\|\cdot\|_1$.

The following theorem, essentially due to Cordes [1], gives a meaning to the expression in (2.3) as an operator on H .

Theorem 3.1. *Let $G \in \mathbf{B}_1(H)$ and $b \in L^\infty(R^n \times R^n)$. Then*

$$(3.1) \quad B = \iint_{R^n \times R^n} b(x, \xi) e^{i\xi X} e^{-ixD} G e^{ixD} e^{-i\xi X} dx d\xi \in \mathbf{B}(H)$$

exists as a strong (improper) integral. The mapping $b, G \mapsto B \equiv b\{G\}$ has the following properties.

- (i) $\|b\{G\}\| \leq (2\pi)^n \|b\|_{L^\infty} \|G\|_1$.
- (ii) $b \geq 0$ and $G \geq 0$ imply $b\{G\} \geq 0$.

- (iii) $1\{G\}=(2\pi)^n(\operatorname{tr} G)I$.
 (iv) $(b_1b_2)\{G\}\ll(|b_1|^2\{|G|\}; |b_2|^2\{|G^*\}|)$.

REMARK. Here $e^{i\xi X}$ and e^{ixD} are regarded as unitary operators on \mathbf{H} . $G\geq 0$ means that G is nonnegative selfadjoint. I is the identity operator. $\operatorname{tr} G$ is the trace of $G\in\mathbf{B}_1(\mathbf{H})$. $|G|$ means $(G^*G)^{1/2}$. $|b|$ is defined by $|b|(x, \xi)=|b(x, \xi)|$. In (iv) we have used the following notation: given three operators T, A, B in $\mathbf{B}(\mathbf{H})$ with $A\geq 0, B\geq 0$, we write

$$(3.2) \quad T\ll(A; B) \text{ if } |(Tu, v)|^2\leq(Au, u)(Bv, v) \quad \text{for } u, v\in\mathbf{H}.$$

The convenience of such a notation is seen from the following lemma, the proof of which is simple and may be omitted.

- Lemma 3.2.** (i) $T\ll(|T|; |T^*|)$ for any $T\in\mathbf{B}(\mathbf{H})$.
 (ii) $T\ll(A; B)$ implies $T^*\ll(B; A)$.
 (iii) $T\ll(A; B)$ implies $S^*TS\ll(S^*AS; S^*BS)$, $S\in\mathbf{B}(\mathbf{H})$.
 (iv) If $T_j\ll(A_j; B_j)$, $j=1, 2, \dots$, then

$$(3.3) \quad \sum_j T_j\ll(\sum_j A_j; \sum_j B_j)$$

in the sense that whenever the series on the right converge in the strong sense, the same is true of the left member and the inequality holds. A similar result holds when the series are replaced by integrals.

Proof of Theorem 3.1. First we consider the case when $G\geq 0$ and $b=1$.
 Let

$$(3.4) \quad G = \sum_k \lambda_k (, f_k) f_k, \quad \lambda_k \geq 0, \quad f_k \in \mathbf{H},$$

$$\sum_k \lambda_k = \operatorname{tr} G, \quad (f_k, f_h) = \delta_{kh},$$

be the spectral decomposition of $G\in\mathbf{B}_1(\mathbf{H})$. Then for $u\in\mathbf{H}$

$$(3.5) \quad \iint (e^{i\xi X} e^{-ixD} G e^{ixD} e^{-i\xi X} u, u) dx d\xi$$

$$= \sum_k \lambda_k \iint |(e^{-i\xi X} u, e^{-ixD} f_k)|^2 dx d\xi$$

$$= \sum_k \lambda_k \int d\xi \int |\hat{u}(\eta+\xi) e^{i\eta x} \overline{\hat{f}_k(\eta)} d\eta|^2 dx$$

$$= (2\pi)^n \sum_k \lambda_k \int d\xi \int |\hat{u}(\eta+\xi) \overline{\hat{f}_k(\eta)}|^2 d\eta \quad (\text{Parseval})$$

$$= (2\pi)^n \|u\|^2 \sum_k \lambda_k = (2\pi)^n (\operatorname{tr} G) \|u\|^2.$$

Since in this case the integrand in (3.1) is a nonnegative operator, this computation establishes not only the strong convergence of the integral but also (iii) and

(i). (The unnecessary detour via \hat{u}, \hat{f}_k in (3.5) was made with a later reference in mind.)

The case of $G \geq 0$ and general b can be reduced to the above case by majorizing b by $\|b\|_{L^\infty}$. The case of a general G can be dealt with as an application of Lemma 3.2 by noting that $U^*GU \ll (U^*|G|U; U^*|G^*|U)$ where $U = e^{ixD}e^{-i\xi X}$. (iii) can be proved by writing G as a linear combination of non-negative operators and using (3.5)

4. Some special symbols. To apply Theorem 3.1 in combination with Lemma 2.1, we need some special symbols g for which $g(X, D)$ has an extension $G \in \mathbf{B}_1(\mathbf{H})$. Such symbols have been constructed by Cordes [1].

Let $\psi = \psi_{n,s}$ be the unique solution (within $\mathcal{S}'(R^n)$) for

$$(4.1) \quad (1 - \Delta)^{s/2} \psi = \delta,$$

where s is a real number, Δ is the Laplacian, and δ is the delta function. ψ can be expressed in terms of the modified Hankel function, but we do not need its precise form. It suffices to know that $\psi \in C^\infty(R^n - \{0\})$, $\psi(x)$ and its derivatives decay exponentially as $|x| \rightarrow \infty$, and that $D^\alpha \psi(x) = O(1 + |x|^{s-n-|\alpha|})$ as $|x| \rightarrow 0$, except when $s-n-|\alpha|=0$ in which case we have to put a logarithmic function in the last estimate.

Lemma 4.1. *Let $g(x, \xi) = \psi_{n,t}(x)\psi_{n,s}(\xi)$, where $s, t > n/2$. Then $g(X, D)$ has an extension $G \in \mathbf{B}_1(\mathbf{H})$. The same is true of $g(X, D)D^\alpha$ for any multi-index α .*

For the proof see [1].

Lemma 4.2. *In Lemma 4.1 assume that $s > n/2$ and $t > n/2 + 1$. Then the operators $D_j g(X, D)$ and $|D| g(X, D)$ have extensions belonging to $\mathbf{B}_1(\mathbf{H})$.*

Proof. We have $D_j g(X, D) = g_1(X, D) + g_2(X, D)$, where

$$g_1(x, \xi) = -i\partial g(x, \xi)/\partial x_j, \quad g_2(x, \xi) = g(x, \xi)\xi_j.$$

Here $g_2(X, D) \subset G_2 \in \mathbf{B}_1(\mathbf{H})$ by Lemma 4.1. Also $g_1(X, D) \subset G_1 \in \mathbf{B}_1(\mathbf{H})$, since $\partial \psi_{n,t}(x)/\partial x_j$ has properties similar to $\psi_{n,t-1}$. For $|D| g(X, D)$, it suffices to note that $|D| = \sum_j D_j |D|^{-1} D_j$ where $D_j |D|^{-1} \in \mathbf{B}(\mathbf{H})$.

5. L^2 -boundedness. We are now able to consider the L^2 -boundedness of certain pseudo-differential operators. First we state the theorems due to Cordes [1].

Theorem 5.1. *If $D_x^\beta D_\xi^\alpha a \in L^\infty(R^n \times R^n)$ for $|\alpha|, |\beta| \leq [n/2] + 1$, then $a(X, D)$ is L^2 -bounded.*

Theorem 5.2. *Let $a \in \mathcal{S}'(R^n \times R^n)$ with $b = (1 - \Delta_x)^{s/2} (1 - \Delta_\xi)^{s/2} a \in L^\infty(R^n \times R^n)$ for some $s > n/2$. Then $a(X, D)$ is L^2 -bounded.*

We shall indicate the proof briefly for later reference. First it is shown that the assumption of Theorem 5.1 implies that of Theorem 5.2. To prove Theorem 5.2, let $g(x, \xi) = \psi(x)\psi(\xi)$ with $\psi = \psi_{n,s}$. Then $g \in L^1$ and $a = b * g$ because g is the elementary solution for the operator $(1 - \Delta_x)^{s/2} (1 - \Delta_\xi)^{s/2}$. Since $g(X, D) \subset G \in \mathcal{B}_1(\mathbf{H})$ by Lemma 4.1, it follows from Theorem 3.1 and Lemma 2.1 that $a(X, D) \subset b \{G\} \in \mathcal{B}(\mathbf{H})$.

Our main result is given by the following theorem.

Theorem 5.3. *Assume that*

$$(5.1) \quad |D_x^\beta D_\xi^\alpha a(x, \xi)| \leq M(1 + |\xi|)^{(\beta| - |\alpha|)\rho}$$

$$\text{for } |\alpha| \leq [n/2] + 1, \quad |\beta| \leq [n/2] + 2,$$

where M and ρ are constants such that $0 < \rho < 1$. Then $a(X, D)$ is L^2 -bounded.

We do not consider here the case $\rho = 0$, since Theorem 5.1 gives a stronger result in this case.

Our proof of Theorem 5.3 is based on the same idea as that of Theorem 5.1 indicated above. But it appears that a preliminary partition of the symbol a into small pieces is necessary (a device suggested by Hörmander [7]).

Proof of Theorem 5.3, Part I. Let $\{\phi_j; j = 1, 2, 3, \dots\}$ be a partition of unity on $[0, \infty)$: $\sum \phi_j = 1$, with the following additional properties. $\phi_1 \in C_0^\infty[0, \infty)$ with $\phi_1(r) = 1$ for $0 \leq r \leq 1$. If $j \geq 2$, $\phi_j \in C_0^\infty(0, \infty)$ with support in $[j-1, j+1]$ and $\phi_j(j+r) = \phi_2(2+r)$. Note that $j=1$ is exceptional. Note also that all the derivatives of the ϕ_j are bounded uniformly in j .

Let $|\xi|_*$ be a C^∞ -function of $\xi \in R^n$ such that $|\xi|_* = |\xi|$ for $|\xi| \geq 1$ and $0 < |\xi|_* < 1$ for $|\xi| < 1$. Then $|\xi|_* = |\xi|$ whenever $|\xi|_* \geq 1$. Set

$$(5.2) \quad \Phi_j(\xi) = \phi_j(|\xi|_*^{1-\rho}).$$

Then $\{\Phi_j\}$ is a partition of unity on R^n , with $\Phi_j \in C_0^\infty(R^n)$ and

$$(5.3) \quad \Phi_j(\xi) = \phi_j(|\xi|^{1-\rho}) \quad \text{for } j \geq 2.$$

An important property of the Φ_j is that

$$(5.4) \quad |D_\xi^\alpha \Phi_j(\xi)| \leq c_1 |\xi|^{-|\alpha|\rho}, \quad |\alpha| \leq [n/2] + 1,$$

where c_1 is a constant independent of j , as is easily seen from (5.3) and the remark above about the derivatives of the ϕ_j .

Also it follows from (5.3) that $\xi \in \text{supp } \Phi_j$ implies

$$(5.5) \quad (j/2)^{1/(1-\rho)} \leq |\xi| \leq (2j)^{1/(1-\rho)}, \quad j \geq 2,$$

$$(5.5') \quad ||\xi| - j^{1/(1-\rho)}| \leq c_2 j^{\rho/(1-\rho)}, \quad j \geq 1,$$

where c_2 is a constant independent of j .

Set

$$(5.6) \quad a_j(x, \xi) = \Phi_j(\xi) a(x, \xi), \quad j = 1, 2, 3, \dots,$$

so that

$$(5.7) \quad a(x, \xi) = \sum_j a_j(x, \xi).$$

In view of (5.4) and (5.5), (5.1) implies that

$$(5.8) \quad |D_x^\beta D_\xi^\alpha a_j(x, \xi)| \leq c_3 j^{(|\beta| - |\alpha|)\sigma} \chi_j(\xi), \quad \sigma = \rho/(1-\rho) > 0,$$

with c_3 independent of j , where χ_j denotes the characteristic function of $\text{supp } \Phi_j$ and α, β range over multi-indices specified in (5.1).

It follows from (5.5') that

$$(5.9) \quad \chi_j(\xi) \leq \chi_j'(j^{-\sigma}\xi),$$

where

$$(5.10) \quad \chi_j'(\xi) = \begin{cases} 1 & \text{for } j - c_2 \leq |\xi| \leq j + c_2, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$(5.11) \quad a_j'(x, \xi) = a_j(j^{-\sigma}x, j^\sigma\xi).$$

Then it follows from (5.8) to (5.10) that

$$(5.12) \quad |D_x^\beta D_\xi^\alpha a_j'(x, \xi)| \leq c_3 \chi_j'(\xi), \\ |\alpha| \leq [n/2] + 1, \quad |\beta| \leq [n/2] + 2.$$

Now define

$$(5.13) \quad b_j'(x, \xi) = (1 - \Delta_x)^{s/2} (1 - \Delta_\xi)^{t/2} a_j'(x, \xi),$$

where $s > n/2$ and $t > n/2 + 1$ are real numbers chosen as

$$(5.14a) \quad s = [n/2] + 1, \quad t = [n/2] + 5/3, \quad \text{if } [n/2] \text{ is odd,}$$

$$(5.14b) \quad s = [n/2] + 2/3, \quad t = [n/2] + 2, \quad \text{if } [n/2] \text{ is even.}$$

Lemma 5.4. *There is a finite positive measure μ on R_ξ^n such that*

$$(5.15) \quad |b_j'(x, \xi)| \leq (\mu * \chi_j')(\xi) \equiv \omega_j'(\xi),$$

$$(5.16) \quad \int |\xi| d\mu(\xi) < \infty.$$

Proof. If $[n/2] = 2k - 1$ is odd, then $s = 2k$ and $t = 2k + 2/3$ by (5.14a). Thus we can write (5.13) in the form

$$\begin{aligned} b_j' &= (1 - \Delta_x)(1 - \Delta_x)^{-2/3}(1 - \Delta_x)^k(1 - \Delta_\xi)^k a_j' \\ &= [(1 - \Delta_x)^{-2/3} - \operatorname{div}_x(1 - \Delta_x)^{-2/3} \operatorname{grad}_x](1 - \Delta_x)^k(1 - \Delta_\xi)^k a_j'. \end{aligned}$$

Here $(1 - \Delta_x)^k(1 - \Delta_\xi)^k a_j'$ and its grad_x are linear combinations of the derivatives of a_j' appearing in (5.12) and are majorized by $\chi_j'(\xi)$. Since, on the other hand, $(1 - \Delta_x)^{-2/3}$ and $D_x(1 - \Delta_x)^{-2/3}$ are convolutions by $\psi_{n,4/3}$ and $D_x\psi_{n,4/3}$ which are integrable (see section 4), it follows that $|b_j'| \leq \operatorname{const} \chi_j'(\xi)$. Thus (5.15) is true with $\mu = \operatorname{const} \cdot \delta$, where δ is the delta function.

If $[n/2] = 2k$ is even, then $s = 2k + 2/3$ and $t = 2k + 2$ by (5.14b). In this case we may write

$$b_j' = [(1 - \Delta_\xi)^{-2/3} - \operatorname{div}_\xi(1 - \Delta_\xi)^{-2/3} \operatorname{grad}_\xi](1 - \Delta_x)^{k+1}(1 - \Delta_\xi)^k a_j'.$$

Again $(1 - \Delta_x)^{k+1}(1 - \Delta_\xi)^k a_j'$ and its grad_ξ are majorized by $\chi_j'(\xi)$ by (5.12). Since $(1 - \Delta_\xi)^{-2/3}$ and $D_\xi(1 - \Delta_\xi)^{-2/3}$ are convolutions on R_ξ^n by $\psi_{n,4/3}$ and $D_\xi\psi_{n,4/3}$, respectively, which are integrable, (5.15) is true if we choose $d\mu(\xi) = m(\xi)d\xi$ where $m(\xi)$ is any positive integrable function that majorizes the convolution kernels involved. Since these kernels decay exponentially at infinity (see section 4), we can also satisfy (5.16).

Part II. (5.13) implies

$$(5.17) \quad a_j' = b_j' * g, \quad g(x, \xi) = \psi_{n,t}(x)\psi_{n,s}(\xi).$$

By a scale transformation, it is easy to obtain (note (5.11))

$$(5.18) \quad \begin{aligned} a_j &= b_j * g_j, \quad \text{where} \\ b_j(x, \xi) &= b_j'(j^\sigma x, j^{-\sigma} \xi), \quad g_j(x, \xi) = g(j^\sigma x, j^{-\sigma} \xi). \end{aligned}$$

It follows from Theorem 3.1 and Lemma 2.1 that

$$(5.19) \quad a_j(X, D) \subset A_j = b_j \{G_j\} \in \mathbf{B}(\mathbf{H}), \quad g_j(X, D) \subset G_j \in \mathbf{B}_1(\mathbf{H});$$

note that $G_j \in \mathbf{B}_1(\mathbf{H})$ because $s, t > n/2$ (Lemma 4.1). Thus

$$(5.20) \quad \begin{aligned} A_j &\ll (|b_j| \{|G_j|\}; |b_j| \{|G_j^*\|}) \\ &\ll (\omega_j \{|G_j|\}; \omega_j \{|G_j^*\|}), \quad \omega_j(\xi) = \omega_j'(j^{-\sigma} \xi), \end{aligned}$$

by Theorem 3.1, Lemma 3.2 and (5.15). Hence

$$(5.21) \quad \sum_j A_j \ll (\sum_j \omega_j \{|G_j|\}; \sum_j \omega_j \{|G_j^*\|})$$

by Lemma 3.2.

To estimate the right member of (5.21), we note that $G_j \supset g_j(X, D)$ is unitarily equivalent to $G \supset g(X, D)$ because g_j and g are related by the scale

transformation given in (5.18). Hence $|G_j|$ is unitarily equivalent to $|G|$. Let

$$(5.22) \quad |G| = \sum_k \lambda_k(\cdot, f_k) f_k$$

be the canonical spectral representation of $|G|$ (see (3.4)). Then we have

$$(5.23) \quad |G_j| = \sum_k \lambda_k(\cdot, f_{kj}) f_{kj}, \quad f_{kj}(x) = j^{n\sigma/2} f_k(j^\sigma x).$$

A computation similar to (3.5) thus gives

$$(5.24) \quad \begin{aligned} & (\omega_j \{|G_j|\} u, u) \\ &= (2\pi)^n \sum_k \lambda_k \int \omega_j(\xi) d\xi \int |\hat{u}(\xi + \eta)|^2 |\hat{f}_{kj}(\eta)|^2 d\eta; \end{aligned}$$

the only difference of (5.24) from (3.5) is the appearance of the factor $\omega_j(\xi)$ in the integrand, which was absent in (3.5) because $b=1$ there. Since (5.23) implies $\hat{f}_{kj}(\eta) = j^{-n\sigma/2} \hat{f}_k(j^{-\sigma}\eta)$, we obtain from (5.24), after changing the integration variable ξ into $\xi - \eta$ and then writing $\eta = j^\sigma \zeta$,

$$(5.25) \quad \begin{aligned} & (\omega_j \{|G_j|\} u, u) \\ &= (2\pi)^n \sum_k \lambda_k \int |\hat{u}(\xi)|^2 d\xi \int \omega_j(\xi - j^\sigma \zeta) |\hat{f}_k(\zeta)|^2 d\zeta. \end{aligned}$$

Now we use the following lemma, which will be proved in the next section.

Lemma 5.5. *There is a constant K such that*

$$(5.26) \quad \sum_j \omega_j(\xi - j^\sigma \zeta) \leq K(1 + |\zeta|), \quad \xi, \zeta \in \mathbb{R}^n.$$

Using this lemma, we see from (5.25) that

$$(5.27) \quad \begin{aligned} & \sum_k (\omega_j \{|G_j|\} u, u) \\ & \leq (2\pi)^n K \|u\|^2 \sum_k \lambda_k \int |\hat{f}_k(\zeta)|^2 (1 + |\zeta|) d\zeta \\ & = (2\pi)^n K \|u\|^2 \sum_k \lambda_k ((1 + |D|) f_k, f_k) \\ & = (2\pi)^n K \|u\|^2 \operatorname{tr}(1 + |D|) |G|. \end{aligned}$$

Let $G = W|G|$ be the polar decomposition of G , where W is a partial isometry. Then $|G| = G^*W$ and

$$(5.28) \quad \operatorname{tr}(1 + |D|) |G| = \operatorname{tr}(1 + |D|) G^*W \leq \|(1 + |D|) G^*\|_1.$$

But $G(1 + |D|)$ has an extension belonging to $\mathcal{B}_1(\mathbf{H})$ (Lemma 4.1). Hence $(1 + |D|)G^* \in \mathcal{B}_1(\mathbf{H})$ and (5.28) is finite. In view of (5.27), we have proved the strong convergence of the first series on the right of (5.21).

The convergence of the second series can be proved in the same way. As above, it reduces to showing that $|D|G \in \mathcal{B}_1(\mathbf{H})$. But this follows from Lemma

4.2; we have chosen $t > n/2 + 1$ for this purpose.

Thus we conclude from (5.21) that $\sum A_j$ converges strongly in $\mathbf{B}(\mathbf{H})$. In view of (5.7), the proof of Theorem 5.3 is complete.

6. Proof of Lemma 5.5. We have by (5.20) and (5.15)

$$(6.1) \quad \begin{aligned} \omega_j(\xi - j^\sigma \zeta) &= \omega_j'(j^{-\sigma} \xi - \zeta) = (\mu * \chi_j')(j^{-\sigma} \xi - \zeta) \\ &= \int \chi_j'(j^{-\sigma} \xi - \zeta - \eta) d\mu(\eta). \end{aligned}$$

We shall now show that

$$(6.2) \quad \sum_j \chi_j'(j^{-\sigma} \xi - \zeta - \eta) \leq \text{const}(1 + |\zeta| + |\eta|),$$

from which the desired result follows by (6.1) because $\int d\mu(\eta)$ and $\int |\eta| d\mu(\eta)$ are finite (Lemma 5.4).

According to (5.10), the left member of (6.2) is equal to the number of positive integers j such that

$$(6.3) \quad ||j^{-\sigma} \xi - \zeta - \eta| - j| \leq c_2.$$

But it is easy to show that (6.3) implies

$$(6.4) \quad |j - j^{-\sigma} |\xi|| \leq c_2 + |\zeta| + |\eta|.$$

Now the function $j \mapsto j - j^{-\sigma} |\xi|$ has derivative larger than 1. Hence the number of integers j satisfying (6.4) does not exceed $2(c_2 + |\zeta| + |\eta|)$. This proves the desired result.

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