Yanagawa, T.
Osaka, J. Math.
6 (1969), 447-464

ON RIBBON 2-KNOTS
THE 3-MANIFOLD BOUNDED BY THE 2-KNOTS

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(Received February 24, 1969)
(Revised May 19, 1969)

1. Introduction

It is known that any locally flat, orientable, closed 2–manifold M^2 in a 4–space R^4 bounds an orientable 3–manifold W^3 in R^4, see [1]. Nevertheless, the question "what type of 3–manifolds can be bounded" seems to be still an open question, for which we will give a partial answer in Theorem (2, 3) in this paper: *If the 2–manifold M^2 is a 2–sphere of a special type of 2–knots, which will be called a ribbon 2–knot, see (2, 2), it bounds a 3–manifold W^3 homeomorphic to $\#(S^1\times S^2)-\mathring{\Delta}^3$.*[0)] Moreover, a little inspection on the 3–manifold W^3 shows that there exists *a trivial system of the 2–spheres in W^3*, see (3, 5), and we can easily prove a converse of the above theorem in Theorem (3, 3). In §4, we will define the following concepts;

$R(3)$: A 2–knot K^2 satisfies that $c(\{K\})=0$,
$R(4)$: A 2–knot K^2 bounds *a 3–ribbon* in R^4,
$R(5)$: A 2–knot K^2 bounds *a monotone 3–ball in H^5_+*.

Since it is easily seen that the concepts $R(4)$ and $R(5)$ are the natural extensions of the definition and the property of the ribbon (1–) knots, we can explain the reason why we denominate *simply knotted 2–knots* defined in [2] as *ribbon 2–knots* in this paper, after we will have accomplished the proof of the equivalence of these three concepts in Theorem (4, 5). In §5, we will introduce *a normal-form for ribbon 2–knots* and an equivalence relation between 2–knots. The equivalence relation is a cobordism relation between 2–knots with the strong restriction, although *ribbon 2–knots are equivalent to a trivial 2–knot under the relation.*

In this paper, everything is considered from the combinatorial stand point of view.

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0) $\#$ means *the connected sum*, and Δ^3 a 3–simplex and $\mathring{\Delta}^3$ is its interior.

2. Ribbon 2-knots

An orthogonal projection p of a 4-space R^4, containing a locally flat 2-sphere K^2 which will be called *a 2-knot,* onto an hyperplane R^3 is called *a regular projection*, or simply *a projection*, if the locally linear map $p|K^2$ of K^2 into R^3 is normal.[1] The homeomorphism class of (K^2, R^4) of 2-knots in R^4 will be called the knot-type containing K^2, and will be denoted by $\{K\}$.

DEFINITION (2.1). $c(K)$ is the minimal number of the triple points and the branch-lines[2] of $p(K)$ in R^3, where the projection p ranges over the set consisting of all the projections for the 2-knot K. $c(\{K\})$ is the minimal number of $c(K)$, where K ranges over the knot-type $\{K\}$. A pair (p, K) will be called *a simple pair for the knot-type* $\{K\}$, if it realizes the number $c(\{K\})$.

DEFINITION (2.2).[3] A 2-knot K^2 will be called *a ribbon 2-knot*, if and only if $c(\{K\})=0$.

Theorem (2.3). *A ribbon 2-knot K^2 bounds a 3-manifold W^3 which is homeomorphic either to a 3-ball or to $\#(S^1\times S^2)-\mathring{\Delta}^3$.*

Proof. According to the result in [2], we can find a 2-knot K' belonging to $\{K\}$ and satisfying the following (1), (2) and (3):

(1). $K'\cap R_0^3=k$ is a ribbon knot in R_0^3.[4]

(2). $K'\cap H_+^4$ and $K'\cap H_-^4$ are symmetric with respect to the hyperplane R_0^3, and necessarily each of them is a locally flat 2-ball.

(3). each saddle point transformation[5] on $K'\cap H_+^4$ increases the number of components of the cross-sections of $K'\cap R_t^3$ as the height t increases; in other words, $K'\cap H_+^4$ has no minimal point.

In the following three-steps, we illustrate the construction of the 3-manifold W^3.

First-step. Since k is a ribbon knot, there is an immersion ψ of a 2-ball $\tilde{D}=\tilde{D}_0\cup\tilde{D}_1\cup\cdots\cup\tilde{D}_n\cup\tilde{B}_1\cup\cdots\cup\tilde{B}_n$ on a plane into R_0^3 such that

(1) $\psi(\partial\tilde{D})=k$, $\psi(\tilde{D})$ is a ribbon,

1) See p. 3 of [8].

2) An arc whose end-points are branch-points is called a branchline.

3) In [2], this type of 2-knots is defined as "simply-knotted 2-sphere".

4) $\mathring{X}$ means the interior, and ∂X the boundary of a set X.

$$\begin{aligned} R_t^3 &= \{(x_1, x_2, x_3, x_4) &&| x_4=t\} \\ H_+^4 &= \{(x_1, x_2, x_3, x_4) &&| x_4\geqq 0\} \\ H_-^4 &= \{(x_1, x_2, x_3, x_4) &&| x_4\leqq 0\} \\ H^4(J) &= \{(x_1, x_2, x_3, x_4) &&| x_4\in J\} \end{aligned}$$

5) See [5], p. 136.

(2) $\tilde{B}_i$ spans $\tilde{D}_0$ and $\tilde{D}_i$ coherently at the segments on $\partial\tilde{B}_i \cap \partial\tilde{D}_0$ and $\partial\tilde{B}_i \cap \partial\tilde{D}_i$ $(i=1, 2, \cdots, n)$,

(3) $\psi; \tilde{D}_0 \cup \tilde{D}_1 \cup \cdots \cup \tilde{D}_n \to D_0 \cup D_1 \cup \cdots \cup D_n$
$\psi; \tilde{B}_1 \cup \cdots \cup \tilde{B}_n \to B_1 \cup \cdots \cup B_n$
are both imbeddings,

(4) $D_0, D_1, \cdots, D_n$ are on a plane, and moreover we may suppose that the visible face of each D_k is the image of the visible-face of $\tilde{D}_k$ $(k=0, 1, 2, \cdots, n)$.

(5) the intersection of B_i and $D_0 \cup D_1 \cup \cdots \cup D_n$, except two segments on ∂B_i, consists of at most the ribbon-type segments $\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{im_i}$, where we indexed these segments in the order from D_0 to D_i on B_i $(i=1, 2, \cdots, n)$.

The segments $\tilde{\alpha}_{i\lambda} = \psi^{-1}(\alpha_{i\lambda}) \cap \tilde{B}_i$, and $\tilde{\beta}_{i\lambda}$, β_{i0} go across the band $\tilde{B}_i$, and $\tilde{B}_{i\lambda}$ is the piece of $\tilde{B}_i$ bounded by $\tilde{\beta}_{i,\lambda-1}$ $(\lambda=1, 2, \cdots, m_i)$ as in Fig. (1).

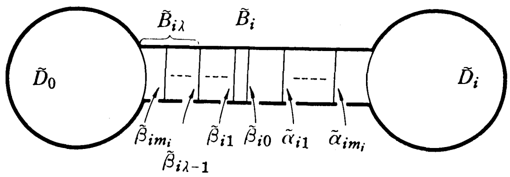


Fig. (1)

If B_i goes through D_k from its visible side to the opposite at α_{i1}, we perform the $(-)$-twist for k, otherwise the $(+)$-twist as in Fig. $(2\pm)$, where the knot-type of k is left fixed.

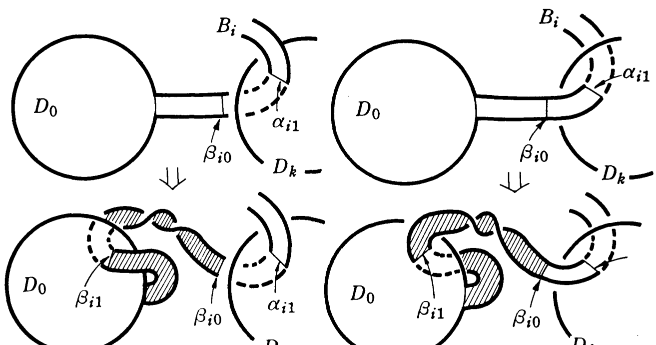


Fig. (2_+) Fig. (2_-)

Now, we have a new immersion ψ' of $\tilde{D}$ into R_0^3 such that

$$\psi' | \tilde{D} - \tilde{B}_{i1} = \psi | \tilde{D} - \tilde{B}_{i1} ,$$
$$\psi'(\tilde{B}_{i1}) \cap D_0 = \psi'(\tilde{\beta}_{i1}) = \beta_{i1} , \quad \text{and}$$

$\psi'(\tilde{B}_{i1})$ is the shaded portion in Fig. $(2\pm)$,
$\psi'(\tilde{D})$ is a ribbon and $\psi'(\partial\tilde{D})$ is the twisted k.

Moreover, if there is α_{i2}, we perform the twist of the same type as before considering $\tilde{B}_{i2}, \tilde{\beta}_{i2}, \beta_{i2}, \beta_{i1}, \psi'$ as $\tilde{B}_{i1}, \tilde{\beta}_{i1}, \beta_{i1}, \beta_{i0}, \psi$. Repeat successively these processes for $\lambda=3, 4, \cdots, m_i$, $i=1, 2, \cdots, n$.

Second-step. In the first-step, we have finally a ribbon $\psi^*(\tilde{D})=D_0\cup D_1\cup \cdots \cup D_n \cup B_1^* \cup \cdots \cup B_n^*$, where $B_i^*\cap(D_0\cup D_1\cup\cdots\cup D_n)=\beta_{i1}\cup\cdots\cup\beta_{im_i}\cup\cdots\cup\alpha_{im_i}$. Remove the mutually disjoint 2-balls $Q_{i\lambda}$ and $Q'_{i\lambda}$ from $D_0\cup D_1\cup\cdots\cup D_n$ such that

$$\mathring{D}_k\supset Q_{i\lambda}\supset\mathring{Q}_{i\lambda}\supset\alpha_{i\lambda},$$
$$\mathring{D}_0\supset Q'_{i\lambda}\supset\mathring{Q}'_{i\lambda}\supset\beta_{i\lambda}\quad(\lambda=1, 2, \cdots, m_i,\ i=1, 2, \cdots, n).$$

Combine $\partial Q_{i\lambda}$ and $\partial Q'_{i\lambda}$ with a tube $T_{i\lambda}$ coherently so that $T_{i\lambda}\cap T_{j\mu}=\phi$ ($i\neq j$ or $\lambda\neq\mu$), and that $T_{i\lambda}\cap(\psi^*(\tilde{D})-\bigcup_{i,\lambda}(\mathring{Q}_{i\lambda}\cup\mathring{Q}'_{i\lambda})=\partial T_{i\lambda}=\partial Q_{i\lambda}\cup\partial Q'_{i\lambda}$. Finally, we have an orientable 2-surface F_0 such that

$$F_0^2=\{(D_0\cup D_1\cup\cdots\cup D_n)-\bigcup_{i,\lambda}(Q_{i\lambda}\cup Q'_{i\lambda})\}\cup\{B_1^*\cup\cdots\cup B_n^*\}\cup\{\bigcup_{i,\lambda}T_{i\lambda}\},$$

see Fig. (3).

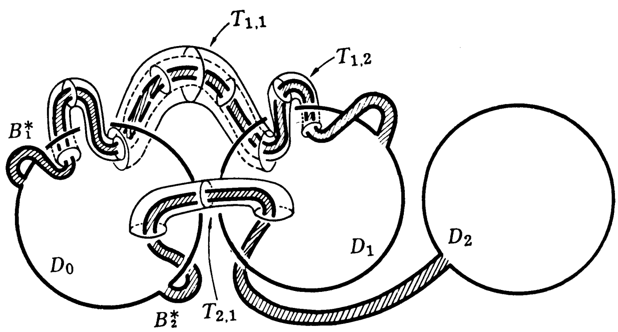


Fig. (3)

Third-step. To construct the 3-manifold W^3 bounded by K', we make use of the method described schematically in Fig. (4), which was already used in the proof of the theorem in [4], p. 267~269, Fig. 5.

$W_+=\bigcup_{0\leq t}F_t^2$ and $W_-=\bigcup_{t\leq 0}F_t^2$ are both homeomorphic to a solid torus perhaps with a large genus, therefore W^3 gained by the natural identification on $F_0^2=W_+\cap R_0^3=W_-\cap R_0^3$ is homeomorphic either to a 3-ball or to $\#(S^1\times S^2)-\mathring{\Delta}^3$, since K' is symmetric with respect to R_0^3.

This completes the proof.

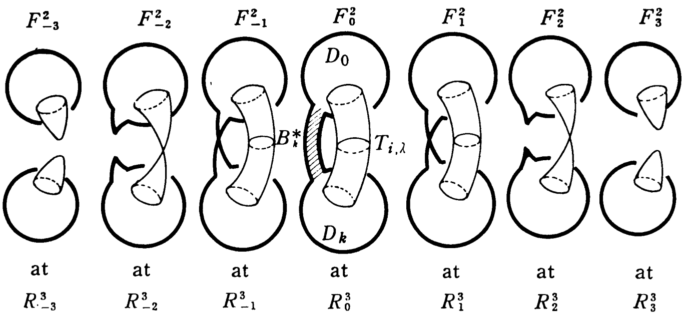


Fig. (4)

3. A fusion of 2-knots[6)]

A collection of the mutually disjoint 2-knots $\{K^2_1, K^2_2, \cdots, K^2_n\}$ is called *a splitted 2-link*, if there exists a collection of the mutually disjoint combinatorial 4-balls $V_1, V_2, \cdots, V_n$ such that $V_i \supset K_i$ $(i=1, 2, \cdots, n)$ in R^4. Especially, a splitted 2-link is called *a trivial 2-link*, if each component K_i is unknotted[7)] in R^4 $(i=1, 2, \cdots, n)$.

DEFINITION (3.1). If there are a collection of the 3-balls $B_1, B_2, \cdots, B_{n-1}$ and a splitted 2-link $\{K^2_1, K^2_2, \cdots, K^2_n\}$ such that, for each B_i, $B_i \cap K_j = \partial B_i \cap K_j$ is a 2-ball $E_{i,j}$ for just two 2-knots K_j of the 2-link $(i=1, 2, \cdots, n-1,\ 1 \leqq j \leqq n)$, and that the 2-sphere $K=(\overline{\bigcup_j K_j - \bigcup_{i,j} E_{i,j}}) \cup (\overline{\bigcup_i \partial B_i - \bigcup_{i,j} E_{i,j}})$ is a 2-knot in R^4, then the 2-knot K is called *a fusion of the splitted 2-link*.

Lemma (3.2). *If a 2-knot K^2 is a fusion of the splitted 2-link $\{S^2_1, S^2_2, \cdots, S^2_n\}$, then $c(\{K\}) \leqq \sum_{j=1}^{n} c(\{S_j\})$.*

Proof. Since the 2-link is splitted, there is an ambient isotopy ξ of R^4 under which the pair $(p, \xi(S_j))$ is a simple pair for the knot-type $\{S_j\}$ by a projection p for all $\{S_j\}$ $(j=1, 2, \cdots, n)$, and moreover $p(\xi(S_j)) \cap p(\xi(S_k)) = \phi$ $(j \neq k)$. For convenience' sake, we denote $\xi(S_j)$, $\xi(B_i)$, $\xi(E_{i,j})$ by S_j, B_i, $E_{i,j}$, again, where the 3-balls B_i $(i=1, 2, \cdots, n-1)$ belong to the collection of the 3-balls in the construction of the fusion K.

Let the 2-balls $E_{i,j} = B_i \cap S_j$ and $E_{i,k} = B_i \cap S_k$ be the intersection of ∂B_i and $\bigcup_i S_j$ and let α_i be the arc in B_i spanning $E_{i,j}$ and $E_{i,k}$, where $\alpha_i \cap \partial B_i = \alpha_i \cap (E_{i,j} \cup E_{i,k}) = \partial \alpha_i$ and α_i is unknotted[8)] in B_i $(i=1, 2, \cdots, n-1)$. We may

6) The concept "fusion" is introduced in [6], p. 364 for 1-knots, but now we will consider an analogy of this concept for 2-knots.

7) K_i bounds a combinatorial 3-ball in R^4.

8) A circle $\alpha_i \cup \alpha$ bounds a 2-ball in B^3_i for an arc α on ∂B_i.

suppose in addition that $p(\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_{n-1})$ contains no double points and each $p(\alpha_i)$ pierces the singular 2–spheres $p(S_j)$ at most at its non-singular points.[9]

Let U_i^3 be a sufficiently fine tubular neighborhood of α_i in B_i, where $U_i^3 \cap \partial B_i = U_1^3 \cap (E_{i,j} \cup E_{i,k})$ are two 2–balls, see Fig. (5).

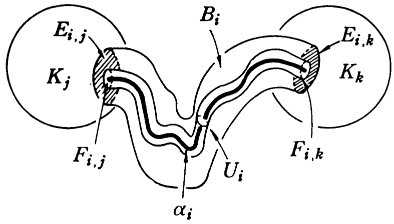


Fig. (5)

According to the properties of the arcs α_i and the tubes U_i^3, the 2–sphere $K' = (\overline{\bigcup_j S_j - \bigcup_{i,j} F_{i,j}}) \cup (\overline{\bigcup_i \partial U_i - \bigcup_{i,j} F_{i,j}})$ is isotopic to $\xi(K) = (\overline{\bigcup_j S_j - \bigcup_{i,j} E_{i,j}}) \cup (\overline{\bigcup_i \partial B_i - \bigcup_{i,j} E_{i,j}})$, where $F_{i,j} = U_i^3 \cap E_{i,j} = U_i^3 \cap \partial B_i$, and under the projection p, $c(K') = \sum_{j=1}^{n} c(S_j)$. Since the pairs (p, S_j) are simple pairs for the knot-type $\{S_j\}$ $(j=1, 2, \cdots, n)$ and $K' \in \{K\}$, we have

$$c(\{K\}) \leqq c(K') = \sum_{j=1}^{n} c(S_j) = \sum_{j=1}^{n} c(\{S_j\})\,.$$

Corollary (3.3). *If a 2–knot K^2 is a fusion of a trivial 2–link, then* $c(\{K\}) = 0$.

Lemma (3.4). *Let a 2–knot K^2 bound a 3–manifold W^3 in R^4 such that W^3 is homeomorphic to a 3–ball or to $\#(S^1 \times S^2) - \mathring{\Delta}^3$, and that, if W^3 is not a 3–ball, W^3 has a trivial system of 2–spheres*[10] *which will be difined as below. Then, K^2 is a fusion of a trivial 2–link in R^4.*

DEFINITION (3.5). A collection of the 2–spheres $S_1^2, S_2^2, \cdots, S_{2n}^2$ in a 3–manifold W^3 in R^4 which is homeomorphic to $\#(S^1 \times S^2) - \mathring{\Delta}^3$, is called *a trivial system of 2–spheres in W^3* if it satisfies the following (1), (2) and (3):

(1) *the collection $\{S_1^2, S_2^2, \cdots, S_{2n}^2\}$ is a trivial 2–link in R^4,*
(2) *$S_i^2 \cup S_{n+i}^2$ bounds a spherical-shell N_i*[11] *in W^3 and $N_i \cap N_j = \phi$ for $i \neq j$, $i, j = 1, 2, \cdots, n$.*

9) At the non-multiple points.
10) The terminology "a trivial system" is due to R.H. Fox in his paper "Ribbon and Slice" (to appear).
11) A combinatorially imbedded $S^2 \times [0, 1]$.

(3) *$W^3-\bigcup_{i=1}^{n}\mathring{N}_i$ is the closure of a combinatorial 3-sphere removed of the mutually disjoint $2n+1$ combinatorial 3-balls.*

Proof of (3.4). Let $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$ be the mutually disjoint arcs in $Q=W-\bigcup_{i=1}^{n}\mathring{N}_i$ such that α_λ spans S_λ and S_0 ($\lambda=1, 2, \cdots, 2n$), where S_0 is a 2-sphere in Q which bounds a combinatorial 3-ball B_0^3 in $\mathring{Q}$, and that for sufficiently fine tubular neighborhoods U_λ of α_λ in $\mathring{Q}$, $\overline{Q-U_1\cup U_2\cup\cdots\cup U_{2n}\cup B_0^3}$ is a combinatorial spherical-shell. Then, a boundary 2-sphere K' of $\overline{Q-U_1\cup U_2\cup\cdots\cup U_{2n}\cup B_0}$ belongs to the knot-type $\{K\}$. Moreover, it is clear that the 2-knot K' is a fusion of a trivial 2-link $\{S_0^2, S_1^2, \cdots, S_{2n}^2\}$ in R^4.

Theorem (3.6). *A 2-knot K^2 bounds a 3-manifold W^3 in R^4 which is homeomorphic to a 3-ball or to $\#(S^1\times S^2)-\mathring{\Delta}^3$ and has a trivial system of 2-spheres*[12]*, if and only if K^2 is a ribbon 2-knot.*

Proof. Remembering the third-step of the construction in the proof of (2, 3), we have easily the imbeddings of the spherical-shell N_i in W^3, see Fig. (6). Thus, we have that if K is a ribbon 2-knot, K bounds the desired 3-manifold. The converse follows from (3.4) and (3.3).

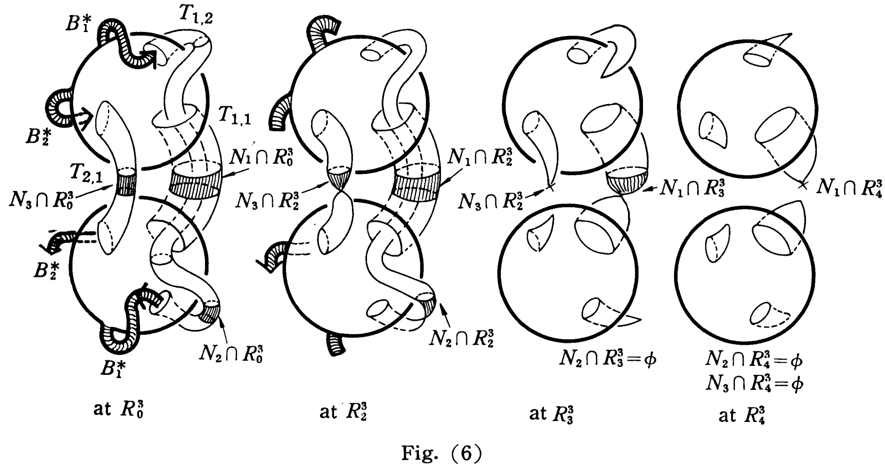


Fig. (6)

4. Equivalence of the definitions

We introduce the following properties.

***R*(3):** A 2-knot K^2 satisfies that $c(\{K\})=0$.

***R*(4):** A 2-knot K^2 bounds a 3-*ribbon* in R^4.

12) If W^3 is a 3-ball, we consider that the system is empty.

$\boldsymbol{R}(5)$: A 2–knot K^2 bounds a *monotone* 3–*ball* in H^5_+.

DEFINITION (4.1). An image of a 3–ball B^3 into R^4 by an immersion φ will be called a 3–*ribbon* bounded by a 2–knot K^2, if it satisfies the following (1), (2) and (3):

(1) $\varphi|\partial B$ is an imbedding and $\varphi(\partial B)=K^2$,

(2) the self-intersection of $\varphi(B)$ consists of a finite number of the mutually disjoint 2–balls $D_1, D_2, \cdots, D_n$,

(3) for each D_i, the inverse image $\varphi^{-1}(D_i)$ consists of a pair of 2–balls D'_i, D''_i, satisfying that

$$D'_i \cap D''_i = \phi,\ D'_i \subset \mathring{B},\ \partial D''_i = D''_i \cap \partial B \quad (i=1, 2, \cdots, n).$$

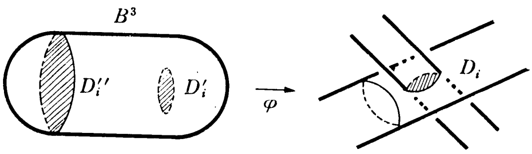


Fig. (7)

DEFINITION (4.2). A 3–ball D^3 will be called *a monotone* 3–*ball* bounded by a 2–knot K^2 in H^5_+, if it satisfies the following (1), (2) and (3):

(1) $K^2=\partial D=D\cap R^4_0$,

(2) D^3 is locally flat and has no minimal point in H^5_+,

(3) in a neighborhood of each (non-maximal) critical point $p_i(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)}, x_5^{(i)})$, D^3 is represented by the equation:

$$\begin{cases} (x_1-x_1^{(i)})^2+(x_2-x_2^{(i)})^2-(x_4-x_4^{(i)})^2 = x_5^{(i)}-x_5 \\ x_3-x_3^{(i)} = 0 \end{cases}$$

For convenience' sake, we will say that

$\boldsymbol{F}$: A 2–knot K^2 is a fusion of a trivial 2–link.

Lemma (4.3). $\boldsymbol{R}(4)$ *is equivalent to* $\boldsymbol{F}$.

Proof. $\boldsymbol{R}(4) \Rightarrow \boldsymbol{F}$. Let V^3_i be 3–balls in B^3 such that $V_i \supset D''_i$, $V_i \cap V_j=\phi$ $(i\neq j)$ and that the annulus $V_i \cap \partial B$ contains $\partial D''_i$ in its interior $(i, j=1, 2, \cdots, n)$. Let $P^2_1, P^2_2, \cdots, P^2_{n+1}$ be the boundary 2–spheres of $\overline{B^3-V_1\cup\cdots\cup V_n}$, then $\{\varphi(P^2_1), \varphi(P^2_2), \cdots, \varphi(P^2_{n+1})\}$ is a trivial 2–link in R^4, and it is clear that K^2 is a fusion of the trivial 2–link.

$\boldsymbol{F} \Rightarrow \boldsymbol{R}(4)$. Remembering the technique in the proof of (3.2), we have a 3–ribbon $J^3_1\cup\cdots\cup J^3_n\cup U^3_1\cup\cdots\cup U^3_{n-1}$ bounded by K^2 in R^4, where $J^3_1, \cdots, J^3_n$ are disjoint 3–balls bounded by the 2–knots $S_1, \cdots, S_n$ respectively and 3–balls

$U_1^3, \cdots, U_{n-1}^3$ are so fine that $U_i^3 \cap J_j^3$ are small 2-balls in J_j^3 ($1 \leqq i \leqq n-1$, $1 \leqq j \leqq n$).

Lemma (4.4). *$\boldsymbol{R}(5)$ is equivalent to $\boldsymbol{F}$.*

Proof. $\boldsymbol{R}(5) \Rightarrow \boldsymbol{F}$. Let D^3 be a monotone 3-ball bounded by K^2 in H_+^5. We may suppose that the coordinates of all (non-maximal) critical points of D^3 satisfy that $x_5^{(i)}=1$ ($i=1, 2, \cdots, n-1$). Then, by the property (3) in (4.2), it is not so difficult to prove the followings: for a sufficiently small positive number ε,

(1) $D^3 \cap R_{1+\varepsilon}^4$ is a trivial 2-link $\{S_1^2, S_2^2, \cdots, S_n^2\}$ in $R_{1+\varepsilon}^4$,

(2) the equations

$$\begin{cases} (x_1-x_1^{(i)})^2+(x_2-x_2^{(i)})^2-(x_4-x_4^{(i)})^2 \leqq \varepsilon \\ x_3-x_3^{(i)}=0\,, \quad x_5=1-\varepsilon \\ |x_4-x_4^{(i)}| \leqq \sqrt{\varepsilon} \qquad (i=1, 2, \cdots, n-1) \end{cases}$$

give the disjoint 3-balls B_i^3 in $R_{1-\varepsilon}^4$,

(3) $D^3 \cap R_{1-\varepsilon}^4$ is a fusion of a trivial 2-link $\{S_1', S_2', \cdots, S_n'\}$ by the 3-balls $B_1^3, \cdots, B_{n-1}^3$, where trivial 2-knots S_i' are the image of S_i^2 by the orthogonal projection of R^5 onto $R_{1-\varepsilon}^4$.

(4) $D^3 \cap R_{1-\varepsilon}^4$ and K^2 belong to the same knot-type.

$\boldsymbol{F} \Rightarrow \boldsymbol{R}(5)$. By the result (4.3), if K^2 satisfies $\boldsymbol{F}$, K^2 satisfies $\boldsymbol{R}(4)$; that is, K^2 bounds a 3-ribbon $\varphi(B)$. Let V_i^3 be 3-balls such that $\mathring{B} \supset V_i^3 \supset \mathring{V}_i^3 \supset D_i'$ and $V_i^3 \cap V_j^3 = \phi$ ($i \neq j$, $i=1, 2, \cdots, n$). Since φ imbeds $\overline{B^3 - V_1^3 \cup \cdots \cup V_n^3}$ into R_0^4 and $\{\varphi(\partial V_1^3), \cdots, \varphi(\partial V_n^3)\}$ is a trivial 2-link in R_0^4, we can suspend these 2-spheres $\varphi(\partial V_1^3), \cdots, \varphi(\partial V_n^3)$ from n points of R_1^4. With a little modification, we have a monotone 3-ball D^3 bounded by K^2 in H_+^5.

Remembering (3.3) and (3.4), we have that $\boldsymbol{R}(3) \Leftrightarrow \boldsymbol{F}$, and with (4.3) and (4.4), finally we have the following

Theorem (4.5). *$\boldsymbol{R}(3)$, $\boldsymbol{R}(4)$ and $\boldsymbol{R}(5)$ are equivalent to $\boldsymbol{F}$.*

We refer to the following results.

All 2-knots are "slice-knot", see [7].

There is a 2-knot which is not "simply-knotted 2-knot", see [2].

Then, we can assert that *the concept "ribbon knot" is different from the concept "slice knot" for 2-knots*, while we have not yet succeeded to distinguish one from another for 1-knots.

5. An equivalence relation

Let K^2 be a ribbon 2-knot which is knotted in R^4. Then, by (3.6) and (3.4), K^2 is a fusion of a trivial 2-link $\{S_0^2, S_1^2, \cdots, S_{2n}^2\}$ in R^4. In the following, we will use the same notations as in the proof of (3.4). We may suppose that

- $\boldsymbol{P}_1$: A 2-sphere S_0^2 and the spherical-shells $N_1, N_2, \cdots, N_n$ are splitted in R^4, where $\partial N_i = S_i^2 \cup S_{n+i}^2$ $(i=1, 2, \cdots, n)$.
- $\boldsymbol{P}_2$: $S_\lambda^2 \cap H^4[-1, 1] = (S_\lambda^2 \cap R_0^3) \times [-1, 1]$ $(\lambda=0, 1, \cdots, 2n)$.
- $\boldsymbol{P}_3$: $N_1 \cap R_0^3, \cdots, N_n \cap R_0^3$ are annuli on a plane in R_0^3.

The 3-balls $B_1^3, \cdots, B_{2n}^3$ and the arcs $\alpha_1, \cdots, \alpha_{2n}$ which cause the fusion have the following properties $\boldsymbol{P}_4$, $\boldsymbol{P}_5$ and $\boldsymbol{P}_6$:

- $\boldsymbol{P}_4$: Since the 2-link $\{S_0^2, S_1^2, \cdots, S_{2n}^2\}$ is trivial, the arcs $\alpha_1, \cdots, \alpha_{2n}$ are moved into R_0^3 by an ambient isotopy of R^4. Since $\mathring{\alpha}_\lambda$ is contained in B_λ^3 $(\lambda=1, 2, \cdots, 2n)$, if we suppose that a finite subcomplex B_λ^3 of R^4 is in a general position with respect to the hyperplane R_0^3, there is a sufficiently narrow band b_λ^2 containing α_λ in $B_\lambda^3 \cap R_0^3$ which spans a circle $c_0 = S_0^2 \cap R_0^3$ and a circle $c_\lambda = S_\lambda^2 \cap R_0^3$ $(\lambda=1, 2, \cdots, 2n)$.
- $\boldsymbol{P}_5$: For a sufficiently small positive number ε, $B_\lambda^3 \cap H^4[-\varepsilon, \varepsilon]$ contains a 3-ball U_λ^3 which is level-preserving-isotopic to $b_\lambda^2 \times [-\varepsilon, \varepsilon]$ leaving the 2-balls $U_\lambda^3 \cap S_0^2$ and $U_\lambda^3 \cap S_\lambda^2$ fixed $(\lambda=1, 2, \cdots, 2n)$.
- $\boldsymbol{P}_6$: In fusing $\{S_0^2, S_1^2, \cdots, S_{2n}^2\}$ to get a 2-knot K^2, we make use of the 3-balls $U_1^3, \cdots, U_{2n}^3$ instead of the 3-balls $B_1^3, \cdots, B_{2n}^3$, and we will denote the new 2-knot belonging to $\{K^2\}$ by $\tilde{K}^2$.

We want to simplify the cross-sections of the 2-knot $\tilde{K}^2$ as follows.

Let θ be an orthogonal projection of R_0^3 onto a plane R^2, and if $\theta(b_\lambda^2) \cap \theta(b_\mu^2) \neq \phi$ $(1 \leqq \lambda, \mu \leqq 2n)$, we may suppose that U_λ^3 and U_μ^3 are in the position as shown in (8_1) in Fig. (8). Move U_λ^3 and U_μ^3 isotopically in R^4 so as to be in the position in (8_2). In the next step, lift up the tube in the level R_ε^3 as shown in (8_3). Replace these in a general position again, and we have the situation in (8_4). Thus, we have the following lemma.

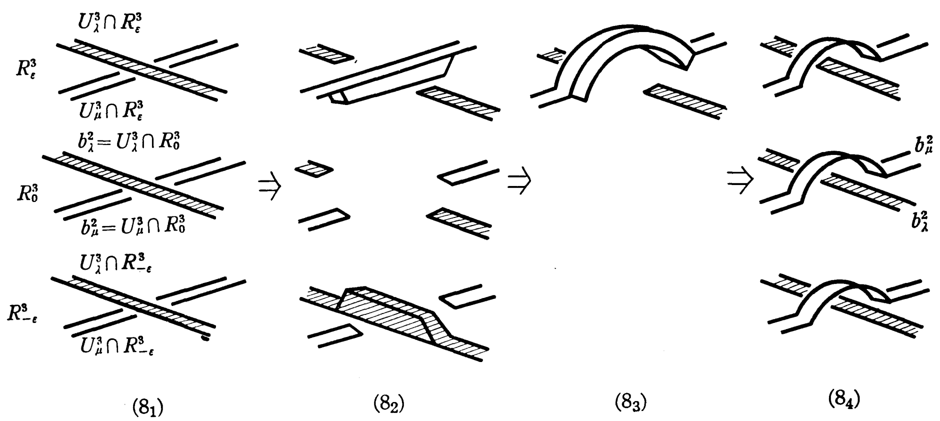


Fig. (8)

Lemma (5.1). *We can exchange the over-and-under passing relation with respect to x_3-coordinate between b^2_λ and b^2_μ $(1\leqq\lambda,\ \mu\leqq 2n)$ preserving the 2–knot type of $\tilde{K}^2$.*

We will consider how to eliminate the twists of the band b^2_λ in the following three steps.

(1) If b^2_λ contains an even number of twists, we perform a modification as follows:

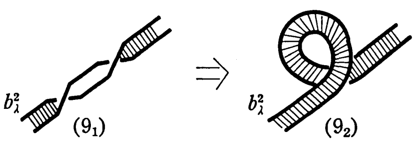


Fig. (9)

This modification is an isotopy only in the subspace R^3_0 in R^4, but in each level R^3_t $(-\varepsilon\leqq t\leqq\varepsilon)$, the similar modification can be performed for $U^3_\lambda\cap R^3_t$, therefore we can understand this modification as an isotopy of R^4.

(2) If b^2_λ contains only one twist, we consider an orientable 2–surface $F=\{(B^3_0\cup N_1\cup\cdots\cup N_n)\cap R^3_0\}\cup b^2_1\cup\cdots\cup b^2_{2n}$. Since the fusion in the present step depends on the 3–manifold W^3, and since $F\cup(\tilde{K}^2\cap H^4_+)$ bounds an orientable 3–manifold $(B^3_0\cup N_1\cup\cdots\cup N_n\cup U^3_1\cup\cdots\cup U^3_{2n})\cap H^4_+$ (a solid torus with a large genus), the surface F should be orientable, see Satz I, §64 in [9]. Therefore, there must be another twist on a band b^2_μ $(\mu\sim\lambda=n)$. Hence, we consider the following replacement of F in R^3_0, see Fig. (10).

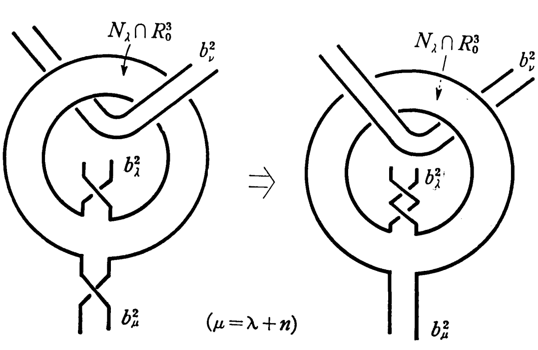


Fig. (10)

After this replacement, the twist on b^2_μ is transferred on b^2_λ. This move can be easily extended to a modification of W^3 in R^4, and the band b^2_ν (, more precisely the tube U^3_ν) is left fixed through the modification, even if b^2_ν links with $N_\lambda\cap R^3_0$ (or $N_\mu\cap R^3_0$) in R^3_0 as shown in Fig. (10).

(3) After the modifications in (1) and (2), each band b^2_λ contains a finite

number of cirri as in (9_2). If we exchange the over-and-under passings of b^2_λ itself by (5.1), we may suppose that the band b^2_λ contains just one cirrus. Here, as in the proof of (5.1), we can pull down U^3_λ onto $R^3_{-\varepsilon}$ isotopically in R^4 as shown in (11_1) and (11_2) in Fig. (11). In $R^3_{-\varepsilon}$, we stretch this solid cylinder and pull up again so that each cross-section contains no cirrus as shown in (11_3) and (11_4).

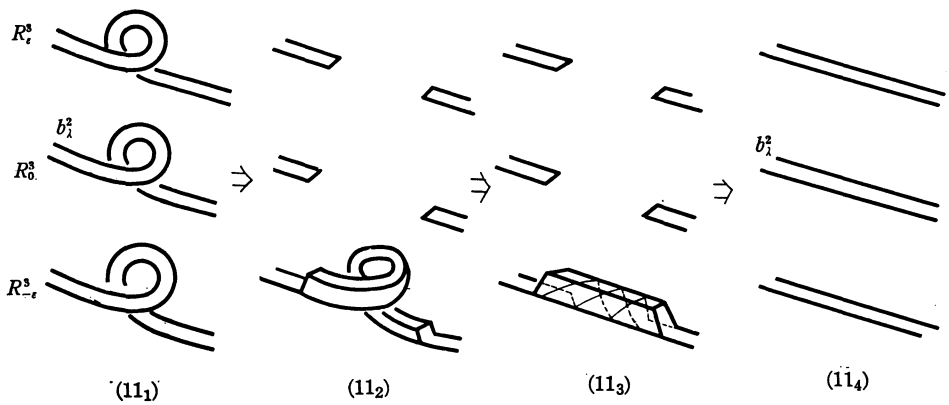


Fig. (11)

Hence, we have

Lemma (5.2). *We can cancel the twists and the cirri of* b^2_λ $(1 \leqq \lambda \leqq 2n)$ *preserving the 2-knot type of* $\tilde{K}^2$.

Theorem (5.3). *Let* K^2 *be a ribbon-2-knot, then there is a 3-manifold* W^3 *and* $\tilde{K}^2$ *in* R^4, *which belongs to* $\{K^2\}$, *satisfying the following* (1), (2), (3) *and* (4):

(1) $\partial W^3 = \tilde{K}^2$, *and* W^3 *is symmetric with respect to* R^3_0

(2) $W^3 \approx B^3$ *or* $W^3 \approx \#(S^1 \times S^2) - \mathring{\Delta}^3$,[13]

and moreover, if $W^3 \not\approx B^3$,

(3) W^3 *has a trivial system of 2-spheres,*

(4) $W^3 \cap R^3_0$ *is an orientable surface F, which has a basis* $\{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n\}$ *of* $H_1(F)$ *such that* both $\alpha_1 \cup \cdots \cup \alpha_n \cup \alpha'_1 \cup \cdots \cup \alpha'_n$ and $\beta_1 \cup \cdots \cup \beta_n \cup \beta'_1 \cup \cdots \cup \beta'_n$ *are trivial links in* R^3_0, *where both* $\alpha_i \cup \alpha'_i$ *and* $\beta_i \cup \beta'_i$ *bound annuli on* F $(i=1, 2, \cdots, n)$.

Corollary (5.4). *Let* K^2 *be a ribbon 2-knot, there is a ribbon 2-knot* $\tilde{K}^2$ *belonging to* $\{K^2\}$ *satisfying the following* (1) *and* (2)

(1) $\tilde{K}^2$ *is symmetric with respect to* R^3_0,

13) $\approx$ means *to be homemorphic to.*

(2) *there is a locally flat 2-ball* $\tilde{D}^2$ *in* H^4_- *such that a 2-knot* $(\tilde{K}^2 \cap H^4_+) \cup \tilde{D}^2$ *is an unknotted 2-knot in* R^4.

Proof of (5.3). If K^2 is unknotted in R^4, the theorem is trivial. Hence we will consider the case that K^2 is knotted in R^4. Let $\tilde{K}^2$ be a 2-knot satisfying $\boldsymbol{P}_1, \cdots, \boldsymbol{P}_6$ described in the beginning of this section. Let W^3 be a 3-manifold $B^3_0 \cup N_1 \cup \cdots \cup N_n \cup U^3_1 \cup \cdots \cup U^3_{2n}$. Then, $W^3 \cap R^3_0 = \{(B^3_0 \cup N_1 \cup \cdots \cup N_n) \cap R^3_0\} \cup b^2_1 \cup \cdots \cup b^2_{2n}$. Let α_1, α'_i, β_i and β'_i $(i=1, 2, \cdots, n)$ be the simple closed curves on F described in Fig. (12), then by (5.1) and (5.2) they satisfy the conditions (4) in (5.3).

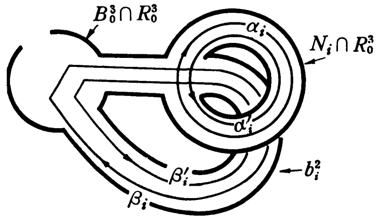


Fig. (12)

Proof of (5.4). If K^2 is unknotted in R^4, it is obvious. If K^2 is knotted in R^4, we consider the 3-manifold W^3 and $\tilde{K}^2$ in R^4 in (5.3). Then, the construction of the 2-ball $\tilde{D}^2$ is described in Fig. (13). Moreover if we apply the method in the proof of Theorem in [4], see Fig. 5, p. 269 in [4], it is not so difficult to construct a 3-ball bounded by $(\tilde{K}^2 \cap H^4_+) \cup \tilde{D}^2$ in R^4.

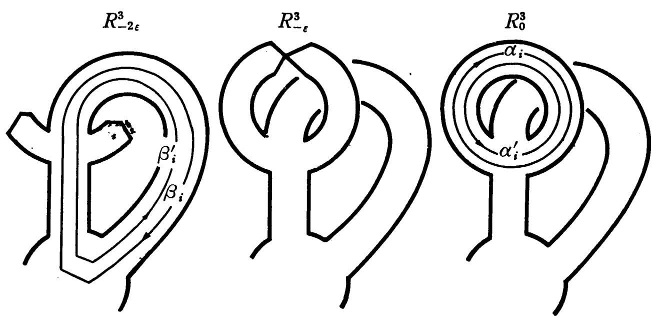


Fig. (13)

Now, we will define an equivalence relation between 2-knots.

DEFINITION (5.5). *Two 2-knots* K^2_0 *and* K^2_1 *will be called cobordant and denoted by* $K^2_0 \sim K^2_1$, *if and only if there exists a 3-manifold* M^3 *satisfying the following* (1), (2), (3) *and* (4):

(1) M^3 *is homeomorphic to* $S^2 \times [0, 1]$,

(2) *M^3 is locally flat in $H^5[0, 1]$,*[14)]

(3) $\partial M^3=K_0^2\cup(-K_1^2)$, *and* $K_i^2=M^3\cap R_i^4$ $(i=0, 1)$,[15)]

(4) $M^3\cap R_t^4$ *is connected for each* t $(0\leqq t\leqq 1)$.

Clearly we have

Theorem (5.6). *The cobordant relation "$\sim$" is an equivalence relation.*

Lemma (5.7). *If a 2-knot K^2 is a ribbon 2-knot, then $K^2\sim 0^2$, where 0^2 is a trivial 2-knot in R^4.*

Proof. Let X^3 be a compact, orientable 3-manifold in R^5. The ordinary cross-section of X^3 by a hyperplane R_t^4 is a compact, orientable 2-manifold. If X^3 is represented by the next equation (*) in a neighborhood of a point $p(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \alpha)$:

$$(*)\qquad \begin{cases} (x_1-\bar{x}_1)^2-(x_2-\bar{x}_2)^2+(x_4-\bar{x}_4)^2 = x_5-\alpha \\ x_3-\bar{x}_3 = 0 \end{cases},$$

the transformation from the ordinary cross-section $X^3\cap R_{\alpha-\varepsilon}^4$ onto the ordinary cross-section $X^3\cap R_{\alpha-\varepsilon}^4$ (for a small number $\varepsilon>0$) is a hyperbolic transformation in R^5.

In the following, we want to construct a 3-manifold M^3 which satisfies not only the conditions (1), (2), (3) and (4) in (5.5) but $M^3\cap R_0^4=K^2$, $M^3\cap R_1^4=0^2$. If K^2 is unknotted in R_0^4, the existence of the 3-manifold is clear, therefore we will suppose that K^2 is knotted in R_0^4. The 3-manifold will be obtained by the following six steps.

(1) Consider a 2-knot $\tilde{K}^2$ belonging to $\{K^2\}$ and bounding the 3-manifold W^3 in (5.3), see (14_1) in Fig. (14).

(2) Between R_0^4 and $R_{1/2}^4$, we perform the hyperbolic transformations as shown schematically in (14_1), (14_2) and (14_3). In (14_2), we show the exceptional cross-section of M^3 by $R_{1/4}^4$ and the cross-section by R_0^3 in $R_{1/4}^4$ is similar to that by $R_{-\varepsilon}^3$ in Fig. (13). The cross-section by $R_{\pm\varepsilon}^3$ in $R_{1/2}^4$ is similar to that by $R_{-\varepsilon}^3$ in Fig. (13), and the cross-section by R_0^3 in $R_{1/2}^4$ is similar to that by $R_{-2\varepsilon}^3$ in Fig. (13). This transformation satisfies the equation (*) in a sufficiently small neighborhood in R^5 of each saddle point at R_0^3 in $R_{1/4}^4$.

---

14) $R_t^4 = \{(x_1, x_2, x_3, x_4, x_5)\,|\,x_5 = t\}$
$H_+^5 = \{(x_1, x_2, x_3, x_4, x_5)\,|\,x_5 \geqq 0\}$
$H^5(J) = \{(x_1, x_2, x_3, x_4, x_5)\,|\,x_5 \in J\}$

The 3-manifold M^3 is *locally flat* in $H^5[0, 1]$ if the pair $(Lk(p, M^3), Lk(p, H^5[0, 1])$ is a trivial sphere pair for $p\in M^3$ and a trivial ball pair for $p\in\partial M^3$.

15) Identify 2-knot (K_i^2, R^4) with a 2-knot (K_i^2, R_i^4). $(-K^2)$ is the reversely-oriented 2-knot for K^2.

(3) As shown in the proof of (5.3), the cross-section of the 2-surface by R^3_0 in $R^4_{1/2}$ is a trivial 1-link, say $\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_n$ with $n+1$ components.

(4) Between $R^4_{1/2}$ and $R^4_{3/4}$, we will contract n circles $\gamma_1, \cdots, \gamma_n$ to points continuously as shown in (14_3) and (14_4) so that in a small neighborhood in R^5 of each pinching point at R^3_0 in $R^4_{3/4}$, the transformation is given by the equation $(**)$:

$$(**) \qquad \begin{cases} (x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2-(x_4-\bar{x}_4)^2 = \bar{x}_5-x_5 \\ x_3-\bar{x}_3 = 0 \\ \bar{x}_5 = 3/4 \quad . \end{cases}$$

(5) We have finally constructed a 2-knot $\hat{K}^2$ in R^4_1 which satisfies the following three properties:

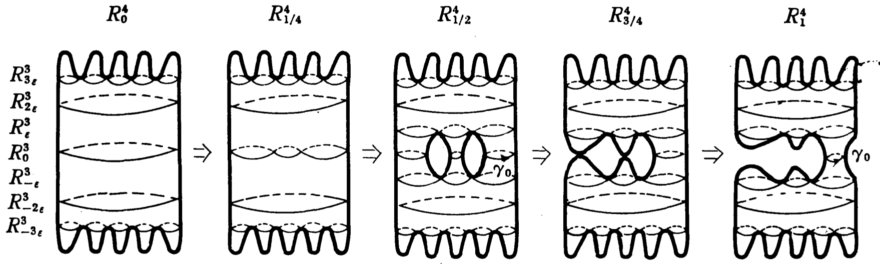
(5_1) $\hat{K}^2$ is symmetric with respect to R^3_0 in R^4_1,

(5_2) $\hat{K}^2 \cap R^3_0$ is a trivial 1-knot γ_0 in R^3_0 in R^4_1,

(5_3) If we remind the proof of (5.4), a 2-knot K^2_1 *is unknotted in* H^4_+, where K^2_1 is a union of a 2-ball $\hat{K}^2 \cap H^4_+$ and a 2-ball D^2 which is bounded by γ_0 in R^3_0.

(6) Then, $\tilde{K}^2 = K^2_1 * (-K^2_1)$[16] is a trivial 2-knot in R^4_1. By the same method as in the proof of theorem in [4], we can construct the desirable 3-manifold M^3 in $H^5[0, 1]$ which is bounded by $\tilde{K}^2$ and the trivial 2-knot $\hat{K}^2$.

This completes the proof of (5.7).



(14_1) (14_2) (14_3) (14_4) (14_5)

Fig. (14)

Lemma (5.8). *For an arbitrary 2-knot* K^2, $K^2 * (-K^2)$ *is cobordant to a ribbon 2-knot.*

Proof. A 2-knot K^2 in R^4_0 can be placed in a position as follows:

(16) $*$ means the *knot-product*; that is, $K^2 = K^2_1 * K^2_2$ if there exists a hyperplane P^3 in R^4 such that $k = K^2 \cap P^3$ is a 1-knot bounding a 2-ball D^2 in P^3 and that a 2-knot $D^2 \cup (P^4_+ \cap K^2)$ belongs to $\{K^2_1\}$ and a 2-knot $D^2 \cup (P^4_- \cap K^2)$ to $\{K^2_2\}$, where $P^4_\pm$ are half 4-spaces bounded by P^3 in R^4. Cf. the argument in §1 in [10].

(1) $K^2 \cap R^3_{3\varepsilon}$ is a knot k in $R^3_{3\varepsilon}$,

(2) $K^2 \cap H^4[3\varepsilon, \infty)$ has no minimal point,

(3) $K^2 \cap H^4[\varepsilon, 3\varepsilon]$ has no maximal point,

(4) all minimal points are at the level R^3_ε.

Place a 2–knot $(-K^2)$ in the symmetric position to K^2 with respect to R^3_0, and product them as shown in (15_1) in Fig. (15). Then, the process from (15_1) to (15_5) follows almost the opposite course of the process from (14_1) to (14_5) in the proof of (5.7). The cross-section by R^3_0 in $R^4_{3/4}$ is the same as that by $R^3_{2\varepsilon}$ in R^4_0. In the final stage (15_5), we have a 2–knot K^2_1 in R^4_1 which satisfies the followings:

(1) K^2_1 is symmetric with respect to R^3_0 in R^4_1,

(2) $K^2_1 \cap H^4_+$ contains no minimal point.

Then, the 2–knot K^2_1 is a ribbon 2–knot, see [2], [3].

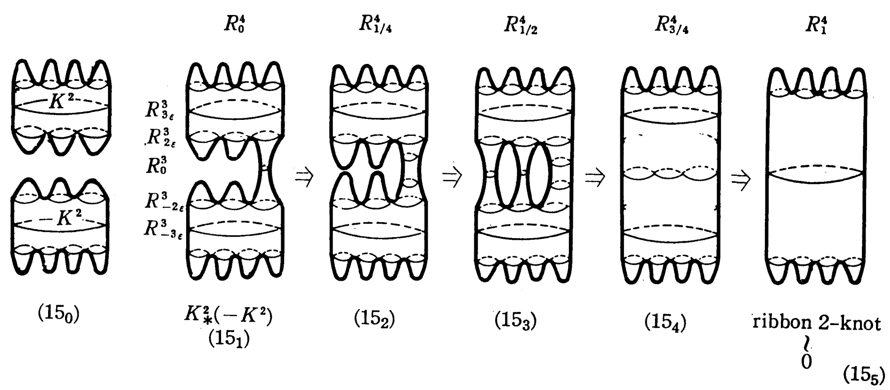


Fig. 15

Concerning the knot-product "$*$", we have

$$K^2_1 * K^2_2 = K^2_2 * K^2_1\,, \quad \text{and}$$
$$K^2_1 * (K^2_2 * K^2_3) = (K^2_1 * K^2_2) * K^2_3\,, \quad \text{see Theorem 1 in [10].}$$

Lemma (5.9). *If $K^2_0 \sim K^2_1$ and $L^2_0 \sim L^2_1$ for 2–knots K^2_0, K^2_1, L^2_0 and L^2_1, then $K^2_0 * L^2_0 \sim K^2_1 * L^2_1$.*

Proof. There exist 3–manifolds M^3_1 and M^3_2 which satisfy the following (1), (2), (3) and (4):

(1) M^3_i is homeomorphic to $S^2 \times [0, 1]$ $(i=0, 1)$,

(2) M^3_i is locally flat in $H^5[0, 1]$,

(3) $\partial M^3_1 = K^2_0 \cup (-K^2_1)$, $\partial M^3_2 = L^2_0 \cup (-L^2_1)$,

$K_j^2 = M_1^3 \cap R_j^4$ and $L_j^2 = M_2^3 \cap R_j^4$ $(j=0, 1)$,

(4) M_1^3 and M_2^3 are splitted by an hyperplane Y^4 in R^5 which is orthogonal to the hyperplane R_t^4 $(0 \leqq t \leqq 1)$.

Then, it is not difficult to see that $K_0^2 * L_0^2 \sim \hat{K}_1^2$, where $\hat{K}_1^3$ is *a fusion of the* 2*-knots* K_1^2 *and* L_1^2 *in* R_1^4 *by a sufficiently fine tube* U^3 for which $U^3 \cap (Y^4 \cap R_1^4)$ is a 2-ball D^2. Since K_1^2 and L_1^2 are splitted by a hyperplane $Y^4 \cap R_1^4$ in R_1^4, the fusion in the present step is surely the product; that is, $\hat{K}_1^2 = K_1^2 * L_1^2$. This completes the proof.

As the consequence of (5.9), the set $\mathfrak{G}$=(all 2-knots)/$\sim$ has an abelian semi-group structure, where the group operation is inherited from the knot-product operation $*$ of 2-knots. Since we can find the inverse element for each element of the semigroup $\mathfrak{G}$ by (5.8), we have the final theorem in this paper:

Theorem (5.10). $\mathfrak{G}$ *is an abelian group.*

In comerison with the result by M. A. Kervaire in [7], we must have a question: Does there exist a 2-knot non-cobordant to 0^2 in the present sense? Nevertheless, it is true that if a 2-knot K^2 is cobordant to 0^2, then there exists a locally flat 3-ball B^3 in H_+^5 satisfying the following (1) and (2):

(1) $B^3 \cap R_0^4 = \partial B^3 = K^2$,

(2) B^3 has only one maximal point but no minimal point.

Therefore, if we conjecture that "*the method of the calculation of* $\pi_1(R^4 - K^2)$ *in* p. 133∼139 *in* [5] *is available for the calculation of* $\pi_1(H_+^5 - B^3)$", then we will be able to conclude the following:

(5.11). $$\pi_1(H_+^5 - B^3) = \boldsymbol{Z}\,.$$

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References

[1] H. Gluck: *The embedding two-spheres in the four-sphere*, Trans. Amer. Math. Soc. **104** (1962), 308–333.

[2] T. Yajima: *On simply-knotted spheres in* R^4, Osaka J. Math. **1** (1964), 133–152.

[3] F. Hosokawa and T. Yanagawa: *Correction to "Is every slice knot a ribbon knot?"* (to appear).

[4] H. Terasaka and F. Hosokawa: *On the unknotted sphere* S^2 *in* E^4, Osaka Math. J. **13** (1961), 265–270.

[5] R.H. Fox: *A quick trip through knot theory*, Topology of 3-manifolds and Related Topics, edited by M.K. Fort Jr. Prentice Hall, 1963.

[6] F. Hosokawa: *A concept of cobordism between links*, Ann. of Math. **86** (1965), 362–373.

[7] M.A. Kervaire: *Les noeds de dimension superioures*, Bull. Soc. Math. France **93** (1965), 225–271.
[8] C.D. Papakyriakopoulos: *On Dehn's Lemma and the asphericity of knots*, Ann. of Math. **66** (1957), 1–26.
[9] H. Seifert und W. Threlfall: Lehrbuch der Topologie, Springer Verlag, 1935.
[10] H. Noguchi: *Obstructions to locally flat embeddings of combinatorial manifolds*, Topology **5** (1966), 203–213.