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On partitions of free products of groups

By Mutuo TAKAHASI

A *partition* of a group G was defined by J. W. Young[1]) to be a system $\{H_\alpha\}$ of subgroups of G such that every element except the identity of G is contained in one and only one component H_α. We may call a partition containing more than one component a *proper partition*. A group G having a proper partition $\{H_\alpha\}$ may be called *decomposable*, because it is represented as a set-theoretical sum of the components H_α, such that any two components have only the identity in common, and especially *completely decomposable* when every component H_α is cyclic subgroup.

The partitions of abelian groups was discussed by J. W. Young in his paper completely. The case of finite groups was studied by G. A. Miller[2]) and P. Kontorovitch.[3])

In this note we consider the case of free products of groups. It can be shown that any free product of groups is always decomposable and especially that any free group is completely decomposable.

Let $G = A * B$ be a free product of groups A and B. The transform $a^r = r a r^{-1}$ of any element $a \varepsilon A$ $(a \neq 1)$ by $r \varepsilon G$ is contained in A if and only if $r \varepsilon A$, and any two elements $a \varepsilon A$ and $b \varepsilon B$ except the identity can not be conjugate in G. Hence the identity is the only common element which is contained in any two of all the conjugate subgroups A^r and B^s.[4])

Now we take an element x which is not contained in any A^r nor in

1) J. W. Young, On the partitions of a group and the resulting clasification, Bulletin of the Amer. Math. Soc. vol. 133 (1927) pp. 453-461.

2) G. A. Miller, Groups in which all the operators are contained in a series of subgroups such that any two have only the identity in common, Bulletin of the Amer. Math. Soc. vol. 12 (1906) pp. 446-449.

3) P. Kontorovitch, Sur la representation d'un groupe fini sous la forme d'une somme directe de sous-groupes, I, Rec. Math. (Mat. Sbornik) 5 (47) (1939) pp. 283-296; II, Rec. Math. 7 (49) (1940) pp. 27-33.

4) $A^r = r A r^{-1}$; the transform of A by r.

any B^s. There exists a maximal cyclic[5]) subgroup M_x of G containing x.

If $$x \varepsilon C_1 \leq C_2 \leq \ldots . \leq C_n \leq \ldots .$$
be an ascending chain of cyclic subgroups of G containing x, the set-theoretical join $x = \sum_{n=1}^{\infty} C_n$ [6]) is an *abelian* subgroup of G. By the subgroup-theorem[7]) in free products we obtain the free-decomposition of X: $X = F * \overset{*}{\Pi} A^t \frown X * \overset{*}{\Pi} B^u \frown X$, where F is a free group and some of the factors may be 1. Since X is abelian, we may conclude that $X = F$ and F is an infinite cyclic subgroup, because $X = A^t \frown X$ or $X = B^u \frown X$ implies $X \leq A^t$ or $X \leq B^u$ respectively and it is contrary to the assumption: $x \notin A^r$, B^s for any r and s. Since X is cyclic, X must coincide with all the C_m for $m \geq n_0$, and the chain is not proper. Therefore there exists a maximal cyclic subgroup M_x containing x. And obviously $M_x \frown A^r = M_x \frown B^s = 1$ for every A^r and B^s.

Let M_x and M'_y be any two distinct maximal cyclic subgroups, where x, $y \notin A^r$, B^s. Then it can be proved that the meet $M_x \frown M'_y$ is the identity.

Let J denote the least subgroup containing M_x and M'_y. Applying the subgroup-theorem to J we obtain: $J = K * \overset{*}{\Pi} A^v \frown J \overset{*}{\Pi} B^w \frown J$.

If J is not free decomposable, either J must be cyclic or $J = A^v \frown J$ or $J = B^w \frown J$ for some v or w. But J can not be cyclic, since M_x and M'_y are maximal cyclic subgroups of G, and x, $y \notin A^v$, B^w for any v, $w \varepsilon G$ implies $J \not\leq A^v$ and $J \not\leq B^w$, so we may conclude that J is free decomposable. Therefore J has the center 1, and $M_x \frown M'_y$ is contained in its center, hence $M_x \frown M'_y = 1$. From this we can also see that the maximal cyclic subgroup M_x containing x is unique.

We have therefore the following

Theorem: *If $G = A * B$ is a free product of groups A and B, then*

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5) A cyclic subgroup which is not contained properly in any cyclic subgroup.

6) The notation $\sum$ or $+$ means the set-theoretical sum.

7) R. Baer und F. Levi, Freie Produkte und ihre Untergruppen, Compositio Math. 3 (1936) p. 391. M. Takahasi, Bemerkungen über den Untergruppensatz in freien Produkte, Proc. Imp. Acad. Tokyo, vol. XX (1945) pp. 589-594.

G is decomposable into the set-theoretical sum of the following form

$$G = \sum A^r + \sum B^s + \sum M_x,$$

where M_x is the (unique) maximal cyclic subgroup containing x for $x \notin \sum A^r + \sum B^s$ and any two distinct subgroups in the sum are disjoint except the identity.

In the case, when G is a free product of arbitrary many (finite or infinite) number of free factors, the theorem can be also proved quite analogously, and free groups are free products of infinite cyclic groups, therefore we have

Corollary: Free groups are completely decomposable.

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Added in proof. After writing this note, I found, in Math. Rev. 9 (No. 9), that the same theorem is given also by P. Kontorovitch in his "Groups with a basis of partitions III", Mat.Sbornik N. S. 22 (64) (1948) pp. 79-100, but I have not yet the chance of reading his original paper.