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ON THE STABLE CLASSIFICATION OF SPIN FOUR-MANIFOLDS

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1. Introduction

The stable classification of closed connected topological respectively smooth four-manifolds (with orientation or spin structure) via bordism theory is a very nice result in topology of manifolds, and can be found in [11] and [23]. Here *stably* means that one allows additional connected sums with copies of $\mathbb{S}^2 \times \mathbb{S}^2$ on both sides. In [19] the closed oriented 4-manifolds with finitely presentable fundamental group π were classified modulo connected sum with simply connected closed 4-manifolds. More precisely, the stable equivalence classes of these manifolds are bijective to the quotient $H_4(B\pi;\mathbb{Z})/(\mathrm{Aut}\,\pi)_*$ via the map $M \to f_*[M]$, where $[M] \in H_4(M)$ is the fundamental class, and $f\colon M \to B\pi$ is the classifying map for the universal covering of M (see [19, Theorem 1]). The proof of this theorem is based on some facts concerning the cobordism groups $\Omega_4(M)$, $\Omega_4(B\pi)$, and Ω_4 (see for example [7] and [28]). Recently, this result has been extended to the non-orientable case in [18] at least for abelian fundamental groups.

The aim of the present paper is to study the stable classification of closed connected oriented spin smooth 4-manifolds by using techniques of Kervaire-Milnor surgery, as explained for example in [4], [5], [6], and [20]. Then we reproduce a nice result of Kurazono and Matumoto [19] for such manifolds under the assumption that the fundamental group is finitely presentable and has vanishing second and third homology with $\mathbb{Z}_2$-coefficients.

Let $\mathcal{M}_\pi$ (resp. $\mathcal{M}_\pi^{\mathrm{Spin}}$) be the set of closed connected oriented smooth (resp. spin) 4-manifolds with finitely presentable fundamental group π, which are considered up to (resp. spin) stable equivalence. We say that two manifolds in $\mathcal{M}_\pi$ (resp. $\mathcal{M}_\pi^{\mathrm{Spin}}$) are (resp. *spin*) *stably equivalent* if they become diffeomorphic (resp. spin preserving diffeomorphic) after taking connected sums with copies of $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{S}^2 \underset{\sim}{\times} \mathbb{S}^2$ (resp. $\mathbb{S}^2 \times \mathbb{S}^2$) on both sides. The first result of the paper is the following

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Theorem A. *There are bijective maps*

$$c\colon \Omega_4(B\pi)/(\mathrm{Aut}\,\pi)_* \to \mathcal{M}_\pi$$

and

$$c^{\mathrm{Spin}}\colon \Omega_4^{\mathrm{Spin}}(B\pi)/(\mathrm{Aut}\,\pi)_* \to \mathcal{M}_\pi^{\mathrm{Spin}}.$$

The inverse maps are given by sending $\{M\} \in \mathcal{M}_\pi$ *and* $\{M\}^{\mathrm{Spin}} \in \mathcal{M}_\pi^{\mathrm{Spin}}$ *to* $\{(M, f)\} \in \Omega_4(B\pi)/(\mathrm{Aut}\,\pi)_*$ *and* $\{(M, \sigma_M, f)\} \in \Omega_4^{\mathrm{Spin}}(B\pi)/(\mathrm{Aut}\,\pi)_*$*, respectively. Here* σ_M *denotes the spin structure on* M*, and* $f\colon M \to B\pi$ *is the classifying map.*

The statement for the map c was proved in [19], while that for the map c^{Spin} follows from the results given in the next section. Then we can consider the Hurewicz homomorphisms

$$\mu\colon \Omega_4(B\pi) \to H_4(B\pi;\mathbb{Z})$$

and

$$\mu^{\mathrm{Spin}}\colon \Omega_4^{\mathrm{Spin}}(B\pi) \to H_4(B\pi;\mathbb{Z})$$

defined by the correspondences $(M, f) \to f_*[M]$ and $(M, \sigma_M, f) \to f_*[M]$, respectively. By [11] and [19] the map μ is surjective with kernel isomorphic to $\mathbb{Z}$ and generated by $\mathbb{C}P^2$. So there is a decomposition

$$\Omega_4(B\pi) \cong \Omega_4 \oplus \widetilde{\Omega}_4(B\pi) \cong \mathbb{Z} \oplus H_4(B\pi;\mathbb{Z})$$

where $\widetilde{\Omega}_4(B\pi) \cong H_4(B\pi;\mathbb{Z})$ is the cokernel of the monomorphism $i\colon \Omega_4 \to \Omega_4(B\pi)$, and the isomorphism $\Omega_4 \cong \mathbb{Z}$ is given by the signature. In §3 we will prove that if $H_2(B\pi;\mathbb{Z}_2) \cong H_3(B\pi;\mathbb{Z}_2) \cong 0$, then the map μ^{Spin} is surjective with kernel isomorphic to $16\mathbb{Z}$ and generated by the Kummer surface K^4. The last result permits to obtain a stable decomposition theorem analogous to that proved in [19] for the class of spin smooth 4-manifolds whose fundamental group satisfies the above homological conditions. For this, we say that two closed connected oriented spin smooth 4-manifolds are *spin weakly stably equivalent* if they become spin preserving diffeomorphic after taking connected sums with copies of the Kummer surface K^4 and $\mathbb{S}^2 \times \mathbb{S}^2$. Then the second result, we will prove, is the following

Theorem B. *Let* π *be a finitely presentable group which has vanishing second and third homology with* $\mathbb{Z}_2$*-coefficients. Then the spin weak stable equivalence classes of closed connected oriented spin smooth* 4*-manifolds* M *with fundamental group* π *one-to-one correspond with the elements of* $H_4(B\pi;\mathbb{Z})/(\mathrm{Aut}\,\pi)_*$ *via the map* $(M, \sigma_M, f) \to f_*[M]$*, where* σ_M *is the spin structure on* M*, and* $f\colon M \to B\pi$ *is the* *classifying map. In particular, if* $f_*[M] = 0$, *then* M *is spin weakly stably equivalent to the boundary of the regular neighborhood of an embedded finite* 2*-complex, realizing* π, *in* 5*-space.*

For the proof we treat with the spin cobordism groups $\Omega_4^{\mathrm{Spin}}(B\pi)$. For the definition of spin cobordism groups we refer to [7] and [28]. A spin structure σ_M on a manifold M is best thought of as a choice of trivialization of the tangent bundle of M over the 2-skeleton [27].

The following corollary is related with some papers concerning the homotopy type and the stable classification of closed 4-manifolds with free fundamental group (see [2], [3], [13], [15], [16] and [17]).

Corollary. *Let* M *be a closed connected oriented spin smooth* 4*-manifold whose fundamental group* $\pi_1(M)$ *is a free product* $G_1 * \cdots * G_p$ *such that* $H_2(BG_i; \mathbb{Z}_2) = H_3(BG_i; \mathbb{Z}_2) = 0$ *for any* $i = 1, \ldots, p$. *Then* $M \# l K^4 \# k(\mathbb{S}^2 \times \mathbb{S}^2)$ *is spin preserving diffeomorphic to a connected sum* $M_1 \# \cdots \# M_p$ *of closed connected oriented spin smooth* 4*-manifolds* M_i *with* $\pi_1(M_i) \cong G_i$ *for some non-negative integers* l *and* k. *The decomposition is spin stably unique.*

2. The map c^{Spin}

In this section we prove Theorem A for the class of closed connected spin smooth 4-manifolds with finitely presentable fundamental group π. We use only simple techniques of Kervaire-Milnor surgery (see for example [4], [5], [6], and [20]).

Lemma 1. *If* π *is finitely presented, any element* ω *in* $\Omega_4^{\mathrm{Spin}}(B\pi)$ *gives a closed oriented spin smooth* 4*-manifold* (N, σ_N) *with* $\pi_1(N) \cong \pi$ *and a map* $g\colon N \to B\pi$ *such that* g *induces an isomorphism on* π_1 *and* $[(N, \sigma_N, g)] = \omega$ *in* $\Omega_4^{\mathrm{Spin}}(B\pi)$.

Proof. The proof goes in the same way as that of Lemma 5 of [19]. We have only to keep the spin structures as in [4], [5], [6] and [20]. We can arrange that f induces an epimorphism on π_1 by redefining M to be $M \# k(\mathbb{S}^1 \times \mathbb{S}^3)$ and redefining f (we continue to use the same notation). It is easy to see that f extends in the desired way as does the spin structure, also denoted σ_M (see for example [6, Proposition 4.2]). Now perform surgery on embedded circles in $\operatorname{Int} M$ which represent elements of the kernel of f_* to get a new spin 4-manifold (N, σ_N) (see [25, Lemma 5]). Indeed, σ_M extends to a spin structure σ_N on the surgery manifold N. Since π is finitely presented, it is possible, by a finite number of surgeries, to obtain a closed oriented spin smooth 4-manifold (N, σ_N) and a map $g\colon N \to B\pi$ which induces an isomorphism on π_1. Furthermore, we have $[(N, \sigma_N, g)] = \omega$ in $\Omega_4^{\mathrm{Spin}}(B\pi)$ since $k(\mathbb{S}^1 \times \mathbb{S}^3)$ represents the trivial class in $\Omega_4^{\mathrm{Spin}}(B\pi)$. □

Corollary 2. *If the pairs* (M, σ_M, f) *and* (N, σ_N, g) *represent the same element of* $\Omega_4^{\mathrm{Spin}}(B\pi)$ *such that the induced maps on* π_1 *are isomorphic, then there exist a compact oriented smooth cobordism* (W, F) *and a spin structure* σ_W *on* W *extending those on* $\partial W = M \cup (-N)$ *such that both inclusions* $M \subset W$ *and* $N \subset W$ *induce isomorphisms on* π_1.

Lemma 3. *Let* (W, σ_W, F) *be a compact oriented smooth spin cobordism between* (M, σ_M, f) *and* (N, σ_N, g) *such that both inclusions* $M \subset W$ *and* $N \subset W$ *induce isomorphisms on* π_1. *Then* $M\#k(\mathbb{S}^2 \times \mathbb{S}^2)$ *is spin preserving diffeomorphic to* $N\#h(\mathbb{S}^2 \times \mathbb{S}^2)$ *for some non-negative integers* k *and* h.

Proof. We can simplify the handle decomposition of W relative to M so that it has only 2-handles and 3-handles as in the usual proof of s-cobordism theorem in higher dimension. Then the feet of 2-handles are isotopic to the trivial one because it should represent the zero element in π_1 by the assumption. So the middle level manifold is a connected sum of M and some copies of $\mathbb{S}^2 \times \mathbb{S}^2$ since the cobordism is spin. By thinking from the other direction, it is also spin preserving diffeomorphic to a connected sum of N and some copies of $\mathbb{S}^2 \times \mathbb{S}^2$. □

These results together imply that the map c^{Spin} is bijective, as claimed.

3. Spin cobordism group

Let (M, σ_M) be a closed connected oriented spin smooth 4-manifold with finite presentable fundamental group π. Then we have a map $f\colon M \to B\pi$ from M to the classifying space $B\pi$. The map is unique up to homotopy if we fix the induced isomorphism on π. The map determines the oriented spin cobordism class $[(M, \sigma_M, f)]$ in $\Omega_4^{\mathrm{Spin}}(B\pi)$. On the other hand, any element ω of $\Omega_4^{\mathrm{Spin}}(B\pi)$ gives a closed connected oriented spin smooth 4-manifold (N, σ_N) and a map $g\colon N \to B\pi$ with $g_*\colon \pi_1(N) \underset{\cong}{\to} \pi$ (see Lemma 1 in §2). The manifolds M and N will be shown to be spin weakly stably equivalent provided $H_2(B\pi; \mathbb{Z}_2) \cong H_3(B\pi; \mathbb{Z}_2) \cong 0$. For this we need some results which describe the properties of the Hurewicz homomorphism μ^{Spin}.

Lemma 4. *Let* X *be a* CW-*complex such that* $H_2(X; \mathbb{Z}_2) = H_3(X; \mathbb{Z}_2) = 0$. *Then the map*

$$\mu^{\mathrm{Spin}}\colon \Omega_4^{\mathrm{Spin}}(X) \to H_4(X; \mathbb{Z}),$$

defined by

$$\mu^{\mathrm{Spin}}[(M, \sigma_M, f)] = f_*[M],$$

is surjective and $\operatorname{Ker}\mu^{\mathrm{Spin}} \cong \Omega_4^{\mathrm{Spin}}$. *Moreover, the restriction of* μ^{Spin} *on*

$$\widetilde{\Omega}_4^{\mathrm{Spin}}(X) = \operatorname{Ker}\bigl(\Omega_4^{\mathrm{Spin}}(X) \to \Omega_4^{\mathrm{Spin}}(*)\bigr)$$

is an isomorphism.

Proof. The Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q}\colon H_p\bigl(X;\Omega_q^{\mathrm{Spin}}\bigr) \Rightarrow \Omega_{p+q}^{\mathrm{Spin}}(X)$$

has vanishing E^2 terms for $p+q \le 4$ except for $E^2_{0,4}$ and $E^2_{4,0}$. In fact, recall that Ω_n^{Spin} is $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_2$, 0, and $\mathbb{Z}$ for $n = 0, 1, 2, 3, 4$ (see [26]), and hence $E^2_{3,1} = H_3(X;\Omega_1^{\mathrm{Spin}})$, $E^2_{2,2} = H_2(X;\Omega_2^{\mathrm{Spin}})$ and $E^2_{1,3} = H_1(X;\Omega_3^{\mathrm{Spin}})$ vanish (under our hypothesis). In general, $E^\infty_{p,q} \cong J_{p,q}/J_{p-1,q+1}$, where

$$J_{p,q} = \operatorname{Im}\bigl(\Omega_{p+q}^{\mathrm{Spin}}(X^{(p)}, X^{(p-1)}) \to \Omega_{p+q}^{\mathrm{Spin}}(X)\bigr).$$

Thus $E^\infty_{0,4}$ is the image of the split monomorphism $\Omega^{\mathrm{Spin}}(*) \to \Omega_4^{\mathrm{Spin}}(X)$ whose cokernel is $E^\infty_{4,0} \subset H_4(X;\mathbb{Z})$. By dimensional reasoning

$$d^r\colon E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

and by comparing with the spectral sequence for $\Omega_{p+q}^{\mathrm{Spin}}(*)$, it follows that every element in $E^2_{0,4}$ and $E^2_{4,0}$ is a permanent cycle. So we have $E^\infty_{4,0} = E^2_{4,0} \cong H_4(X;\mathbb{Z})$ and $E^\infty_{0,4} = E^2_{0,4} \cong H_0(X;\mathbb{Z}) \cong \mathbb{Z} \cong \Omega_4^{\mathrm{Spin}}(*)$. Then we get the exact sequence

$$0 \longrightarrow E^2_{0,4} \cong \Omega_4^{\mathrm{Spin}}(*) \longrightarrow \Omega_4^{\mathrm{Spin}}(X) \longrightarrow E^2_{4,0} \cong H_4(X;\mathbb{Z}) \longrightarrow 0.$$

The map $\mu^{\mathrm{Spin}}\colon \Omega_n^{\mathrm{Spin}}(X) \to H_n(X;\mathbb{Z})$ induces a map from the spectral sequence for $\Omega_{p+q}^{\mathrm{Spin}}(X)$ to the spectral sequence for $H_{p+q}(X;\mathbb{Z})$ and coincides with the map $\Omega_4^{\mathrm{Spin}}(X) \to E^2_{4,0} \cong H_4(X;\mathbb{Z})$ of the sequence above for $n = 4$. Finally, we note that the kernel of this map is $E^2_{0,4} \cong \Omega_4^{\mathrm{Spin}} \cong \mathbb{Z}$, which is generated by the Kummer surface K^4. □

Corollary 5. *If* $H_2(B\pi;\mathbb{Z}_2) = H_3(B\pi;\mathbb{Z}_2) = 0$, *then the map*

$$\mu^{\mathrm{Spin}}\colon \Omega_4^{\mathrm{Spin}}(B\pi) \to H_4(B\pi;\mathbb{Z})$$

is an epimorphism, and $\operatorname{Ker}\mu^{\mathrm{Spin}} \cong \Omega_4^{\mathrm{Spin}}$ *is generated by the Kummer surface. Then there is a decomposition*

$$\Omega_4^{\mathrm{Spin}}(B\pi) \cong \Omega^{\mathrm{Spin}} \oplus \widetilde{\Omega}_4^{\mathrm{Spin}}(B\pi) \cong 16\mathbb{Z} \oplus H_4(B\pi;\mathbb{Z})$$

where $\widetilde{\Omega}_4^{\mathrm{Spin}}(B\pi) \cong H_4(B\pi;\mathbb{Z})$ *denotes the cokernel of the split monomorphism*

$$i^{\mathrm{Spin}}\colon \Omega_4^{\mathrm{Spin}} \to \Omega_4^{\mathrm{Spin}}(B\pi),$$

and the isomorphism $\Omega_4^{\mathrm{Spin}} \cong 16\mathbb{Z}$ *is given by the signature.*

As a consequence of Corollary 5, we get the following useful results first proved in [5, Theorem 5.2] and [6, Proposition 5.1], respectively.

Corollary 6. *If* $H_2(B\pi;\mathbb{Z}_2) = H_3(B\pi;\mathbb{Z}_2) = 0$, *then an oriented spin cobordism class* $[(M,\sigma_M, f)]$ *is zero in* $\Omega_4^{\mathrm{Spin}}(B\pi)$ *if and only if the signature of* M *vanishes, and* $f_*[M] = 0$ *in* $H_4(B\pi;\mathbb{Z})$.

Corollary 7. *Suppose that* $H_2(B\pi;\mathbb{Z}_2) = H_3(B\pi;\mathbb{Z}_2) = 0$. *Then* $\widetilde{\Omega}_4^{\mathrm{Spin}}(B\pi)$ *is trivial if and only if* $H_4(B\pi;\mathbb{Z}) = 0$.

Now we are going to prove Theorem B. Let π be a finitely presented group which has vanishing second and third homology with $\mathbb{Z}_2$-coefficients. A closed connected oriented spin 4-manifold (M,σ_M) with fundamental group π carries a classifying map $f\colon M \to B\pi$. The triple (M,σ_M, f) determines an oriented spin cobordism class $[(M,\sigma_M, f)]$ in $\Omega_4^{\mathrm{Spin}}(B\pi)$, and an element $\mu^{\mathrm{Spin}}[(M,\sigma_M, f)] = f_*[M]$ in $H_4(B\pi;\mathbb{Z})$. Of course, spin weakly stably equivalent 4-manifolds determine the same element of $H_4(B\pi;\mathbb{Z})/(\mathrm{Aut}\,\pi)_*$. Conversely, take any element of $H_4(B\pi;\mathbb{Z})$. Then it gives an element of

$$\widetilde{\Omega}_4^{\mathrm{Spin}}(B\pi) = \mathrm{Ker}\big(\Omega_4^{\mathrm{Spin}}(B\pi) \to \Omega_4^{\mathrm{Spin}}(*)\big)$$

by Corollary 5. It comes from a closed connected spin smooth 4-manifold (N,σ_N) with $\pi_1(N) \cong \pi$ and a map $g\colon N \to B\pi$ by Lemma 1. Let (M,σ_M, f) be another triple with $\pi_1(M) \cong \pi$ and a map $f\colon M \to B\pi$ such that $f_*[M] = g_*[N]$. Then for some l and m we have

$$\big[(M\#lK^4, \sigma'_M, f')\big] = \big[(N\#mK^4, \sigma'_N, g')\big]$$

in $\Omega_4^{\mathrm{Spin}}(B\pi)$ by Corollary 5, and the fact that $\Omega_4^{\mathrm{Spin}}(*)$ is generated by the Kummer surface K^4 (Here f' and g' are maps sending K^4 's to one point). Therefore the manifolds M and N are spin weakly stably equivalent by Corollary 2, and Lemma 3, i.e. $M\#lK^4\#k(\mathbb{S}^2\times\mathbb{S}^2)$ is spin preserving diffeomorphic to $N\#mK^4\#h(\mathbb{S}^2\times\mathbb{S}^2)$ for some l, m, h and k.

4. Some applications

(1). If π is a free group of rank p, then $B\pi \simeq \bigvee_p \mathbb{S}^1$, so we get in particular $H_i(B\pi;\mathbb{Z}_2) \cong 0$ for $i = 2, 3$, and $H_4(B\pi;\mathbb{Z}) \cong 0$. Thus we have $\Omega_4^{\mathrm{Spin}}(B\pi) \cong 16\mathbb{Z}$, and the isomorphism is given by the signature. Theorem A implies that if M is a closed connected oriented spin 4-manifold with signature zero and $\pi_1(M) \cong \pi$, then M is spin stably homeomorphic to $\#p(\mathbb{S}^1 \times \mathbb{S}^3)$ (see [2], [3], [13], and [15]). Theorem B says that a closed connected oriented spin 4-manifold M with $\pi_1(M) \cong \pi$ becomes homeomorphic to $\#p(\mathbb{S}^1 \times \mathbb{S}^3)$ after taking connected sums with copies of K^4 and $\mathbb{S}^2 \times \mathbb{S}^2$. We recall that there exists a closed oriented topological 4-manifold with fundamental group $\mathbb{Z}$ which is not the connected sum of $\mathbb{S}^1 \times \mathbb{S}^3$ with a simply connected 4-manifold (see [12]).

(2). Let π be a group with a presentation of deficiency one which is an extension of $\mathbb{Z}$ by a finitely generated normal subgroup. It was shown in [14] that the canonical 2-complex corresponding to that presentation is aspherical, hence π has geometric dimension at most 2. Furthermore, the Euler characteristic of $B\pi$ vanishes. Suppose that $H_1(B\pi;\mathbb{Z}_2) \cong \mathbb{Z}_2$ (examples are given by *knot like groups*, i.e., groups having abelianization $\mathbb{Z}$ and deficiency one). Since $\chi(B\pi) = 0$, it follows that $H_i(B\pi;\mathbb{Z}_2) = 0$ for $i = 2, 3$, and $H_4(B\pi;\mathbb{Z}) = 0$. Thus we obtain $\Omega_4^{\mathrm{Spin}}(B\pi) \cong 16\mathbb{Z}$, as before. We recall that an algebraic characterization of certain 4-manifolds (called *exact manifolds*) with infinite cyclic first homology was given in nice recent papers of Kawauchi (see [16] and [17]).

(3). If $\pi \cong \mathbb{Z}_p \oplus \mathbb{Z}$ where p is a prime number, $p > 2$, then $H_4(B\pi;\mathbb{Z}) \cong \mathbb{Z}_p$. Since $\mathrm{Aut}\,\pi$ identifies all the non-zero elements of $H_4(B\pi;\mathbb{Z})$, we get that $H_4(B\pi;\mathbb{Z})/(\mathrm{Aut}\,\pi)_*$ is isomorphic to $\mathbb{Z}_2$ (see [19]). Further, we have $H_i(B\pi;\mathbb{Z}_2) \cong 0$ for $i = 2, 3$, hence $\Omega_4^{\mathrm{Spin}}(B\pi) \cong 16\mathbb{Z} \oplus \mathbb{Z}_p$. Let Y^4 be the boundary of a regular neighbourhood of an embedded finite 2-complex X^2 realizing π in the standard 5-space. The induced homomorphism $H_4(Y;\mathbb{Z}) \to H_4(B\pi;\mathbb{Z})$ is trivial since it factorizes through $H_4(X;\mathbb{Z}) = 0$. Thus $[Y]$ goes to zero in $H_4(B\pi;\mathbb{Z})/(\mathrm{Aut}\,\pi)_* \cong \mathbb{Z}_2$. Of course, Y^4 is spin and has trivial signature since it embeds smoothly in $\mathbb{R}^5$. Let Σ_p be the product $L(p,1) \times \mathbb{S}^1$, where $L(p,1)$ is the usual lens space. Then $[\Sigma_p]$ goes to a nontrivial element of $H_4(B\pi;\mathbb{Z})$. Theorem B says that any closed connected oriented spin smooth 4-manifold M becomes spin stably equivalent to either Σ_p or Y^4.

(4). If π is a cyclic group $\mathbb{Z}_p$ of odd order, then $H_i(B\pi;\mathbb{Z}_2) = 0$ for $i = 2, 3$, and $H_4(B\pi;\mathbb{Z}) = 0$, hence $\Omega_4^{\mathrm{Spin}}(B\pi) \cong 16\mathbb{Z}$. Let $\bar{\Sigma}_p$ be the closed spin 4-manifold obtained from Σ_p by killing the generator of $\mathbb{Z} \subset \pi_1(\Sigma_p) = \mathbb{Z}_p \oplus Z$. By Theorem B any closed connected oriented spin 4-manifold M with $\pi_1(M) \cong \mathbb{Z}_p$ becomes diffeomorphic to $\bar{\Sigma}_p$ after stabilization with copies of K^4 and $\mathbb{S}^2 \times \mathbb{S}^2$ (compare with Theorem 2.5 of [11]). Further examples of smooth 4-manifolds with cyclic fundamental groups were constructed in [8] by using the knot surgery construction.

(5). Let π be the fundamental group of a closed aspherical 4-manifold Q^4 which is a rational homology 4-sphere. The existence of such a manifold was proved for example in [24]. If further $H_2(B\pi;\mathbb{Z}_2) = 0$, then the condition $\chi(B\pi) = 2$ implies that the Betti numbers β_i vanish (mod 2) for $i = 1, 3$, hence $H_3(B\pi;\mathbb{Z}_2) = 0$. Of course, we also have $H_4(B\pi;\mathbb{Z}) \cong \mathbb{Z}$, hence $H_4(B\pi;\mathbb{Z})/(\mathrm{Aut}\,\pi)_*$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/\{\pm 1\}$

(see [19]). Finally, we obtain $\Omega_4^{\mathrm{Spin}}(B\pi) \cong 16\mathbb{Z} \oplus \mathbb{Z}$.

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