TENSOR FIELDS AND THEIR PARALLELISM

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Much has been studied about an almost complex structure these ten years. One of the problems about the structure is to find an affine connection which makes a given almost complex tensor field parallel. A Riemannian connection is a one without torsion for which the fundamental tensor field of a Riemannian manifold is parallel. Affine connections on the group manifold were investigated fully by E. Cartan in [1]. In this paper we treat in general some tensor fields and affine connections which make the fields parallel. Moreover some studies about certain tensor fields are given.

1. Affine connections associated with a given tensor field

We assume in this paper that M is an n-dimensional connected separable differentiable manifold of class C^∞ with a tensor field A of class C^∞. In the first we have the following theorem.

THEOREM 1. *We assume that M has an affine connection for which a given tensor field A is parallel. Then for any point $p \in M$ we can take in each tangent space of points of a suitably chosen neighborhood U_p differentiable frames with respect to which components of A are constant each and the constants are equal on the whole manifold M.*

Proof. We take an arbitrary point $p \in M$ and a coordinate neighborhood U_p. Then U_p can be covered simply by curves starting from the point p. Now we take in the tangent space at p a frame R which by means of the given connection we translate along every curve above stated in such a way that resulting frames are parallel. Then the forms (ω^i_j) of the affine connection vanish along the curve and for the components of absolute differentials of the tensor $A = (a^{i_1 \cdots i_r}_{j_1 \cdots j_s})$ we have

$$\nabla a^{i_1 \cdots i_r}_{j_1 \cdots j_s} = da^{i_1 \cdots i_r}_{j_1 \cdots j_s}.$$

As A is parallel, we have $da^{i_1 \cdots i_r}_{j_1 \cdots j_s} = 0$ along the curve and $a^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are constant and are equal to the values at p. Thus they are constant on U_p for the frames

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chosen above. Now M can be covered by neighborhoods U_p of every point p, and neighborhoods U_0 and U_p of two points p_0 and p can be joined by a finite chain of neighborhoods $U_1, \ldots, U_k$. Firstly we take a frame R_0 on the tangent space at p_0 and determine frames for all points of U_0 in the way discussed above. We take a point $p_1 \in U_0 \cap U_1$ and similarly determine frames for the points of U_1 starting from the one at p_1 already determined. Next we take $p_2 \in U_1 \cap U_2$ and proceed in the same manner. In this way frames for points of U_p are determined, and we have frames at every point of M for which the components of the tensor A are each the same. Of course the frame at a same point is not unique, because U_0 and U_p may be connected by different chains of neighborhoods, and for two frames R and $\bar{R}$ at a point p frame transformation preserves the constant components of A.

Before we take up a converse of theorem 1 we define a reductive decomposition of a Lie group $G = GL(n)$. Elements of G can be represented by matrices P with respect to a frame taken in the vector space. Each coefficient ρ_j^i of $dPP^{-1} = (\rho_j^i)$ (j denotes a number of row and i a number of column) is an invariant form of G. Independent linear combinations

$$\rho_p = \sum_{ij} k_{pij} \rho_j^i \qquad (p = 1, \ldots, n^2;\ i, j = 1, \ldots, n) \qquad (1.1)$$

with constant coefficients k_{pij} are called relative components by E. Cartan. For a closed subgroup H of G we can take such a system that

$$\rho_a = 0 \qquad (a = 1, \ldots, h) \qquad (1.2)$$

hold good for $P \in H$. These ρ_a are called principal relative components of G/H. We take variable $P \in G$ and constant $S \in G$ and denote relative components induced from dPP^{-1} and $d(SP)(SP)^{-1}$ by ρ_p and σ_p respectively. Then we have

$$\sigma_p = \sum_q s_{pq} \rho_q \qquad (p, q = 1, \ldots, n^2)$$

and (s_{pq}) is an element of a linear adjoint group corresponding to S. If for any $S \in H$ we have

$$\sigma_a = \sum_b s_{ab} \rho_b, \qquad \sigma_u = \sum_v s_{uv} \rho_v, \qquad (1.3)$$

$$(a, b = 1, \ldots, h\ ;\ u, v = h+1, \ldots, n^2)$$

we say that G is reductive with respect to H. Relations $s_{au} = 0$ ($a = 1, \ldots, h$; $u = h+1, \ldots, n^2$) hold good for $S \in H$ if H is connected, and $s_{ua} = 0$ for $S \in H$

hold good if H is connected and the Lie algebra $\mathfrak{g}$ of G has a decomposition $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ such that $[\mathfrak{h}\mathfrak{h}]\subset\mathfrak{h}$, $[\mathfrak{h}\mathfrak{m}]\subset\mathfrak{m}$, where $\mathfrak{h}$ denotes the subalgebra corresponding to H. (cf. [7] p. 4, 7. [11] p. 41)

Now a converse of theorem 1 can be stated as follows.

THEOREM 2. *We assume that M is an n-dimensional differentiable manifold and has a tensor field A and in each neighborhood of any $p\in M$ differentiable frames in the tangent spaces can be so chosen that the components $(a^{i_1\cdots i_r}_{j_1\cdots j_s})$ of A are constant each and the same over the whole manifold M. We denote by H a subgroup of $G=GL(n)$ which preserves all the constant components $(a^{i_1\cdots i_r}_{j_1\cdots j_s})$ and we assume that G is reductive with respect to H. Then we have an affine connection on M for which the given tensor field is parallel.*

Proof. Any differentiable manifold has an affine connection (without torsion) induced by a Riemannian metric which always exists. We take an advantage of such a connection on M. We take a neighborhood U_p of an any point $p\in M$ and for the frames already chosen in U_p by assumption we denote the connection forms of the above connection by (π^i_j) and put $\pi_p=\sum_{ij}k_{pij}\pi^i_j$ with k_{pij} as in (1.1) for which (1.2) holds good. (1.1) can be written conversely as $\rho^i_j=\sum_p l_{ijp}\rho_p$ $(i,\ j=1,\ldots,n\ ;\ p=1,\ldots,n^2)$. With these l_{ijp} we have

$$\pi^i_j=\sum_p l_{ijp}\pi_p. \tag{1.4}$$

We drop terms containing π_a $(a=1,\ldots,h)$ and we get

$$\omega^i_j=\sum_u l_{iju}\pi_u. \qquad (u=h+1,\ldots,n^2) \tag{1.5}$$

Then (ω^i_j) is a $\mathfrak{h}$-valued differentiable form, and for the constant components $a^{i_1\cdots i_r}_{j_1\cdots j_s}$ of the given tensor field A we have

$$\sum_k\sum_i \omega^{i_k}_i a^{i_1\cdots i\cdots i_r}_{j_1\cdots\cdots j_s}-\sum_l\sum_j \omega^j_{j_l}a^{i_1\cdots\cdots i_r}_{j_1\cdots j\cdots j_s}=0, \tag{1.6}$$
$$(i,\ j=1,\ldots,n;\ \ k=1,\ldots,r;\ \ l=1,\ldots,s)$$

and for a covariant differential ∇A of the given tensor field we have

$$\nabla a^{i_1\cdots i_r}_{j_1\cdots j_s}=da^{i_1\cdots i_r}_{j_1\cdots j_s}+R, \tag{1.7}$$

where R denotes the term on the left side of (1.6). As $a^{i_1\cdots i_r}_{j_1\cdots j_s}$ are constant, we get $\nabla A=0$. Thus we have obtained a required connection in a neighborhood U_p of any point p. Now we will show that our process is consistent and our connection is defined on the whole manifold M.

To each point in $U_p \cap U_q$ two differentiable frames are attached, namely a frame R defined in U_p and $\bar{R}$ defined in U_q by assumption. We denote by $\pi = (\pi_j^i)$ and $\bar{\pi} = (\bar{\pi}_j^i)$ forms of the connection taken in the first with respect to frames R and $\bar{R}$ respectively and by T a frame transformation from R to $\bar{R}$. Then we have

$$\bar{\pi} = T\pi T^{-1} + dTT^{-1}. \tag{1.8}$$

When we put $\pi_p = \sum_{ij} k_{pij}\pi_j^i$, $\bar{\pi}_p = \sum_{ij} k_{pij}\bar{\pi}_j^i$, (1.8) can be represented as

$$\bar{\pi}_p = \sum_q t_{pq}\pi_q + \tau_p, \tag{1.9}$$

where (t_{pq}) is a linear adjoint transformation and τ_p are relative components corresponding to T. Constant components $a_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ are same for frames R and $\bar{R}$ and so T keeps each $a_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ invariant. Thus T belongs to the group H and we have

$$t_{ab} = 0,\ t_{uv} = 0,\ \tau_a = 0 \qquad (a, b = 1, \ldots, h:\ u, v = h+1, \ldots, n^2)$$

and so

$$\bar{\pi}_a = \sum_b t_{ab}\pi_b, \qquad \bar{\pi}_u = \sum_v t_{uv}\pi_v + \tau_u. \tag{1.10}$$

Now a required connection (ω_j^i) and $(\bar{\omega}_j^i)$ were constructed from

$$\pi_j^i = \sum_p l_{ijp}\pi_p, \qquad \bar{\pi}_j^i = \sum_p l_{ijp}\bar{\pi}_p$$

by dropping terms containing π_a and $\bar{\pi}_a$ respectively. (1.10) shows that if we take $\omega_a = 0$, $\omega_u = \pi_u$ instead of π_a, π_u and $\bar{\omega}_a = 0$, $\bar{\omega}_u = \bar{\pi}_u$ instead of $\bar{\pi}_a$, $\bar{\pi}_u$ respectively we have

$$\bar{\omega}_a = \sum_b t_{ab}\omega_b, \qquad \bar{\omega}_u = \sum_v t_{uv}\omega_v + \tau_u \tag{1.11}$$

and putting $\omega = (\omega_j^i)$, $\bar{\omega} = (\bar{\omega}_j^i)$ we get $\bar{\omega} = T\omega T^{-1} + dTT^{-1}$, and ω and $\bar{\omega}$ define the same connection in $U_p \cap U_q$. Thus our proof concludes.

Remark 1. As the proof of theorem 1 shows, a local existence of an affine connection for which a given tensor field A is parallel can be assured under a condition that components of A are constant for suitably chosen frames. Symmetric tensor fields of type (0, 2) and antisymmetric tensor fields of type (0, 2) satisfy our condition and a local existence is assured for them. Tensor fields of type (1, 1) do not satisfy our condition because they have eigenvalues which are not constant in general. If they are all constant, our connections exist locally.

Remark 2. As an application of our theorem 2 we have the case of an almost complex tensor field and more generally a tensor field of type $(1, 1)$ whose Jordan's canonical form is diagonal and whose eigenvalues are all constant. We have also the case of an antisymmetric tensor field A of type $(0, 2)$ of rank $n=2k$ on the differentiable manifold M of even dimension n. In the latter case we can verify that the conditions of theorem 2 hold good. Verification runs as follows. For suitable chosen frames the components of $A=(a_{ij})$ and form $\Omega = dPP^{-1} = (\rho^i_j)$ can be put as

$$A = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{pmatrix}, \tag{1.12}$$

where E is a unit matrix of degree k and Ω_1, Ω_2, Ω_3, Ω_4 are $k \times k$ matrices. For a linear transformation T in the vector space of dimension n which preserves A we have ${}^tTAT = A$ and for a group H of such T relations ${}^t\Omega_1 = -\Omega_4$, ${}^t\Omega_2 = \Omega_2$, ${}^t\Omega_3 = \Omega_3$ hold good. In general we decompose Ω into such a sum as

$$\Omega = \Omega^{(1)} + \Omega^{(2)}, \tag{1.13}$$

where $$\Omega^{(1)} = \frac{1}{2}\begin{pmatrix} \Omega_1 + {}^t\Omega_4 & \Omega_2 - {}^t\Omega_2 \\ \Omega_3 - {}^t\Omega_3 & {}^t\Omega_1 + \Omega_4 \end{pmatrix}, \qquad \Omega^{(2)} = \frac{1}{2}\begin{pmatrix} \Omega_1 - {}^t\Omega_4 & \Omega_2 + {}^t\Omega_2 \\ \Omega_3 + {}^t\Omega_3 & -{}^t\Omega_1 + \Omega_4 \end{pmatrix}.$$

Then by a transformation of a linear adjoint transformation corresponding to $T \in H$ we have

$$T\Omega T^{-1} = T\Omega^{(1)}T^{-1} + T\Omega^{(2)}T^{-1} \tag{1.14}$$

and we can show that the decomposition (1.14) is a one for $T\Omega T^{-1}$ corresponding to (1.13), and the decomposition (1.13) is reductive with respect to H.

Remark 3. Theorem 1 can be extended to the case of a Euclidean connection (especially Riemannian connection). We replace in theorem 1 'affine connection' by 'Euclidean connection' and 'differentiable frames' by 'differentiable rectangular frames' and then the theorem holds good, which is evident from the proof. As an example we have a symmetric Riemannian manifold on which Riemannian curvature tensor is parallel.

Our next interest is an existence of an affine connection without torsion, for which a tensor field is parallel. For a non degenerate symmetric tensor of type $(0, 2)$ a unique existence is wellknown, namely Riemannian connection. It seems hard to find a general theory. We treat some cases in the following sections.

2. Antisymmetric tensor field of type (0, 2)

We assume that M is an n-dimensional differentiable manifold with an antisymmetric tensor field A of type $(0, 2)$. We denote by $\omega^1, \ldots, \omega^n$ base in the dual tangent spaces and components of tensor A by (a_{ij}). Then we have a form

$$\alpha = \frac{1}{2} a_{ij}\omega^i \wedge \omega^j.$$

Next we assume that an affine connection exists. We denote connection forms by (ω_j^i) and torsion forms by $\tau^i = d\omega^i - \omega^j \wedge \omega_j^i$. (Hereafter we obey a usual rule of tensor calculus and omit a summation symbol except when specially mentioned to.) Then we have

$$2\, d\alpha = \nabla a_{ij} \wedge \omega^i \wedge \omega^j + a_{ij}\tau^i \wedge \omega^j - a_{ij}\omega^i \wedge \tau^j.$$

If $A = (a_{ij})$ is parallel and torsion vanishes, we have $d\alpha = 0$. Thus we get

THEOREM 3. *When a differentiable manifold M has an antisymmetric tensor field of type $(0, 2)$ and an affine connection without torsion for which A is parallel, a quadratic differential form α induced by A is closed.*

Now we take up a converse.

THEOREM 4. *We assume that M is a $2k$-dimensional differentiable manifold and has an antisymmetric tensor field A of maximal rank, for which an induced quadratic differential form α is closed. Then there exists on M an affine connection without torsion for which a given tensor field A is parallel.*

Proof. For any point $p \in M$ we take a neighborhood U of p. Then by virtue of $d\alpha = 0$ we have a 1-form β on U such that $\alpha = d\beta$. Owing to a fundamental theorem about 1-form, β can be written as

$$\text{either} \qquad \beta = dx^0 + y_a dx^a \qquad \text{or} \quad \beta = y_a dx^a,$$

where a ranges from 1 to a certain integer l, and x^0, x^a, y_a are independent functions. In each case we have $\alpha = d\beta = dy_a \wedge dx^a$. By our assumption α has a maximal rank $2k$ and we must have

$$\beta = y_a dx^a, \quad \text{and so} \quad \alpha = dy_a \wedge dx^a \qquad (a = 1, \ldots, k).$$

We take $x^1, \ldots, x^k, x^{k+1} = y_1, \ldots, x^{2k} = y_k$ as local coordinates. Thus components of our tensor $A = (a_{ij})$ are each constant for a natural frame attached to coordinates $x^1, \ldots, x^n (n = 2k)$, namely

$$d\alpha = dx^{a+k} \wedge dx^a \qquad \text{(summed for } a\text{)}.$$

In the following we use coordinrtes $x = (x^1, \ldots, x^n)$ only, tor which all the components a_{ij} are constant. Now there exists on M an affine connection without torsion (for example Riemannian connection) and we denote the connection forms by $\pi^i_j = \Gamma^i_{jk} dx^k (\Gamma^i_{jk} = \Gamma^i_{kj})$. We take an antisymmetric tensor $B = (b^{ij})$ of type (2, 0) such that $a_{ik} b^{kj} = \delta^j_i$ (which exists owing to non singularity of (a_{ij})) and put

$$\Gamma^l_{ij} a_{lk} = \Gamma_{kij} \tag{2.1}$$

and also
$$L_{ijk} = \frac{1}{3}(\Gamma_{ijk} + \Gamma_{jki} + \Gamma_{kij}). \tag{2.2}$$

L_{ijk} is symmetric with respect to indices i, j, k. We put

$$L^k_{ij} = b^{lk} L_{lij}, \qquad \omega^i_j = L^i_{jk} dx^k. \tag{2.3}$$

L^i_{jk} is symmetric with respect to indices j, k and for the connection (ω^i_j)

$$\nabla a_{ij} = da_{ij} - a_{ik}\omega^k_j - a_{kj}\omega^k_i = da_{ij} + (L_{ijl} - L_{jil}) dx^l.$$

These vanish because a_{ij} are constant and $L_{ijl} = L_{jil}$.

Now we will show that a connection thus defined is consistent throughout the manifold M. We cover M by coordinate neighborhoods and take frames on them for which a_{ij} are each the same constant on the whole manifold M, which is possible. We assume that U and $\bar{U}$ are intersecting neighborhoods in which coordinates are given by $x = (x^1, \ldots, x^n)$ and $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$. x and $\bar{x}$ are related differentiably in $U \cap \bar{U}$, and we put

$$p^i_j = \frac{\partial \bar{x}^i}{\partial x^j}, \qquad p^i_{jk} = p^i_{kj} = \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k}.$$

Then we have

$$a_{hl} p^h_i p^l_j = a_{ij} \tag{2.4}$$

and by differentiation of both sides $\qquad a_{hl} p^h_{ik} p^l_j + a_{hl} p^h_i p^l_{jk} = 0.$

Hence by putting
$$Q_{ijk} = a_{hl} p^h_i p^l_{jk} \tag{2.5}$$

we have
$$Q_{jik} = Q_{ijk}.$$

Thus Q_{ijk} is symmetric with respect to indices i, j, k.

We take connection forms $\pi^i_j = \Gamma^i_{jk}dx^k$ and $\bar{\pi}^i_j = \bar{\Gamma}^i_{jk}d\bar{x}^k$ of the same connection with respect to coordinates x and $\bar{x}$ respectively and construct ω^i_j and $\bar{\omega}^i_j$ in the way stated above. We have in the first

$$\Gamma^h_{ij}p^k_h = p^h_i p^m_j \bar{\Gamma}^k_{hm} + p^k_{ij}$$

and by multiplying with $p^m_l a_{km}$ we obtain by virtue of (2.1), (2.4), (2.5)

$$\Gamma_{lij} = p^h_i p^m_j p^k_l \bar{\Gamma}_{khm} - Q_{lij}.$$

Hence for $L_{lij} = \frac{1}{3}(\Gamma_{lij} + \Gamma_{ijl} + \Gamma_{jli})$, $\bar{L}_{lij} = \frac{1}{3}(\bar{\Gamma}_{lij} + \bar{\Gamma}_{ijl} + \bar{\Gamma}_{jli})$

we have
$$L_{lij} = p^h_i p^m_j p^k_l \bar{\Gamma}_{khm} - Q_{lij}.$$

We get by multiplying with $b^{ln}p^k_n$

$$L^l_{ij}p^k_l = p^k_i p^l_j \bar{\Gamma}^k_{hl} + p^k_{ij}.$$

Thus L^l_{ij} and $\bar{L}^l_{ij}$ define a same connection in $U \wedge \bar{U}$.

Remark. An affine connection without torsion, for which a given antisymmetric tensor of type (0, 2) is parallel, is not unique, because the connection taken at the beginning of the proof is not unique. When we restrict to a local existence of a required connection we can take arbitrary L_{ijk} which are symmetric with respect to indices and our connection can be given by (2.3).

3. Tensor fields of type (1, 1)

We assume that M is a differentiable manifold with a tensor field A of type (1, 1). In the first we investigate a Nijenhuis tensor of A whose eigenvalues are not necessarily constant, and then an affine connection for which A with constant eigenvalues is parallel.

1. We take two arbitrary tangent vector fields X and Y and construct a vector such as

$$Z = -A^2[X, Y] - [AX, AY] + A[AX, Y] + A[X, AY].$$

Then a mapping $(X, Y) \to Z$ is bilinear and antisymmetric in X and Y, and define a tensor N which was introduced by A. Nijenhuis and others. (cf. [10]) We take a neighborhood U and differentiable base $X_1, \ldots, X_n$ in the tangent space of each point of U, and denote by $\omega^1, \ldots, \omega^n$ dual base in the space of tangent covectors. We put as usual

$$d\omega^i = \frac{1}{2} c^i_{jk}\omega^j \wedge \omega^k, \qquad [X_j, X_k] = X_j X_k - X_k X_j = -c^i_{jk} X_i.$$

We take component (a^i_j) of A with respect to the base, and for $X = u^i X_i$ we have $AX = (a^i_j u^j)\, X_i$. Putting $X_k a^i_j = a^i_{jk}$, namely $da^i_j = a^i_{jk}\omega^k$, we get for $N = (N^i_{jk})$

$$\begin{aligned} N^i_{jk} = &-a^h_j a^i_{kh} + a^h_k a^i_{jh} + a^i_h (a^h_{kj} - a^h_{jk}) \\ &+ a^l_j a^m_k c^i_{lm} - a^i_l a^h_j c^l_{hk} + a^i_l a^h_k c^l_{hj} + a^i_l a^l_h c^h_{jk}. \end{aligned} \tag{3.1}$$

For an l-th power $B = A^l = (b^i_j)$ of A we have

$$b^i_j = a^i_{k_1} a^{k_1}_{k_2} \cdots a^{k_{l-1}}_j.$$

Putting $\mathrm{tr.}\, B = S^{(l)}$, which is a sum of l-th power of eigenvalues of A, and also $dS^{(l)} = S^{(l)}_k \omega^k$, we get by contraction of (3.1)

$$N_k = N^i_{ik} = a^h_k a^i_{ih} - a^i_h a^h_{ik} = a^h_k S^{(1)}_h - \frac{1}{2} S^{(2)}_k.$$

If $S^{(1)}$ and $S^{(2)}$ are constant (which is true when eigenvalues of A are constant), a vector $N = (N_k)$ vanishes. Next we assume that A is non singular and put $A^{-1} = (d^i_j)$. Then we have by contraction

$$M_k = d^j_i N^i_{jk} = d^j_i a^h_k a^i_{jh} - a^i_{ik}.$$

If we put $\varDelta = \det. A$ and $d\varDelta = \varDelta_k \omega^k$, we have $\varDelta_k = a^i_{jk} d^j_i \varDelta$ and so

$$M_k = a^h_k \varDelta^{-1} \varDelta_h - S^{(1)}_k.$$

If $S^{(1)}$ and $\varDelta$ are constant (which is true when eigenvalues of A are constant), a vector $M = (M_k)$ vanishes.

2. Hereafter we assume that Jordan's canonical form of a matrix $A = (a^i_j)$ is diagonal and the dimensions of the eigenspaces are each constant on M. We take a neighborhood U of any point $p \in M$ and take complex base in the tangent space of each point of U. Then formal tensor algebra holds good in the space. We have a decomposition

$$A = \sum_i \mu_i E_i \qquad (i = 1, \ldots, r) \tag{3.2}$$

such that $\mu_1, \ldots, \mu_r$ are all different and

$$\sum_i E_i = E \ (\text{unit}), \quad E_i^2 = E_i, \quad E_i E_j = 0 \qquad (i \neq j).$$

We take suitable complex base $\omega^1, \ldots, \omega^n$ in the dual tangent space and we get

$$a^i_j = \lambda_i \text{ for } i = j \qquad \text{and} \qquad a^i_j = 0 \text{ for } i \neq j,$$

where λ_j is each equal to some μ_i. We put $d\lambda_i = \lambda_{ij}\omega^j$ and then (3.1) reduces to

$$N^i_{jk} = -\delta^i_k(\lambda_j - \lambda_i)\lambda_{kj} + \delta^i_j(\lambda_k - \lambda_i)\lambda_{jk} + (\lambda_i - \lambda_j)(\lambda_i - \lambda_k)c^i_{jk} \tag{3.3}$$
(not summed for i, j, k)

Thus we get

$$\begin{aligned} &N^i_{jk} = 0 \text{ for } \lambda_i = \lambda_j = \lambda_k \\ &N^i_{jk} = \delta^i_j(\lambda_k - \lambda_i)\lambda_{jk} \text{ for } \lambda_i = \lambda_j \neq \lambda_k \\ &N^i_{jk} = (\lambda_i - \lambda_j)(\lambda_i - \lambda_k)c^i_{jk} \text{ for } \lambda_i \neq \lambda_j,\ \lambda_i \neq \lambda_k. \end{aligned} \tag{3.4}$$
(not summed for i, j, k)

Thus the condition $N^i_{jk} = 0$ is equivalent to

$$c^i_{jk} = 0 \text{ for } \lambda_i \neq \lambda_j,\ \lambda_i \neq \lambda_k, \text{ and } \lambda_{jk} = 0 \text{ for } \lambda_j \neq \lambda_k. \tag{3.5}$$

From this we get a following theorem.

THEOREM 5. *When a Nijenhuis tensor of A whose decomposition is given by* (3.2) *with μ_i all different, then a Nijenhuis tensor of $B = \sum_i \nu_i E_i$ with constant ν_i (not necessarily different) vanishes.*

The condition (3.5) means that the following relations hold good for $j = i_1, \ldots, i_h$, where $\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_h}$ exhaust eigenvalues equal to μ_i:

$$d\omega^j \equiv 0 \qquad (\text{mod } \omega^{i_1}, \ldots, \omega^{i_h}) \tag{3.6}$$

$$d\lambda_j \equiv 0 \qquad (\text{mod } \omega^{i_1}, \ldots, \omega^{i_h}). \tag{3.7}$$

As our tensor $A = (a^i_j)$ is real, eigenvalues are real or complex. We assume $\lambda_1 = \cdots = \lambda_h$ are real and different from others. Then a part of basic tangent covectors $\omega^1, \ldots, \omega^h$ corresponding to the eigenvalues λ_1 is real and by (3.6) local coordinates $x^1, \ldots, x^n$ can be taken in such a way that for $i = 1, \ldots, h$ we have

$$\omega^i \equiv 0 \qquad (\text{mod } dx^1, \ldots, dx^h)$$

and by (3.5)

$$d\lambda_i \equiv 0 \qquad (\text{mod } dx^1, \ldots, dx^h).$$

Thus λ_i is a function of $x^1, \ldots, x^h$ and moreover we can take $dx^1, \ldots, dx^h$ as a part of base instead of $\omega^1, \ldots, \omega^h$. The same process can be taken for any other real eigenvalues.

Next we assume that

$$\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_{p+b},\ \lambda_{p+b+1} = \lambda_{p+b+2} = \cdots = \lambda_{p+2b}$$

are complex conjugate and are different from others. Then as a corresponding part of our basic vectors we can take such $\omega^{p+1}, \ldots, \omega^{p+2b}$ that ω^i and ω^{b+i} are complex conjugate $(i=p+1, \ldots, p+b)$. By (3.6) we have for $j=p+1, \ldots, p+2b$

$$d\omega^j \equiv 0 \qquad (\text{mod } \omega^{p+1}, \ldots, \omega^{p+2b}),$$

and if we take real base $\pi^i = \omega^i + \omega^{b+i}$, $\pi^{b+i} = \sqrt{-1}\ (\omega^i - \omega^{b+i})$ $(i=p+1, \ldots, p+b)$, we have

$$d\pi^i \equiv 0, \quad d\pi^{b+i} \equiv 0 \qquad (\text{mod } \pi^{p+1}, \ldots, \pi^{p+2b})$$

and a part $x^{p+1}, \ldots, x^{p+2b}$ of real coordinates $x^1, \ldots, x^n$ can be taken in such a way that $\pi^j \equiv 0$ (mod $dx^{p+1}, \ldots, dx^{p+2b}$) for $j=p+1, \ldots, p+2b$. By virtue of (3.5) we have

$$d\lambda_j \equiv 0 \qquad (\text{mod } dx^{p+1}, \ldots, dx^{p+2b}).$$

We take up a submanifold V of U with local coordinates $x^{p+1}, \ldots, x^{p+2b}$. We put $\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_{p+b}$ $(=\mu_s)$. Then $\mu_s E_s$ and its conjugate $\bar{\mu}_s \bar{E}_s$ are contained in the decomposition (3.2) and a tensor defined by $\sqrt{\frac{-1}{2}}\,(E_s - \bar{E}_s)$ induces a real and almost complex tensor on V and its Nijenhuis tensor vanishes by theorem 5. Thus by a wellknown theorem of N. Newlander and L. Nirenberg [9] we can take on V complex analytic coordinates $z^1, \ldots, z^b$ and we can take as basic covectors $dz^1, \ldots, dz^b, d\bar{z}^1, \ldots d\bar{z}^b$. E_s reduces to a unit matrix with respect to $dz^1, \ldots, dz^b$ and so does $\bar{E}_s$ with respect to $d\bar{z}^1, \ldots, d\bar{z}^b$. Then we have by (3.7)

$$d\mu_s \equiv 0 \qquad (\text{mod } dz^1, \ldots, dz^b),$$

which means that μ_s is an analytic function. Thus we have got

THEOREM 6. *We assume that A is a tensor field of type* (1, 1) *on a differentiable manifold M and Jordan's canonical matric form of A is diagonal and the multiplicities of eigenvalues of A are each constant on M. Then vanishing of a Nijenhuis tensor of the tensor A means the following: M decomposes locally into a product of submanifolds $V_1, \ldots, V_k$ and $V_{k+1}, \ldots, V_{k+l}$, where V_a $(a=1, \ldots, k)$ correspond each to a real eigenvalues μ_a of A and μ_a is a function on V_a, while $V_s(s=k+1, \ldots, k+l)$ are real forms of complex manifolds on which eigenvalues μ_s, $\bar{\mu}_s$ are complex analytic functions on V_s and their* *complex conjugates.*

The vanishing of a Nijenhuis tensor has already been studied by him and A. Frölicher [5], and theorem 6 overlaps partly their results.

3. Next we investigate an affine connection for which a given tensor field of type (1, 1) is parallel. For that purpose it is necessary that the eigenvalues of A are constant, as is clear from the theorem 1. In the first place we prove

THEOREM 7. *When a tensor of type* (1, 1) *decomposes into tensors* E_i ($i=1, \dots, k$) *in such a way that in matric form*

$$A=\sum_i \mu_i E_i, \quad \textit{where } \sum_i E_i = E \ (\textit{unit matrix}), \quad E_i^2 = E_i, \quad E_i E_j = 0 \ (i \neq j)$$

with constant μ_i *all different. Then* A *is parallel with respect to an affine connection when and only when all* E_i *are so.*

Proof. We take a neighborhood of any point and frames in the tangent spaces of points of U in such a way that $A=(a^i_j)$ reduces to a diagonal form. If μ_i, μ_j are complex conjugate, corresponding base ω^i, ω^j in dual tangent spaces are complex conjugate. We denote by $\Omega=(\omega^i_j)$ connection forms and assume that

$$A=\begin{pmatrix} \mu_1 E^{(1)} & & \\ & \ddots & \\ & & \mu_r E^{(r)} \end{pmatrix} \quad \text{and} \quad \Omega=\begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1r} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \Omega_{r1} & \Omega_{r2} & \cdots & \Omega_{rr} \end{pmatrix}$$

correspond, where $E^{(1)}, \dots, E^{(r)}$ are unit matrices of degree $d_1, \dots, d_r$ respectively and Ω_{ij} is a $d_i \times d_j$ matrix. As μ_i are constant, parallelism $\nabla A = dA + A\Omega - \Omega A = 0$ of A reduces to $\Omega A = A\Omega$, and so $\Omega_{ij}=0$ for $i \neq j$. This is also a condition in order that E_i are all parallel.

We assume that M is an n-dimensional differentiable manifold with an affine connection, which makes a tensor field A of type (1, 1) parallel. We denote in a coordinate neighborhood of any point $p \in M$ the connection forms by $\omega^i_j = \Gamma^i_{jk} dx^k$, where $x^1, \dots, x^n$ are local coordinates. Then we have

$$da^i_j + a^k_j \omega^i_k - a^i_k \omega^k_j = 0.$$

Putting $da^i_j = a^i_{jk} dx^k$ we have

$$a^i_{jl} = -a^k_j \Gamma^i_{kl} + a^i_k \Gamma^k_{jl}. \tag{3.8}$$

As we have taken natural frames, c^i_{jk} vanish in (3.1) and putting $T^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}$

(torsion tensor) we get by (3.1) and (3.8)

$$N^i_{jk} = a^l_j a^h_k T^i_{lh} + a^i_l a^l_h T^h_{jk} + a^i_h a^l_j T^h_{kl} + a^i_l a^h_k T^l_{hj}.$$

If our connection is without torsion, then $N^i_{jk} = 0$. Thus we get

THEOREM 8. *We assume that a differentiable manifold M has an affine connection without torsion for which a tensor field A of type* (1.1) *is parallel. Then eigenvalues of A are all constant and a Nijenhuis tensor of A vanishes.*

Now we prove a converse of theorem 8.

THEOREM 9. *We assume that a differentiable manifold M has a tensor A of type* (1, 1) *for which eigenvalues are all constant and Jordan's canonical form is diagonal and moreover a Nijenhuis tensor vanishes. Then there exists locally on M an affine connection without torsion for which A is parallel.*

Proof. We denote by $\mu_1, \ldots, \mu_k$ real eigenvalues which are all different and by $\mu_{k+1}, \ldots, \mu_{k+l}, \mu_{k+l+1} = \bar{\mu}_{k+1}, \ldots, \mu_{k+2l} = \bar{\mu}_{k+l}$ complex eigenvalues which are all different. We take a neighborhood U of any point of M. Then by theorem 6 U decomposes into a product of real submanifolds $V_1, \ldots, V_k$ and of complex manifolds $V_{k+1}, \ldots, V_{k+l}$ (in real form), where V_a correspond to μ_a $(a = 1, \ldots, k)$, and V_s correspond to μ_s and $\mu_{s+l} = \bar{\mu}_s$ $(s = k+1, \ldots, k+l)$. Now A decomposes into a direct sum

$$\begin{aligned} A &= \sum_i \mu_i E_i \\ &= \sum_a \mu_a E_a + \sum_s (\mu_s E_s + \bar{\mu}_s \bar{E}_s) \qquad (3.9) \\ &(i = 1, \ldots, k+l;\ a = 1, \ldots, k; \\ &\ s = k+1, \ldots, k+l), \end{aligned}$$

where $\sum E_i = E$ is a unit and

$$E_i^2 = E_i, \qquad E_i E_j = 0 \ (i \neq j).$$

Each E_a can be considered as a tensor on V_a. As E_a corresponds to a unit matrix with respect to a tangent space of each point of V_a, a tensor $\mu_a E_a$ (μ_a const) is parallel for any affine connetion on V_a. A tensor $\mu_s E_s + \bar{\mu}_s \bar{E}_s$ can be considered as one on a complex manifold V_s. As a Nijenhuis tensor of A vanishes, that of $B = \sqrt{\frac{-1}{2}}(E_s - \bar{E}_s)$ vanishes by theorem 5. The tensor B is a real almost complex tensor on V_s, because $B^2 = -(E_s + \bar{E}_s)$. It is already known that there exists an affine connection without torsion for which an almost complex tensor field B is parallel. (cf. [4] and [6]. This will be proved more generally in theorem 10.) For such an conneotion $\mu_s E_s + \bar{\mu}_s \bar{E}_s$ is also parallel.

We take an arbitrary affine connection without torsion on each manifold V_a *and on* V_s *the connections above stated. Then the totality of the affine* connections defines one without torsion on U for which A is parallel.

4. Now we investigate an affine connection for which a given tensor A of type (1.1) is parallel and whose torsion is closely related to a Nijenhuis tensor. (cf. [15])

THEOREM 10. *We assume that a differentiable manifold M has a tensor field A of type* (1.1), *whose eigenvalues are each one of two constants* μ_1, μ_2, *and whose Jordan's canonical form is diagonal. Then there exists on M an affine connection for which A is parallel and a torsion tensor is a constant multiple of a Nijenhuis tensor of A.*

Proof. We assume that M is n-dimensional and eigenvalues of A are

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k (= \mu_1)$$
$$\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n (= \mu_2)$$

and throughout the proof we use indices which run as follows:

$$a, b, c, d, e = 1, \ldots, k; \quad p, q = k+1, \ldots, n.$$

We take a neighborhood U of any point of M and frames in the tangent spaces of every point of U in such a way that $A = (a^i_j)$ has a diagonal form with respect to the frames and denote by $\omega^1, \ldots, \omega^n$ dual base corresponding to $\lambda_1, \ldots, \lambda_n$ respectively. If μ_1 and μ_2 are complex numbers (naturally conjugate), the forms $\omega^1, \ldots, \omega^n$ are complex. In this case n is even $(n = 2k)$ and we can take base $\omega^1, \ldots, \omega^n$ in such an order as

$$\omega^1, \ldots, \omega^k, \omega^{k+1} = \bar{\omega}^1, \ldots, \omega^{2k} = \bar{\omega}^k.$$

Thus we have by (3.4)

$$N^a_{bc} = 0, \quad N^a_{pq} = (\mu_1 - \mu_2)^2 c^a_{pq}, \quad N^a_{bp} = 0$$

owing to the relations $\lambda_a = \lambda_b = \lambda_c$, $\lambda_a \neq \lambda_p = \lambda_q$, and $\lambda_{bp} = 0$, which is a consequence of the assumption that λ_b is constant. If we put

$$\nu^i = \frac{1}{2}(\mu_1 - \mu_2)^{-2} N^i_{jk}\omega^j \wedge \omega^k, \tag{3.10}$$

we have
$$\nu^a = \frac{1}{2} c^a_{pq}\omega^p \wedge \omega^q$$

and also
$$\nu^p = \frac{1}{2} c^p_{ab}\omega^a \wedge \omega^b.$$

Hence
$$\nu^a = d\omega^a - \frac{1}{2} c^a_{bc}\omega^b \wedge \omega^c - c^a_{bp}\omega^b \wedge \omega^p \tag{3.11}$$
$$\nu^p = d\omega^p - \frac{1}{2} c^p_{qr}\omega^q \wedge \omega^r - c^p_{qa}\omega^q \wedge \omega^a.$$

Now we shall show that there exists an affine connection for which A is parallel and whose torsion forms are equal to ν^i. We take in the first place an affine connection (which always exists) and denote by $\pi^i_j = \Gamma^i_{jk}\omega^k$ connection forms defined for frames in U given above. We put

$$H^a_{bc} = \frac{1}{2}(\Gamma^a_{bc} + \Gamma^a_{cb}), \qquad H^p_{qr} = \frac{1}{2}(\Gamma^p_{qr} + \Gamma^p_{rq}). \tag{3.12}$$

Hence we have
$$H^a_{bc} = H^a_{cb}, \qquad H^p_{qr} = H^p_{rq}. \tag{3.13}$$

Next we put

$$\omega^a_b = \frac{1}{2} c^a_{bc}\omega^c + c^a_{bp}\omega^p + H^a_{bc}\omega^c$$
$$\omega^p_q = \frac{1}{2} c^p_{qr}\omega^r + c^p_{qa}\omega^a + H^p_{qr}\omega^r \tag{3.14}$$
$$\omega^a_p = 0, \qquad \omega^p_a = 0.$$

Then (ω^i_j) gives a required connection as is shown in the following. In the first we have by (3.11), (3.13), (3.14)

$$\nu^a = d\omega^a - \omega^b \wedge \omega^a_b - \omega^p \wedge \omega^a_p$$
$$\nu^p = d\omega^p - \omega^a \wedge \omega^p_a - \omega^q \wedge \omega^p_q$$

and so
$$\nu^i = d\omega^i - \omega^j \wedge \omega^i_j.$$

Thus ν^i $(i=1, \ldots, n)$ are torsion forms of our connection. $\nabla a^i_j = 0$ can be easily verified on account of (3.14) and

$$a^a_b = \delta^a_b\mu_1, \quad a^p_q = \delta^p_q\mu_2, \quad a^a_p = 0, \quad a^p_a = 0.$$

Next we will prove that connections given by (3.14) on each neighborhoods are consistent on M and give a required one. We take two intersecting neighborhoods U and $\overline{U}$ (here—does not mean complex conjugate), and on U we construct a connection (ω_j^i) in the way above stated, and on $\bar{U}$ a connection $(\bar{\omega}_j^i)$ in the same way. As we have taken the same canonical form of A, a frame transformation on $U \cap \bar{U}$ is such that

$$\omega^a = t_b^a \bar{\omega}^b, \quad \omega^p = t_q^p \bar{\omega}^q \qquad (t_p^a = 0,\ t_a^p = 0). \tag{3.15}$$

In advance we have

$$d\omega^i = \frac{1}{2} c_{jk}^i \omega^j \wedge \omega^k, \quad d\bar{\omega}^i = \frac{1}{2} \bar{c}_{jk}^i \bar{\omega}^j \wedge \bar{\omega}^k$$

and for a transformation $\omega^i = t_j^i \bar{\omega}^j$ we have

$$t_j^h t_k^l c_{hl}^i = -t_{jk}^i + t_{kj}^i + t_h^i \bar{c}_{jk}^h, \tag{3.16}$$

where we have put $dt_j^i = t_{jk}^i \bar{\omega}^k$. We get by (3.14), (3.15)

$$\begin{aligned} t_a^c \omega_c^b &= \frac{1}{2} t_a^c c_{cd}^b \omega^d + t_a^c c_{cp}^b \omega^p + t_a^c H_{cd}^b \omega^d \\ &= \frac{1}{2} t_a^c t_e^d c_{cd}^b \bar{\omega}^e + t_a^c t_q^p c_{cp}^b \bar{\omega}^q + t_a^c t_e^d H_{cd}^a \bar{\omega}^e. \end{aligned}$$

By taking (3.16) into account we have

$$t_a^c \omega_c^b = \frac{1}{2}(-t_{ae}^b + t_{ea}^b + t_c^b \bar{c}_{ae}^c)\bar{\omega}^e + (-t_{aq}^b + t_e^b \bar{c}_{aq}^e)\bar{\omega}^q + t_a^c t_e^d H_{cd}^a \bar{\omega}^e.$$

By adding $dt_a^b = t_{ae}^b \bar{\omega}^e + t_{aq}^b \bar{\omega}^q$ we get

$$\begin{aligned} t_a^c \omega_c^b + dt_a^b &= t_c^b \Big(\frac{1}{2} \bar{c}_{ae}^c \bar{\omega}^e + \bar{c}_{aq}^c \bar{\omega}^q + H_{ae}^c \bar{\omega}^e \Big) \\ &\quad + (t_a^c t_e^d H_{cd}^a - t_c^b H_{ae}^c + \frac{1}{2} t_{ae}^b + \frac{1}{2} t_{ea}^b)\bar{\omega}^e. \end{aligned} \tag{3.17}$$

Next for forms $\pi_j^i = \Gamma_{jk}^i \omega^k$, $\bar{\pi}_j^i = \bar{\Gamma}_{jk}^i \bar{\omega}^k$, on U and $\bar{U}$ respectively, of the connection taken at the beginning, we have

$$t_a^c \pi_c^b + dt_a^b = t_c^b \bar{\pi}_a^c.$$

Hence by (3.15) $\qquad t_e^d t_a^c \Gamma_{cd}^b + t_{ae}^b = t_c^b \bar{\Gamma}_{ae}^c$

and so $\qquad t_e^d t_a^c \Gamma_{dc}^b + t_{ea}^b = t_c^b \bar{\Gamma}_{ea}^c.$

Hence for $H_{bc}^a = \frac{1}{2}(\Gamma_{bc}^a + \Gamma_{cb}^a)$ and $\bar{H}_{bc}^a = \frac{1}{2}(\bar{\Gamma}_{bc}^a + \bar{\Gamma}_{cb}^a)$ we have

$$t_e^d t_a^c H_{cd}^b + \frac{1}{2}(t_{ae}^b + t_{ea}^b) = t_c^b \bar{H}_{ae}^c, \tag{3.18}$$

and by (3.14), (3.17), (3.18) $\quad t^c_a \omega^b_c + dt^b_a = t^b_c \overline{\omega}^c_a$.
The same is true for ω^p_q and $\overline{\omega}^p_q$.
Thus ω^i_j and $\overline{\omega}^i_j$ define the same connection in $U \cap \overline{U}$.

Hitherto we have dealt with complex base $\omega^1, \dots, \omega^n$ if μ_1 and μ_2 are complex conjugate. But in that case our connection (ω^i_j) determined by (π^i_j) in U defines a real one if we take real base in the tangent spaces. This fact can be verified as follows. When eigenvalues μ_1 and μ_2 are complex, their multiplicities are the same as A is real on M. When we take conjugate base $\omega^1, \dots, \omega^k, \omega^{k+1} = \overline{\omega}^1, \dots, \omega^{2k} = \overline{\omega}^k$ (here—means complex conjugate) corresponding to $\lambda_1 = \cdots = \lambda_k (= \mu_1)$ and $\lambda_{k+1} = \cdots = \lambda_{2k} (= \mu_2)$, affine connections determined by forms (ω^i_j) are real in real coordinates when and only when

$$\overline{\omega}^a_b = \omega^{a+k}_{b+k}, \quad \overline{\omega}^a_{b+k} = \omega^{a+k}_b.$$

This can be verified by taking real frames $\pi^a = \omega^a + \omega^{a+k}$, $\pi^{a+k} = \sqrt{-1}(\omega^a - \omega^{a+k})$. Hence for an affine connection $\pi^i_j = \Gamma^i{}_{jk}\omega^k$, which is real in real frames, we have in conjugate frames $\omega^1, \dots, \omega^n$

$$\overline{\pi}^a_b = \pi^{a+k}_{b+k}, \quad \text{hence} \quad \overline{\Gamma}^a_{bc} = \Gamma^{a+k}_{b+k,c+k} \tag{3.19}$$

and so

$$\overline{H}^a_{bc} = H^{a+k}_{b+k,c+k}. \tag{3.20}$$

Next

$$d\omega^a = \frac{1}{2} c^a_{bc}\omega^b \wedge \omega^c + c^a_{b,c+k}\omega^b \wedge \omega^{c+k} + \frac{1}{2} c^a_{b+k,c+k}\omega^{b+k} \wedge \omega^{c+k}$$

$$d\omega^{a+k} = \frac{1}{2} c^{a+k}_{bc}\omega^b \wedge \omega^c + c^{a+k}_{b,c+k}\omega^b \wedge \omega^{c+k} + \frac{1}{2} c^{a+k}_{b+k,c+k}\omega^{b+k} \wedge \omega^{c+k}.$$

As $\omega^{a+k} = \overline{\omega}^a$, we have

$$\overline{c}^a_{b,c+k} = c^{a+k}_{b+k,c}, \quad \overline{c}^a_{bc} = c^{a+k}_{b+k,c+k} \tag{3.21}$$

and by (3.14), (3.20), (3.21) $\quad \overline{\omega}^a_b = \omega^{a+k}_{b+k}, \ \overline{\omega}^a_{b+k} = \omega^{a+k}_b = 0$
and so our connection is real.

Remark. For an almost complex tensor A we havs $A^2 = -E$ (unit) and so $\mu_1 = \sqrt{-1}$, $\mu_2 = -\sqrt{-1}$. Hence by (3.10) we have $\nu^i = -\frac{1}{8} N^i_{jk}\omega^j \wedge \omega^k$. For an almost product tensor A we have $A^2 = E$. Then $\mu_1 = 1$, and $\mu_2 = -1$ with multiplicities h and l ($h + l = n$, h arbitrary). Hence by (3.10) we have $\nu^i = \frac{1}{8} N^i_{jk}\omega^j \wedge \omega^k$. Thus in the cases of an almost complex tensor and an almost product tensor we have an affine connection without torsion when and only when $N^i_{jk} = 0$.

4. Group manifold

We consider a space M of a connected Lie group. We take an arbitrary differentiable frame $\omega^1, \ldots, \omega^n$ in the dual tangent spaces and put

$$d\omega^i = \frac{1}{2} c^i_{jk}\omega^j \wedge \omega^k \qquad (c^i_{jk} = -c^i_{kj}).$$

Then (c^i_{jk}) are not components of a tensor. But if we take invariant differential forms $\omega^1, \ldots, \omega^n$ as a base we have structure constants (c^i_{jk}) and they are invariant under a linear adjoint transformation. (cf. [7] p. 3 and [12] p. 220) They are components of a tensor of type (1, 2) in somuch as we take invariant differential forms as a base. The tensor C so defined satisfies the condition for a local existence of an affine connection for which C is parallel. In fact, if we take an affine connection defined by forms

$$\omega^i_j = ac^i_{jk}\omega^k \qquad (a \text{ constant}) \tag{4.1}$$

with respect to the base chosen above, the tensor $C = (c^i_{jk})$ is parallel. This can be verified as follows.

$$\nabla c^i_{jk} = dc^i_{jk} + \omega^i_h c^h_{jk} - \omega^h_j c^i_{hk} - \omega^h_k c^i_{jh}$$

and these vanish by the relation

$$c^i_{hl}c^h_{jk} + c^i_{hk}c^h_{lj} + c^i_{hj}c^h_{kl} = 0. \tag{4.2}$$

Next as a torsion form of the connection we have

$$\tau^i = d\omega^i - \omega^j \wedge \omega^i_j = (1-2a)d\omega^i. \tag{4.3}$$

When we put $a = \pm\frac{1}{2}$, 0, we get $+$, $-$, 0-connection of E. Cartan [1]. We denote by $c = (c_k)$ a vector obtained from $C = (c^i_{jk})$ by a contraction with respect to i and j. By contracting (4.2) with respect to i and l, we get $c_h c^h_{jk} = 0$ and this means that a 1-form $\alpha = c_k\omega^k$ is closed, namely $d\alpha = 0$. Vanishing of the vector $c = (c_k)$ is equivalent to a unimodularity of a linear adjoint group, and also to an existence of a both side invariant volume on our group manifold G (cf. [2] and [8])

We may define an almost group structure by such a tensor field (a^i_{jk}) of type (1,2) that for suitably chosen frames a^i_{jk} reduce to structure constants c^i_{jk} of a certain Lie group. For such a frame $\omega^1, \ldots, \omega^n$ we define

$$\rho^i = d\omega^i - \frac{1}{2} c^i_{jk}\omega^j \wedge \omega^k.$$

For a frams transformation $\bar{\omega}^i = t^i_j \omega^j$, where (t^i_j) is a transformation of a linear adjoint group of G, we have $\bar{\rho}^i = t^i_j \rho^j$ for $\bar{\rho}^i = d\bar{\omega}^i - \frac{1}{2} c^i_{jk}\bar{\omega}^j \wedge \bar{\omega}^k$ (with the same c^i_{jk}). Vanishing of a vector valued differential form (ρ^i) characterizes a group manifold locally.

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