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THE JACOBIAN OF A CYCLIC QUOTIENT OF A FERMAT CURVE

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§ 0. Introduction

Fix a positive integer m. Let F_m denote the Fermat curve over $\mathbf{Q}$ of degree m, given by the projective equation

$$X^m + Y^m + Z^m = 0.$$

Let $\mu_m \subseteq \overline{\mathbf{Q}}$ be the group of m-th roots of unity, Δ be the image of μ_m in μ_m^3 under the diagonal embedding, and let $G_m = \mu_m^3/\Delta$. Then G_m acts on F_m as follows:

$$(\xi_1, \xi_2, \xi_3) \bmod \Delta\colon (X, Y, Z) \longrightarrow (\xi_1 X, \xi_2 Y, \xi_3 Z).$$

The group ring $\mathbf{Z}[G_m]$ acts on the Jacobian J_m of F_m. Let $K = \mathbf{Q}(\mu_m)$. Then J_m/K has CM by $\mathbf{Z}[G_m]$ [4].

Let $a, b, c \in \mathbf{Z}$, with $a + b + c = 0$, $(a, b, c, m) = 1$, and none of a, b, c divisible by m. Let $\Gamma^m_{a,b,c}$ be the following subgroup of G_m:

$$\{(\xi_1, \xi_2, \xi_3) \in \mu_m^3 \mid \xi_1^a \xi_2^b \xi_3^c = 1\}/\Delta.$$

Then the quotient curve

$$F^m_{a,b,c} = \Gamma^m_{a,b,c} \backslash F_m$$

is defined over $\mathbf{Q}$, and has equation $y^m = (-1)^c x^a (1 - x)^b$. Its Jacobian $J^m_{a,b,c}$ has CM by

$$\mathbf{Z}[G_m/\Gamma^m_{a,b,c}].$$

Let g be a generator of the cyclic group $G_m/\Gamma^m_{a,b,c}$, and let $f_m(x)$ denote the m-th cyclotomic polynomial. Then the sum of the images of the maps

$$J^d_{a,b,c} \longrightarrow J^m_{a,b,c}$$

induced from $F^m_{a,b,c} \to F^d_{a,b,c}$, $(x, y) \to (x, y^{m/d})$, as d varies over the set of

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proper divisors of m, generates the abelian subvariety $f_m(g)J^m_{a,b,c}$ of $J^m_{a,b,c}$. We define $(J^m_{a,b,c})^{\text{new}}$ to be the quotient of $J^m_{a,b,c}$ by $f_m(g)J^m_{a,b,c}$.

In [8], Koblitz-Rohrlich determined the necessary and sufficient conditions for $(J^m_{a,b,c})^{\text{new}}$ to be non-simple and its decomposition into simple factors up to isogeny in the case when $(m, 6) = 1$. Aoki [1] has solved this problem for all sufficiently large m. In § 2, we use the above mentioned results to determine the ring of rational endomorphisms of some non-simple $(J^m_{a,b,c})^{\text{new}}$.

In the rest of this paper, we let p be an odd prime, fix a cyclic quotient curve of F_p and denote its Jacobian by A. From the work of Koblitz-Rohrlich [8] and Schmidt [12], we know that A is either absolutely simple or isogeneous to a cube of an absolutely simple abelian variety over the p-th cyclotomic field $\mathbf{Q}(\mu_p)$. When A is simple, $\operatorname{End}(A)$ is isomorphic to the ring of integers in $\mathbf{Q}(\mu_p)$. In § 4, we shall completely characterize the endomorphism ring of A whenever it is non-simple. We then use this information to show in § 6 that A is in fact isomorphic over $\mathbf{Q}(\mu_p)$ to a cube of a simple abelian variety. A special case of this result ($p = 7$) is that the Jacobian $\operatorname{Jac}(C)$ of the Klein curve

$$C\colon X^3Y + Y^3Z + Z^3X = 0$$

is isomorphic to a cube of an elliptic curve [10] (in fact, the elliptic modular curve $J_0(49)$).

§ 1. Preliminaries

For the Fermat curve F_m, let $x = X/Z$ and $y = Y/Z$. Now let $r, s, t \in \mathbf{Z}$, $0 < r, s, t < m$ and $r + s + t \equiv 0 \pmod m$. Then

$$w_{r,s,t} = x^{r-1}y^{s-1}\frac{dx}{y^{m-1}}$$

is a differential form of the second kind on F_m. G_m is generated by $\sigma = (\zeta, 1, 1)$ and $\tau = (1, \zeta, 1)$, where ζ is a fixed primitive m-th root of unity, and the forms $w_{r,s,t}$ are eigenforms for the action of G_m: $(\sigma^j\tau^k)^*w_{r,s,t} = \zeta^{rj+sk}w_{r,s,t}$. Since the characters on $(\mathbf{Z}/m\mathbf{Z})^2$ are mutually distinct,

$$\Omega = \{w_{r,s,t} \mid 0 < r, s, t < m, r + s + t \equiv 0 \pmod m\}$$

is a basis of the de Rham cohomology $H^1_{\mathrm{DR}}(F_m)$. $\Omega_1 = \{w_{r,s,t} \in \Omega \mid r + s + t = m\}$ is a basis for $H^0(F_m, \Omega^1)$ in the Hodge splitting of $H^1_{\mathrm{DR}}(F_m)$.

The set of elements of Ω invariant under the action of $\Gamma^m_{a,b,c}$ descends to a basis of eigenforms for $H^1_{\mathrm{DR}}(J^m_{a,b,c})$ under the action of $\mathbf{Z}[G_m/\Gamma^m_{a,b,c}]$. $(J^m_{a,b,c})^{\mathrm{new}} = J^{\mathrm{new}}$ has CM (in the sense of Shimura-Taniyama) by the ring of integers

$$\mathbf{Z}[G_m/\Gamma^m_{a,b,c}]/(f_m(g)) \approx \mathcal{O}_K$$

of $K = \mathbf{Q}(\mu_m)$, with CM type

$$H^m_{a,b,c} = \{h \in (\mathbf{Z}/m\mathbf{Z})^* \mid \langle ha \rangle + \langle hb \rangle + \langle hc \rangle = m\},$$

where $\langle h \rangle$ denotes the unique representative of h modulo m between 0 and $m-1$.

Let $\mathscr{E}$ denote the set of positive integers m which are different from each of the following numbers:

2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 30,
36, 39, 40, 42, 48, 54, 60, 66, 72, 78, 84, 90, 120, 156, 180.

Then from the works of Koblitz-Rohrlich (for the cases where m is relatively prime to 6) [8] and Aoki [1], for $m \in \mathscr{E}$, J^{new} is non-simple if and only if

(1) (a, b, c) is equivalent to $(1, r, -(1+r))$, where $1 + r + r^2 \equiv 0 \pmod m$, or

(2) (a, b, c) is equivalent to $(1, s, -(1+s))$, where $s^2 \equiv 1 \pmod m$ and $s \not\equiv \pm 1 \pmod m$, and $s \neq m/2 + 1$ if $2^3 \mid m$, or

(3) (a, b, c) is equivalent to $(1, 1, -2)$, with $2^2 \mid m$, or

(4) (a, b, c) is equivalent to $(1, m/2+1, m/2-2)$, with $2^3 \mid m$.

In case (1), J^{new} is isogeneous to a cube of an absolutely simple abelian variety. In cases (2) and (3), J^{new} is isogeneous to a square of a simple abelian variety. Finally in case (4), J^{new} is isogeneous to X^4 for some simple abelian variety X.

We shall denote J^{new} by A and B in the first and second cases respectively.

Let ρ be the automorphism of F_m given by

$$(X, Y, Z) \longrightarrow (Z, X, Y).$$

Let Γ_A and J_A denote the $\Gamma^m_{a,b,c}$ and $J^m_{a,b,c}$ associated with A. Since

$$\rho \Gamma_A \rho^{-1} \subseteq \Gamma_A,$$

ρ induces an automorphism of G_m/Γ_A by conjugation. We note that $f_m(x^l)$

is divisible by $f_m(x)$ if l and m are relatively prime. Hence, if g is a generator of G_m/Γ_A, then

$$\rho f_m(g)J_A = f_m(\rho g \rho^{-1})J_A \subseteq f_m(g)J_A .$$

So ρ induces an automorphism ρ of A such that the following diagram commutes:

$$\begin{tikzcd} J_m \arrow[r, "\rho"] \arrow[d] & J_m \arrow[d] \\ J_A \arrow[r, "\rho"] \arrow[d] & J_A \arrow[d] \\ A \arrow[r, "\rho"] & A \end{tikzcd} .$$

Let $\iota \in \mathrm{Aut}(F_m)$ be given by

$$\iota\colon (X, Y, Z) \longrightarrow (Y, X, Z) .$$

Then we have a similar commutative diagram to the one above with (A, ρ) replaced by (B, ι).

Since

$$H^{1,0}(J^{\text{new}}, \mathbf{C}) = \bigoplus_{h \in H^m_{a,b,c}} V(\langle ha \rangle, \langle hb \rangle, \langle hc \rangle) ,$$

where

$$V(a, b, c) = \{\eta \in H^1(F_m, \mathbf{C}) \mid g^*\eta = \xi_1^a \xi_2^b \xi_3^c \eta \text{ for all } g = (\xi_1, \xi_2, \xi_3) \in G_m\} ,$$

a basis of holomorphic differential forms for $H^0(J^{\text{new}}, \Omega^1)$ is

$$\{w_{\langle ha \rangle, \langle hb \rangle, \langle hc \rangle} \mid h \in H^m_{a,b,c}\} .$$

The following lemma shows that the abelian varieties A and B are isogeneous to

$$\prod_{l=0}^{2} A/\langle g_l \rangle \quad \text{and} \quad \prod_{l=0}^{1} B/\langle h_l \rangle$$

respectively, where g_l and h_l denote $\sigma^l \rho \sigma^{-l}$ and $\sigma^l \iota \sigma^{-l}$ respectively.

LEMMA 1.1. *$H^0(J_A, \Omega^1)^{\langle g_l \rangle}$ is spanned by*

$$g_l^*\{w_{r,s} \mid w_{r,s} \in H^0(J_A, \Omega^1)\} ,$$

and $H^0(J_A, \Omega^1) = \bigoplus_{l=0}^{2} H^0(J_A, \Omega^1)^{\langle g_l \rangle}$. Similar statements hold for $H^0(J_B, \Omega^1)$, h_0 and h_1.

Proof. Let V_l and W_l denote $(1 + g_l + g_l^2)^* H^0(J_A, \Omega^1)$ and $H^0(J_A, \Omega^1)^{\langle g_l \rangle}$ respectively. Then $V_l \subseteq W_l$ and $\dim V_l = \dim H^0(J_A, \Omega^1)/3$ by definition.

We claim that $W_j \cap (W_k + W_l) = \{0\}$ when $\{j, k, l\} = \{0, 1, 2\}$. We verify this for $j = 0$, $k = 1$ and $l = 2$. The other cases are treated similarly.

Let $w_0 = w_1 + w_2$, where $w_l \in W_l$ $(l = 0, 1, 2)$. Then $w_1 = (\sigma\rho\sigma^{-1})^* w_0 - (\sigma\rho\sigma^{-1})^* w_2 = (\sigma^{-(r+2)})^* w_0 - (\sigma^{r+2})^* w_2$. Therefore, $(\sigma^{-(r+2)} - 1)^* w_0 = (1 - \sigma^{r+2})^* w_2$. Applying $(\sigma^{r+2})^*$ to both sides of the latter equation, we obtain $(1 - \sigma^{r+2})^* (w_0 - (\sigma^{r+2})^* w_2) = 0$. In particular,

$$w_0 - (\sigma^{r+2})^* w_2 \in H^0(F_A/\langle \sigma \rangle, \Omega^1) \approx H^0(\mathbf{P}^1, \Omega^1)\,.$$

Hence, $w_0 = \rho^* w_0 = \rho^* (\sigma^{r+2})^* w_2 = (\sigma^{r+2}\rho)^* w_2 = (\sigma^2)^* (\sigma^2 \rho \sigma^{-2})^* w_2 = (\sigma^2)^* w_2$, and $(\sigma^r)^* w_2 = w_2$. So, $w_2 = 0$, and $w_0 = w_1 \in W_0 \cap W_1$, which we can show to be $\{0\}$, as before. $\square$

Let $A_l = A/\langle g_l \rangle$ and $B_l = B/\langle h_l \rangle$. Then each A_l and B_l is simple, and admits CM by the ring of integers in $L = K^{\langle r \rangle}$ and $M = K^{\langle s \rangle}$ respectively. To be precise, the endomorphisms $\sigma + \sigma^r + \sigma^{r^2}$ and $\sigma + \sigma^s$ of A and B descend to endomorphisms on A_0 and B_0 respectively. We identify the products $\prod_{l=0}^2 A_l$ and $\prod_{l=0}^1 B_l$ with $(A_0)^3$ and $(B_0)^2$ respectively through fixed isomorphisms $A_l \xrightarrow{\approx} A_0$ and $B_l \xrightarrow{\approx} B_0$.

Let us fix some terminology. (1) If R is a ring, let $\Delta_n(R)$ be the subspace of the ring of $n \times n$-matrices $M_n(R)$ with entries in R consisting of all the diagonal elements. If $\alpha_1, \cdots, \alpha_n \in R$, let $\Delta(\alpha_1, \cdots, \alpha_n)$ be the matrix $(\alpha_{i,j})$ in $\Delta_n(R)$ with $\alpha_{i,j} = \delta_{i,j}\alpha_j$.

(2) If X is an abelian variety, we associate to an endomorphism ϕ of X^n, the matrix U_ϕ in $M_n(\mathrm{End}(X))$, if on points, $\phi\colon \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} \to U_\phi \cdot \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}$.

(3) Let $\phi\colon X \to Y$ be an isogeny of degree N. Let $\bar{\phi}\colon Y \to X$ be such that $\bar{\phi}\phi$ is multiplication by N on X. Let $F_\phi\colon \mathrm{End}^0(X) \to \mathrm{End}^0(Y)$ map α in $\mathrm{End}(X)$ to $N^{-1}(\phi\alpha\bar{\phi})$ in $\mathrm{End}^0(Y)$.

§ 2. Rational endomorphisms

Let Σ_l be a basis for $H^0(A_l, \Omega^1)$ consisting of forms of the type $(1 + g_l + g_l^2)^* w_{r,s}$. Then $\Sigma = \bigcup_{l=0}^2 \Sigma_l$ is a basis for $H^0(A, \Omega^1)$. The main result in this section is

PROPOSITION 2.1. *Let $m \in \mathscr{E}$. Then the following sequences are exact*:

$$0 \longrightarrow (f_m(\sigma)) \longrightarrow \mathbf{Q}[\sigma, \rho] \longrightarrow \mathrm{End}^0(A) \longrightarrow 0\,,$$
$$0 \longrightarrow (f_m(\sigma)) \longrightarrow \mathbf{Q}[\sigma, \iota] \longrightarrow \mathrm{End}^0(B) \longrightarrow 0\,.$$

Proof. We will prove that $F\colon \mathbf{Q}[\sigma, \rho] \to \mathrm{End}^0(A_0^3) = M_3(L)$ is surjective. Since $f_m(\sigma) \in \mathrm{Ker}(F)$, a dimension argument shows that the first sequence is exact. We omit the proof of exactness of the second sequence.

The matrices for $(1 + g_l + g_l^2)^*$ on $H^0(A, \Omega^1)$, with respect to the basis Σ are:

$$\begin{pmatrix} 3 & 0 & 0 \\ M_0 & 0 & 0 \\ N_0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 0 \\ 0 & M_1 & 0 \\ 0 & N_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & M_2 \\ 0 & 0 & N_2 \end{pmatrix}$$

for $l = 0, 1, 2$ respectively.

Now $w_{1,r} \in H^0(A, \Omega^1)$ and

$$(1 + g_0 + g_0^2)^*(1 + g_1 + g_1^2)^* w_{1,r} = (1 + \zeta^{r^2+1} + \zeta^{r^2+2})(1 + g_0 + g_0^2)^* w_{1,r}\,.$$

Let $l \in (\mathbf{Z}/m\mathbf{Z})^* - \{1, (r^2+1)(r^2+2)^{-1}, (r^2+1)(r^2+2)^{-1}\}$. Since $\{\zeta^a \mid a \in (\mathbf{Z}/m\mathbf{Z})^*\}$ is a $\mathbf{Z}$-basis for $\mathcal{O}_K$, ζ^{r^2+1}, $\zeta^{(r^2+1)l}$, $\zeta^{(r^2+2)}$, $\zeta^{(r^2+2)l}$ are linearly independent over $\mathbf{Q}$. Thus $\zeta^{r^2+1} + \zeta^{r^2+2}$ is not in $\mathbf{Q}$, and $1 + \zeta^{r^2+1} + \zeta^{r^2+2} \neq 0$. This shows that the matrix M_0 is not the null matrix. In a similar way, we can prove that N_0, M_1, N_1, M_2 and N_2 are not zero. Then, in $\mathrm{End}(A_0^3) = M_3(\mathcal{O}_L)$, the matrices for $(1 + g_l + g_l^2)$ are:

$$\begin{pmatrix} 3 & 0 & 0 \\ \alpha_0 & 0 & 0 \\ \beta_0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & \beta_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & \alpha_2 \\ 0 & 0 & \beta_2 \end{pmatrix}$$

for $l = 0, 1, 2$ respectively, where each α_j, β_j are in $\mathcal{O}_L$.

Let $X, Y, Z \in \mathbf{Q}[\sigma]$. In the group ring $\mathbf{Q}[\sigma, \rho]$, we have the following:

$$(1 + g_l + g_l^2)(X + \rho Y + \rho^2 Z) = (1 + g_l + g_l^2)(X + Y\sigma^{l(1-r^2)} + Z\sigma^{l(1-r)})$$

by using the relations $\rho\sigma\rho^{-1} = \sigma^r$ and $\rho^{-1}\sigma\rho = \sigma^{r^2}$ in $\mathrm{Aut}(A)$.

The determinant of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^{1-r^2} & \sigma^{1-r} \\ 1 & \sigma^{2-2r^2} & \sigma^{2-2r} \end{pmatrix}$ is $D = f(\sigma) \in \mathbf{Q}[\sigma]$, where

$$f(x) = x^{\langle 4-r\rangle} - x^{\langle 4-r^2\rangle} + x^{\langle 1-r\rangle} - x^{\langle 1-r^2\rangle} + x^{\langle 2-2r\rangle} - x^{\langle 2-2r^2\rangle} \in \mathbf{Q}[x]\,.$$

Since $r^2 + r + 1 \equiv 0 \pmod m$, the exponents $4 - r$, $4 - r^2$, $1 - r$, $1 - r^2$, $2 - 2r$, $2 - 2r^2$ are pairwise distinct $(\mathrm{mod}\, m)$ except possibly when $m \mid 3^2$

or $m = 13$. Hence, $D \neq 0$ (the exceptional case $m = 13$ is taken care of by inspection). In particular, there are $X, Y, Z \in \mathbf{Z}[\sigma]$ and a positive integer N such that

$$X + Y + Z = ND, \quad X + Y\sigma^{1-r^2} + Z\sigma^{1-r} = 0, \quad X + Y\sigma^{2-2r^2} + Z\sigma^{2-2r} = 0.$$

With the latter choice of X, Y and Z, let the matrix of $(X + \rho Y + \rho^2 Z)$ in $M_3(\mathcal{O}_L)$ be $(\alpha_{i,j})$. From $(1 + g_1 + g_1^2)(X + \rho Y + \rho^2 Z) = 0$, we conclude that $\alpha_{2,j} = 0$ for all j. On the other hand, $\alpha_{3,j} = 0$ for all j, follows from $(1 + g_2 + g_2^2)(X + \rho Y + \rho^2 Z) = 0$. Then the matrix of $(X + \rho Y + \rho^2 Z)(1 + g_0 + g_0^2)$ is

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 & 0 \\ \alpha_0 & 0 & 0 \\ \beta_0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \delta_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\delta_0 = 3\alpha_{1,1} + \alpha_0\alpha_{1,1} + \beta_0\alpha_{1,3} \in \mathcal{O}_L$.

CLAIM. $\delta_0 \neq 0$.

Suppose, on the contrary, that $\delta_0 = 0$. Then

$$\begin{aligned} N^{-1}(1 + g_0 + g_0^2)(X + \rho Y + \rho^2 Z)(1 + g_0 + g_0^2) \\ = (1 + \rho + \rho^2)D(1 + g_0 + g_0^2) = 0. \end{aligned}$$

We note that

$$D^*(1 + \rho + \rho^2)^* w_{1,r} = f(\zeta)w_{1,r} + f(\zeta^r)w_{r,m-r-1} + f(\zeta^{r^2})w_{m-r-1,1}$$

and if $\lambda_m = f(\zeta^r) + f(\zeta^r) + f(\zeta^{r^2})$,

$$(1 + g_0 + g_0^2)^* D^*(1 + \rho + \rho^2)^* = \lambda_m(w_{1,r} + w_{r,m-r-1} + w_{m-r-1,1}).$$

We will show that $\lambda_m \neq 0$.

First, consider the prime case $m = p$. If $\lambda_p = 0$, then the polynomial $g(x) = f(x) + f_1(x) + f_2(x)$, where $f_j(x)$ is the polynomial obtained by replacing each exponent $\langle a \rangle$ in $f(x)$ by $\langle ar^j \rangle$, has degree at most $p - 1$, and ζ as a root. We note that $4 - r$, $4 - r^2$, $1 - r$, $1 - r^2$, $2 - 2r$, $2 - 2r^2$ are distinct elements in $(\mathbf{Z}/p\mathbf{Z})^*/\{1, r, r^2\}$ for $p \neq 7, 19, 31$. Thus, with the above exceptions, $g(x) \neq 0$ and therefore, $g(x) = \pm f_p(x)$. This is a contradiction, since $g(1) = 0$ but $f_p(1) = p$. Inspection shows that $\lambda_p \neq 0$ for $p = 7, 19, 31$.

Now we treat the composite case.

Suppose that l is a prime divisor of m and $r - 4$. Then $r^2 + r + 1 \equiv 0 \pmod{l^k}$ and $r \equiv 4 \pmod{l^k}$ imply that $l^k | 21$. Thus $(m, r^2 - 4) | 21$.

Similarly $(m, r-4) \mid 21$. However, 7 can divide at most one of the two numbers $(m, r-4)$ and $(m, r^2-4) = (m, m-r-5)$. Furthermore, it is not difficult to verify that $(1-r, m) = (1-r^2, m) \mid 3$.

Case (1). First suppose that each of the integers $1-r$, $1-r^2$, $2-2r$, $2-2r^2$ are relatively prime to m (this is the case when $(m, 6)=1$).

Case (1a). Both $(m, r-4)$ and (m, r^2-4) are co-prime to 7.

For $\beta \in K = \mathbf{Q}(\mu_m)$, let $\beta^{1+r+r^2} = \beta + \beta^r + \beta^{r^2}$, where $\{1, r, r^2\} \subseteq \mathrm{Gal}(K/\mathbf{Q})$. We note that if two of the integers $4-r$, $4-r^2$, $1-r$, $1-r^2$, $2-2r$, $2-2r^2$ represent the same class in $(\mathbf{Z}/m\mathbf{Z})^*/\{1, r, r^2\}$, then $m \in S$, where S is a finite set of integers whose elements can be easily found using the congruence relation $r^2+r+1 \equiv 0 \pmod{m}$. If $m \in S \cap \mathscr{E}$, inspection shows that $\lambda_m \neq 0$. If m is not in S, a $\mathbf{Z}$-basis for $\mathcal{O}_L$ is

$$\{\zeta^{a(1+r+r^2)} \mid a \in (\mathbf{Z}/m\mathbf{Z})^*/\{1, r, r^2\}\},$$

and we conclude that λ_m is non-zero since it is a linear combination of elements of a subset of a $\mathbf{Z}$-basis for $\mathcal{O}_L$.

Case (1b). $7 \| (m, r^2-4)$.

The elements of $\mathrm{Gal}(K/\mathbf{Q})$ which fix $\mathbf{Q}(\zeta^7)$ elementwise are the units $j \in (Z/m\mathbf{Z})^*$ such that $j \equiv 1 \pmod{m/7}$. We fix one such $j = 1 + k(m/7) \neq 1$ in $(\mathbf{Z}/m\mathbf{Z})^*$. We make the following observation: if a and bj are equal in $(\mathbf{Z}/m\mathbf{Z})^*/\{1, r, r^2\}$, then $a \equiv r^l bj \pmod{m}$ implies $a \equiv r^l b \pmod{m/7}$, and so a and b are equal in $(\mathbf{Z}/(m/7)\mathbf{Z})^*/\{1, r, r^2\}$.

The calculations for case (1a) show that $1-r$, $1-r^2$, $2-2r$, $2-2r^2$, $4-r$ are distinct in $(\mathbf{Z}/m\mathbf{Z})^*/\{1, r, r^2\}$ (hence in $(\mathbf{Z}/(m/7)\mathbf{Z})^*/\{1, r, r^2\}$), except possibly when $m/7 \in S$. For these exceptional values of m, $\lambda_m \neq 0$ by inspection. For the other values of m, the observation in the previous paragraph shows that $\bar{\lambda}_m = \lambda_m - \zeta^{(4-r^2)(1+r+r^2)}$ is such that $\bar{\lambda}_m^j \neq \bar{\lambda}_m$, since $\{\zeta^{a(1+r+r^2)} \mid a \in (\mathbf{Z}/m\mathbf{Z})^*/\{1, r, r^2\}\}$ is a $\mathbf{Z}$-basis for $\mathcal{O}_L$. Thus $\bar{\lambda}_m \notin \mathbf{Q}(\zeta^7)$, and $\lambda_m \neq 0$.

Case (1c). $7 \| (m, r-4)$.

This is case (1b), with the roles of r and r^2 reversed.

Case (2). Suppose now that $(1-r, m) = 3$. If m is odd, then we have that

$$(1-r, m) = (1-r^2, m) = (2-2r, m) = (2-2r^2, m) = 3$$
$$\text{and} \quad 9 \mid (4-r, m) \cdot (4-r^2, m) \mid 9.7 .$$

We apply the arguments in case (1) applied to $(1-r)/3$, $(1-r^2)/3$, $(2-2r)/3$, $(2-2r^2)/3$, $(4-r)/3$, $(4-r^2)/3$ in $(\mathbf{Z}/(m/3)\mathbf{Z})^*/\{1, r, r^2\}$.

If m is even, we look at $(1-r)/3$, $(1-r^2)/3$, $(2-2r)/6$, $(2-2r^2)/6$, $(4-r)/3$, $(4-r^2)/3$ instead. The calculations are similar to the ones above.

This proves that $\lambda_m \neq 0$, and hence our claim that $\delta_0 \neq 0$. We have shown that $F((X+\rho Y+\rho^2 Z)(1+g_0+g_0^2)) = \Delta(\delta_0, 0, 0)$, with $\delta_0 \neq 0$. Similarly, we can show the existence of $X_l, Y_l, Z_l \in \mathbf{Z}[\sigma]$ such that $(X_l+\rho Y_l+\rho^2 Z_l)(1+g_l+g_l^2)$ are mapped onto

$$\Delta(0, \delta_1, 0) \quad \text{and} \quad \Delta(0, 0, \delta_2) \qquad \text{for } l = 1, 2 \text{ respectively.}$$

In particular, since $L \to \mathrm{End}^0(A_0^3) = M_3(L)$ (in which ζ^{1+r+r^2} is mapped to $(\sigma+\sigma^r+\sigma^{r^2})^3)$ is the diagonal embedding by the theory of complex multiplication, we conclude that

$$\Delta_3(L) \subseteq \mathrm{Im}(F) \subseteq M_3(L).$$

We observe that

$$\sigma^*(\sigma(1+\rho+\rho^2)\sigma^{-1})^* w_{a,b} = (1+\rho+\rho^2)^*\sigma^* w_{a,b}, \quad \text{and}$$
$$\sigma^*(\sigma^2(1+\rho+\rho^2)\sigma^{-2})^* w_{a,b} = (\sigma(1+\rho+\rho^2)\sigma^{-1})^*\sigma^* w_{a,b}.$$

Thus the matrix for σ in $M_3(\mathcal{O}_L)$ is of the form: $\begin{pmatrix} a & b & c \\ d & 0 & 0 \\ 0 & e & 0 \end{pmatrix}$, for some a, b, c, d and e in $\mathcal{O}_L$ with $cde \in (\mathcal{O}_L)^*$ (this follows from $\det(\sigma)^m = 1$). Therefore the image of F contains the following matrices:

$$\begin{pmatrix} 0 & b & c \\ d & 0 & 0 \\ 0 & e & 0 \end{pmatrix}, \quad \begin{pmatrix} bd & ce & 0 \\ 0 & bd & cd \\ de & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & c \\ d & 0 & 0 \\ 0 & e & 0 \end{pmatrix}^2, \quad \begin{pmatrix} 0 & ce & 0 \\ 0 & 0 & cd \\ de & 0 & 0 \end{pmatrix}.$$

This completes the proof that F is surjective. □

§ 3. Homology groups

Let I: $[0, 1] \to F_m(\mathbf{C})$ denote the one-simplex

$$I(t) = (t^{1/m}, (1-t)^{1/m}, \alpha), \qquad t \in [0, 1],$$

where $\alpha = -1$ if m is odd and a primitive $2m$-th root of unity if m is even. Let g be the one-cycle:

$$g = (\sigma\tau)^{(m-1)/2}(1-\sigma)(1-\tau)I \qquad \text{if } m \text{ is odd, and}$$
$$g = (1-\sigma^{-1})(1-\tau^{-1})I \qquad \text{if } m \text{ is even.}$$

The homology group $H_1(F_m(\mathbf{C}), \mathbf{Z})$ is generated by g [11]. Moreover by the period calculations in [11], we have that $\rho(g) = g$ and $\iota(g) = -g$ [9].

PROPOSITION 3.1. *$H_1(F_m(\mathbf{C}), \mathbf{Z})$ is a cyclic $\mathbf{Z}[G_m]$-module, with g as a generator such that $\rho(g) = g$ and $\iota(g) = -g$ in homology.*

For the rest of this paper, let p be a fixed prime congruent to 1 (mod 6), let r be a fixed cube root of unity modulo p, $K = \mathbf{Q}(\mu_p)$, ζ be a fixed p-th root of unity, and A be the Jacobian variety of the curve F_A:

$$y^p = x(1-x)^r .$$

A has CM by $\mathcal{O}_K$: we fix the embedding

$$\mathcal{O}_K \longrightarrow \mathrm{End}_K(A), \qquad \zeta \longrightarrow \sigma = (\zeta, 1, 1) .$$

Let $\varphi_A\colon F_p \to F_A$ denote the canonical projection, and let I_A be the one simplex $\varphi_A I$ on F_A. Fix a base point e_0 in $F_p(\mathbf{C})$, and let x_0 be its image in $F_A(\mathbf{C})$ under φ_A. The cyclic covering φ_A gives rise to a monomorphism

$$H = \pi_1(F_p(\mathbf{C}), e_0) \longrightarrow \pi_1(F_A(\mathbf{C}), x_0) = G$$

of fundamental groups. G/H is a cyclic group of order p since φ_A has degree p. So H contains the commutator subgroup of G, and the homomorphism

$$H_1(F_p) = H_1(F_p(\mathbf{C}), \mathbf{Z}) \longrightarrow H_1(F_A(\mathbf{C}), \mathbf{Z}) = H_1(F_A)$$

factors as follows:

$$\begin{tikzcd} H/[H,H] \arrow[rr] & & G/[G,G] \\ & H/[G,G] \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Thus, the index of the image T of $H_1(F_p)$ in $H_1(F_A)$ is p. T, by definition, is a cyclic $\mathbf{Z}[\sigma]$-module with $(\sigma - 1)(\sigma^r - 1)I_A$ as a generator by Proposition 3.1.

Let $\bar{T}$ be the $\mathbf{Z}[\sigma]$-submodule of $H_1(F_A)$ generated by $\alpha = (\sigma - 1)I_A$. Then $T \subseteq \bar{T} \subseteq H_1(F_A)$. We claim that $T \neq \bar{T}$, from which it follows that $H_1(F_A) = \bar{T}$.

Identifying

$$\mathbf{Q}[\sigma]/(f_p(\sigma)) \xrightarrow{\approx} K, \quad \sigma \longrightarrow \zeta ,$$

$H_1(F_A)\otimes\mathbf{Q}$ is a vector space over K. Hence the annihilator of $H_1(F_A)\otimes\mathbf{Q}$ as a $\mathbf{Q}[\sigma]$-module is $(f_p(\sigma))$, and the annihilator of $H_1(F_A)$, as a $\mathbf{Z}[\sigma]$-module is

$$(f_p(\sigma))\mathbf{Q}[\sigma]\cap\mathbf{Z}[\sigma]=(f_p(\sigma))\mathbf{Z}[\sigma].$$

Since $H_1(F_A)$ is torsion-free over $\mathbf{Z}$, and $[H_1(F_A)\colon \bar{T}]<\infty$, $\mathrm{Ann}_{\mathbf{Z}[\sigma]}(\bar{T})=(f_p(\sigma))\mathbf{Z}[\sigma]$.

Suppose, on the contrary, that $T=\bar{T}$. Then $\alpha=a(\sigma)(\sigma-1)\alpha$ for some $a(x)\in\mathbf{Z}[x]$. Therefore, $(a(\sigma)(\sigma-1)-1)\alpha=0$ implies $a(x)(x-1)-1=b(x)f_p(x)$ for some $b(x)\in\mathbf{Z}[x]$. Then $-1=b(1)p$ in $\mathbf{Z}$, a contradiction. Thus, $H_1(F_A)=\bar{T}$.

Let $\bar{I}=\rho I$ and $\bar{I}_A=\varphi_A\bar{I}$. From $\rho(g)=g$ in $H_1(F_p)$, we obtain

$$(\sigma-1)(\sigma^r-1)I_A=\sigma^{1+r((p+1)/2)}(\sigma^r-1)(\sigma^{p-r-1}-1)\bar{I}_A$$

in $H_1(F_A)$.

Let $v\in H_1(F_A)$ be such that $(\sigma^r-1)v=0$. Passing to $\mathcal{O}_K\subseteq\mathrm{End}_K(A)$, we have $(\zeta^r-1)v=0$. Then $pv=\pm N_{\mathbf{Q}}^K(\zeta^r-1)v=0$, and $v=0$. Thus, we have proved

PROPOSITION 3.2. *$H_1(F_A)$ is a cyclic $\mathcal{O}_K$-module with $g_A=(1-\sigma)I_A$ as a generator. Moreover,*

$$\rho(g_A)=\zeta^{r((p-1)/2)}\left(\frac{\zeta^r-1}{\zeta^{r^2}-1}\right)g_A.$$

§ 4. Endomorphisms

In the present section, we prove the following theorem. Let $\pi=\zeta-1\in\mathbf{Z}[\zeta]\subseteq\mathrm{End}(A)$ and $W=p^{-1}(1+r\rho+r^2\rho^2)(\sigma-1)^{p-3}\in\mathbf{Q}[\sigma,\rho]$.

THEOREM 4.1. $\mathrm{End}(A)=\mathrm{Im}(\mathbf{Z}[\sigma,\rho,W])$ *has group index* p^3 *over* $\mathrm{Im}(\mathbf{Z}[\sigma,\rho])$.

Proof. By Proposition 2.1, $F\colon Q[\sigma,\rho]\to\mathrm{End}^0(A)$ is surjective, and by Proposition 3.2, $H_1(F_A)$ is a cyclic $\mathbf{Z}[\zeta]$-module with a generator g_A such that $\rho(g_A)=\eta g_A$, $\rho^2(g_A)=\xi g_A$, where

$$\eta=\zeta^{r((p-1)/2)-1}\frac{(\zeta^r-1)}{(\zeta^{r^2}-1)}\quad\text{and}\quad\xi=\zeta^{r^2+(p+1)/2}\frac{(\zeta^r-1)}{(\zeta-1)}.$$

We will use the following to determine $\mathrm{End}(A)$:

$$\mathrm{End}(A)=\{\alpha\in\mathrm{End}^0(A)\mid\alpha(H_1(F_A))\subseteq H_1(F_A)\}.$$

Let $X, Y, Z \in K$. Then $\alpha = X + Y\rho + Z\rho^2 \in \mathrm{End}(A)$ if and only if $\alpha(\zeta^a g_A) \subseteq H_1(F_A)$ for all $a \in \mathbf{Z}$, or equivalently, for all $a \in \mathbf{Z}$,

$$X\zeta^a + Y\zeta^{ar}\eta + Z\zeta^{-a(r+1)}\xi \in \mathbf{Z}[\zeta]. \tag{4.1}$$

Let $\tilde{X} = X$, $\tilde{Y} = Y\eta$ and $\tilde{Z} = Z\xi$. Then (4.1) reads as

$$\tilde{X}\zeta^a + \tilde{Y}\zeta^{ar} + \tilde{Z}\zeta^{-a(r+1)} \in \mathbf{Z}[\zeta]. \tag{4.2}$$

Using $\tilde{X} + \tilde{Y} + \tilde{Z} \in \mathbf{Z}[\zeta]$ and (4.2) to eliminate $\tilde{X}$, we obtain for all $a \in (\mathbf{Z}/p\mathbf{Z})^*$,

$$\tilde{Y}(\zeta^{ar} - \zeta^a) + \tilde{Z}(\zeta^{-a(r+1)} - \zeta^a) \in \mathbf{Z}[\zeta]. \tag{4.3}$$

For such a, $\zeta^{ar} - \zeta^a$ and $\zeta^{-a(r+1)} - \zeta^a$ are elements of the ideal (π) of $\mathbf{Z}[\zeta]$.

Let $D_{a,b}$ be the determinant of the following matrix:

$$\begin{pmatrix} \zeta^{ar} - \zeta^a & \zeta^{-a(r+1)} - \zeta^a \\ \zeta^{br} - \zeta^b & \zeta^{-b(r+1)} - \zeta^b \end{pmatrix}.$$

Then

$$D_{a,b} = \{\zeta^{ar-b(r+1)} + \zeta^{br+a} + \zeta^{b-a(r+1)}\} - \{\zeta^{ar+b} + \zeta^{a-b(r+1)} + \zeta^{br-a(r+1)}\},$$

and (4.3) implies that

$$D_{a,b}\tilde{Y}, \qquad D_{a,b}\tilde{Z} \in (\pi) \tag{4.4}$$

for all $a, b \in (\mathbf{Z}/p\mathbf{Z})^*$.

If we set $(a, b) = (r+1, 1)$ and $(a, b) = (1, -r)$ in (4.4), we obtain, after simplification,

$$(\zeta^{3r+3} + \zeta^3 + 1 - 3\zeta^{r+2})\tilde{Z} \in (\pi) \quad \text{and} \quad (\zeta^{3r+3} + \zeta^{3r} + 1 - 3\zeta^{2r+1})\tilde{Z} \in (\pi)$$

respectively. By subtracting one from the other, we obtain

$$\zeta^3(\zeta^{r-1} - 1)^2\tilde{Z} \in (\pi).$$

Since $(p, r-1) = 1$, $\pi^2\tilde{Z} \in \mathbf{Z}[\zeta]$. By symmetry, $\pi^2\tilde{Y} \in \mathbf{Z}[\zeta]$.

We write $Y_0 = \tilde{Y}\pi^2$ and $Z_0 = \tilde{Z}\pi^2$. Then $Y_0, Z_0 \in \mathbf{Z}[\zeta]$, and (4.3) can be rewritten as

$$Y_0\frac{(\zeta^r - \zeta)^h}{(\zeta - 1)^2} + Z_0\frac{(\zeta^{-(r+1)} - \zeta)^h}{(\zeta - 1)^2} \in \mathbf{Z}[\zeta],$$

where h ranges over $H = \mathrm{Gal}(K/\mathbf{Q})$, or equivalently,

$$Y_0 + \varepsilon_h \cdot Z_0 \in (\pi) \qquad \text{for all } h \in H, \tag{4.5}$$

where

$$\varepsilon_h = \frac{(\zeta^{r^2} - \zeta)^h}{(\zeta^r - \zeta)^h} = \left(\sum_{j=0}^{r} \zeta^{j(r-1)}\right)^h \in (\mathbf{Z}[\zeta])^* .$$

Clearly, (4.5) may be rewritten as

$$Y_0 \equiv r^2 Z_0 \pmod{\pi}. \tag{4.6}$$

We have proved that $\alpha = X + Y\rho + Z\rho^2$ is in $\mathrm{End}(A)$ if and only if
(*) $X + \eta Y + \xi Z \in \mathbf{Z}[\zeta]$, and
(**) $Y_0 \equiv r^2 Z_0 \pmod{\pi}$, where $Y_0 = \pi^2 \eta Y$ and $Z_0 = \pi^2 \xi Z$.
We write

$$Y_0 \equiv a_0 + a_1\pi \pmod{\pi^2}, \qquad Z_0 \equiv b_0 + b_1\pi \pmod{\pi^2},$$

where $a_0, a_1, b_0, b_1 \in \mathbf{Z}$. By (**), $a_0 \equiv r^2 b_0 \pmod{p}$. Thus, we find that α is congruent to

$$b_0 \frac{1}{\pi^2}\{-(r^2+1) + r^2\eta^{-1}\rho + \xi^{-1}\rho^2\} + a_0 \frac{1}{\pi}(-1 + \eta^{-1}\rho) + b_1 \frac{1}{\pi}(-1 + \xi^{-1}\rho^2) \tag{4.7}$$

modulo $\mathrm{Im}(\mathbf{Z}[\sigma, \rho])$.

By inspection,

$$v_0 = \frac{1}{\pi^2}\{-(r^2+1) + r^2\eta^{-1}\rho + \xi^{-1}\rho^2\},$$

$$v_1 = \frac{1}{\pi}(-1 + \eta^{-1}\rho), \qquad v_2 = \frac{1}{\pi}(-1 + \xi^{-1}\rho^2)$$

satisfy (*) and (**). Hence, they are in $\mathrm{End}(A)$, and we conclude that

$$\mathrm{End}(A) = \mathrm{Im}(\mathbf{Z}[\sigma, \rho]) + Zv_0 + Zv_1 + Zv_2 . \tag{4.8}$$

From (4.8), the quotient group

$$Q = \mathrm{End}(A)/\Lambda \qquad \text{where } \Lambda = \mathrm{Im}(\mathbf{Z}[\sigma, \rho])$$

is an elementary p-abelian group. So Q is an $\mathbf{F}_p$-vector space, and $\dim_{\mathbf{F}_p}(Q) \leq 3$.

The theorem follows from the next few lemmas. □

LEMMA 4.2. *Let*

$$w = (1 + r\rho + r^2\rho^2)\frac{1}{\pi^2} = \frac{1}{\pi^2} + \frac{r}{(\zeta^r - 1)^2}\rho + \frac{r^2}{(\zeta^{r^2} - 1)^2}\rho^2 \in \mathrm{End}^0(A).$$

Then $w \in \mathrm{End}(A)$.

Proof. We verify (**) for w. We have $Y_0 = (r\pi^2\eta)/(\zeta^r - 1)^2$ and $Z_0 = (r^2\pi^2\xi)/(\zeta^{r^2} - 1)^2$ in the notation of the proof of Theorem 4.1. Since

$$Y_0 \equiv r\zeta^{r(p-1)/2-1}\frac{(\zeta - 1)}{(\zeta^r - 1)}\frac{(\zeta - 1)}{(\zeta^{r^2} - 1)} \equiv r \pmod{\pi}$$

and

$$Z_0 \equiv r^2\zeta^{r^2+(p+1)/2}\frac{(\zeta - 1)}{(\zeta^{r^2} - 1)}\frac{(\zeta^r - 1)}{(\zeta^{r^2} - 1)} \equiv r^2 \pmod{\pi},$$

we have $Y_0 \equiv r^2 Z_0 \pmod{\pi}$. Likewise, (*) can be verified for w. This completes the proof of the lemma. □

LEMMA 4.3. *Let* $\Sigma = \mathrm{Im}(\mathbf{Z}[\sigma, \rho, W])$. *Then* $\Sigma \subseteq \mathrm{End}(A)$, *and the following are elements of* Σ:

$$w,\ w_0 = \{1 + (r + 1)\rho\}\frac{1}{\pi}, \quad w_1 = (r\rho - \rho^2)\frac{1}{\pi}.$$

Proof. Let $u \in (\mathbf{Z}[\zeta])^*$ be the endomorphism of A such that $p = u\pi^{p-1}$. As an element of $\mathrm{End}^0(A)$, $W = wu^{-1}$. Hence the image of w is in Σ, and $\Sigma \subseteq \mathrm{End}(A)$.

From $w\sigma = (\sigma + r\sigma^r\rho + r^2\sigma^{r^2}\rho^2)1/\pi^2$ and $\sigma w = (\sigma + r\sigma\rho + r^2\sigma\rho^2)1/\pi^2$, we have

$$\sigma w - w\sigma \equiv (r - 1)\rho\{1 + (r + 1)\rho\}\frac{1}{\pi} \pmod{\Lambda}.$$

Since p does not divide $r - 1$ and $\rho \in \mathrm{Aut}(A)$, there is a $\lambda \in \mathbf{Z}$ such that

$$\{1 + (r + 1)\rho\}\frac{1}{\pi} \equiv \lambda\rho^2(\sigma w - w\sigma) \pmod{\Lambda}.$$

Hence, $w_0 \in \Sigma$. Since $w_1 \equiv r\rho w_0 \pmod{\Lambda}$, we have $w_1 \in \Sigma$ also. □

LEMMA 4.4. *The mapping* f: $(\mathbf{Z}[\zeta])^3 \to \Lambda$, $(X, Y, Z) \to X + \rho Y + \rho^2 Z$ *is a right* $\mathbf{Z}[\zeta]$*-module isomorphism.*

Proof. By definition, f is surjective. By Proposition 2.1, $f \otimes 1$: $K^3 = (\mathbf{Q}(\mu_p))^3 \to \Lambda \otimes \mathbf{Q}$ is an isomorphism. Hence f is injective. □

LEMMA 4.5. *Let* V *be the subspace of* Q *spanned by* w, w_0 *and* w_1. *Then* $\dim_{\mathbf{F}_p}(V) = 3$.

Proof. Let $\lambda, \lambda_0, \lambda_1 \in \mathbf{Z}$ be such that

$$\lambda w + \lambda_0 w_0 + \lambda_1 w_1 \in \Lambda . \tag{4.9}$$

Multiplying by π on the right, $\lambda(1 + r\rho + r^2\rho^2) \in \pi\Lambda$. Using Lemma 4.4, $\lambda/\pi \in \mathbf{Z}[\zeta]$. Hence $\lambda \in (\pi) \cap \mathbf{Z} = p\mathbf{Z}$. Since $p/\pi^2 \in \mathbf{Z}[\zeta]$, we have

$$\lambda_0 w_0 + \lambda_1 w_1 \in \Lambda . \tag{4.10}$$

Another application of Lemma 4.4 to (4.10) gives $\lambda_0, \lambda_1 \in p\mathbf{Z}$. Therefore $\{w, w_0, w_1\}$ is an $\mathbf{F}_p$-basis for V. □

Combining Lemmas 4.3 and 4.5,

$$\dim_{\mathbf{F}_p}(\Sigma/\Lambda) \geq 3 .$$

Since $\dim_{\mathbf{F}_p}(Q) \leq 3$, we have the desired equality: $\mathrm{End}(A) = \Sigma$, and $\mathrm{End}(A)$ has group index p^3 over Λ. This completes the proof of Theorem 4.1.

COROLLARY 4.6. *A free* $\mathbf{Z}$*-basis for* $\mathrm{End}(A)$ *is given by*:

$$\{\rho^j\pi^k \mid 0 \leq j \leq 2,\ 0 \leq k \leq p-4\} \cup \{\rho\pi^{p-3}, \rho^2\pi^{p-3}, \rho\pi^{p-2}\} \cup \{w, w_0, w_1\} .$$

Proof. Let M be the $\mathbf{Z}$-submodule of $\mathrm{End}(A)$ spanned by the above elements. Inspection shows that $\Lambda \subseteq M$. By Lemma 4.5, the corollary follows. □

Remarks. Let k be a proper subfield of K, and let h be a generator of $\mathrm{Gal}(K/k) \subseteq (\mathbf{Z}/p\mathbf{Z})^*$. Then the subring of endomorphisms of A defined over k is

$$\mathrm{End}(A) = \mathrm{Im}\left(\mathbf{Z}\left[\sum_{j=1}^{t-1} \sigma^{ah^j}, \rho \mid a \in \mathbf{Z}\right]\right),$$

where t is the order of h. $\mathrm{End}_k(A)$ is commutative if and only if k is $\mathbf{Q}$ or $L = K^{\langle r \rangle}$. In the latter cases, $\mathrm{End}_k(A)$ are contained in $\mathbf{Z} \times \mathbf{Z}[(1+\sqrt{-3})/2]$ and $\mathcal{O}_K \times \mathcal{O}_{K(\sqrt{-3})}$ respectively.

§ 5. Action of rho on some division points

Let P_1, P_2 and P_3 be any 3 points on F_p where $X = 0$, $Y = 0$ and $Z = 0$ respectively. Recall that φ_A: $F_p \to F_A$ is the canonical projection. Set

$$\infty_2 = \varphi_A(P_1), \quad \infty_3 = \varphi_A(P_2), \quad \text{and} \quad \infty_1 = \varphi_A(P_3) .$$

Then the group of $A[\pi]$ of π-division points on A has order p, and contains all the divisor classes of degree zero supported on the set of cusps $\{\infty_1, \infty_2, \infty_3\}$ of F_A.

For each integer $a \geq 1$,

$$\pi^a \rho = \rho(\zeta^{r^2} - 1)^a = \rho \frac{\zeta^{r^2} - 1}{\zeta - 1} \pi^a$$

in End(A), so that ρ induces an automorphism of $A[\pi^a]$ by restriction.

LEMMA 5.1. *ρ acts on $A[\pi]$ as multiplication by r.*

Proof. Recall that the equation of F_A is $v^p = u(1-u)^r$. The divisor of the rational function v on F_A is $\infty_2 - (r+1)\infty_1 + r\infty_3$. Hence, on A, $\infty_2 - (r+1)\infty_1 + r\infty_3 = 0 = \infty_1 - (r+1)\infty_3 + r\infty_2$ (the latter equality is obtained by applying ρ to the former). In particular,

$$\rho(\infty_1 - \infty_2) = \infty_2 - \infty_3 = (r+1)(\infty_1 - \infty_3) = r(\infty_1 - \infty_2)\,. \qquad \square$$

LEMMA 5.2. *There is an element $Q \in A[\pi^2] - A[\pi]$ such that $\rho(Q) = Q$.*

Proof. Let us fix a Q in $A[\pi^2] - A[\pi]$. Then $A[\pi^2] = \{(a + b\pi)Q \mid a, b \in \mathbf{F}_p\}$ is a vector space of dimension 2 over $\mathbf{F}_p$. Let $f(x)$ be the minimal polynomial of ρ restricted to $A[\pi^2]$. Since ρ has order 3, we have $f(x) \mid (x-1)(x-r)(x-r^2)$ in $\mathbf{F}_p[x]$. Since ρ can have at most two distinct eigenvalues, and $f(x)$ splits completely, we have $f(x) = x - \lambda_1$ or $f(x) = (x-\lambda_1)(x-\lambda_2)$, where $\lambda_1, \lambda_2 \in \{1, r, r^2\}$ and $\lambda_1 \neq \lambda_2$.

Suppose that $f(x) = x - \lambda_1$. Then $\lambda_1(\pi Q) = \rho(\pi Q) = (\zeta^r - 1)\pi Q = \lambda_1\{(\zeta^r - 1)/\pi\}\pi Q = \lambda_1\{r + (r(r-1)/2)\pi + \cdots\}\pi Q = \lambda_1 r(\pi Q)$, whence $\lambda_1 = \lambda_1 r$ and $\lambda_1 = 0$, a contradiction. Hence, $f(x) = (x-\lambda_1)(x-\lambda_2)$, and there is an $\mathbf{F}_p$-basis Q_1, Q_2 of $A[\pi^2]$ such that the matrix of ρ with respect to $\{Q_1, Q_2\}$ is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Since at least one of Q_1, Q_2 is not in $A[\pi]$, we have found a Q in $A[\pi^2] - A[\pi]$ and a $\lambda \in \{1, r, r^2\}$ such that $\rho(Q) = \lambda Q$. By Lemma 5.1, $r(\pi Q) = \rho(\pi Q) = \lambda r(\pi Q)$, and $\lambda = 1$. This completes the proof of the lemma. $\square$

Remarks. (1) In the same way as above, we can show that there is a $Q \in A[\pi^3] - A[\pi^2]$ such that $\rho(Q) = r^2 Q$. We also remark that the annihilator, in End(A), of $A[\pi]$ is

$$\mathbf{Z}[\zeta]\pi + \mathbf{Z}[\zeta](\rho - r) + \mathbf{Z}[\zeta](\rho^2 - r^2) + \mathbf{Z}(1 + r\rho - (r+1)\rho^2)\frac{1}{\pi}\,.$$

(2) If $^-$ denotes complex conjugation, then for $Q \in A[\pi^2] - A[\pi]$, $\bar{Q} = -Q \Leftrightarrow \rho(Q) = Q$.

§ 6. The kernel of an isogeny

Let $X_j = F_A/\langle \sigma^j \rho \sigma^j \rangle$, $(j = 0, 1, 2)$, and we denote the canonical projection $F_A \to X_j$ by φ_j. Let φ be the isogeny

$$\varphi = \prod_{j=0}^{2} (\varphi_j)_*\colon A \longrightarrow \prod_{j=0}^{2} \mathrm{Jac}\,(X_j)\,.$$

LEMMA 6.1. $\mathrm{Ker}\,(\varphi) \subseteq A[\pi^2]$.

Proof. The composition $A \xrightarrow{(\varphi_j)_*} \mathrm{Jac}\,(X_j) \xrightarrow{(\varphi_j)^*} A$ is $\zeta^j(1 + \rho + \rho^2)\zeta^{-j} \in \mathrm{End}\,(A)$, so that $\mathrm{Ker}\,(\varphi_j)_* \subseteq A[\zeta^j(1 + \rho + \rho^2)\zeta^{-j}]$. Let N be $\bigcap_{j=0}^{2} A[\zeta^j(1 + \rho + \rho^2)\zeta^{-j}]$. Then

$$\mathrm{Ker}\,(\varphi) = \mathrm{Ker}\,(\varphi_0)_* \cap \mathrm{Ker}\,(\varphi_1)_* \cap \mathrm{Ker}\,(\varphi_2)_* \subseteq N\,.$$

We claim that $N \subseteq A[\pi^2]$. Let $D \in N$. Then we have

$$(1 + \rho + \rho^2)D = 0\,, \tag{6.1}$$

$$(1 + \zeta^{1-r}\rho + \zeta^{1-r^2}\rho^2)D = 0\,, \tag{6.2}$$

and

$$(1 + \zeta^{2-2r}\rho + \zeta^{2-2r^2}\rho^2)D = 0\,, \tag{6.3}$$

using the relations $\rho\sigma\rho^{-1} = \sigma^r$ and $\rho^{-1}\sigma\rho = \sigma^{r^2}$ in $\mathrm{Aut}\,(F_A)$. From (6.1) and (6.2), we obtain that

$$\{(\zeta^{1-r^2} - 1) + (\zeta^{1-r^2} - \zeta^{1-r})\rho\}D = 0\,. \tag{6.4}$$

From (6.2) and (6.3),

$$\{(\zeta^{1-r^2} - 1) + (\zeta^{2-r-r^2} - \zeta^{2-2r})\rho\}D = 0\,. \tag{6.5}$$

From (6.4) and (6.5),

$$\zeta^r(1 - \zeta^{1-r})(1 - \zeta^{2r+1})\rho D = \{(\zeta^{1-r^2} - \zeta^{1-r}) - (\zeta^{2-r-r^2} - \zeta^{2-2r})\}\rho D = 0\,.$$

Hence, $\pi^2(\rho D) = 0$ and $\rho((\zeta^{r^2} - 1)/(\zeta - 1))^2\pi^2 D = 0$. Since ρ and $(\zeta^{r^2} - 1)/(\zeta - 1)$ are in $\mathrm{Aut}\,(A)$, we have $\pi^2(D) = 0$. □

THEOREM 6.2. *Let* $N = \bigcap_{j=0}^{2} A[\zeta^j(1 + \rho + \rho^2)\zeta^{-j}]$. *Then we have* $\mathrm{Ker}\,(\varphi) = N = A[\pi]$.

Proof. Under the canonical projection φ_0: $F_A \to X_0 = F_A/\langle\rho\rangle$, ∞_1 and ∞_2 are mapped onto the same point. Thus, $\operatorname{Ker}(\varphi_0)_*$ contains $A[\pi]$. Likewise, $A[\pi]$ is contained in $\operatorname{Ker}(\varphi_j)_*$. Thus

$$A[\pi] \subseteq \operatorname{Ker}(\varphi) \subseteq N \subseteq A[\pi^2]\,.$$

Let $D \in N$. Applying the endomorphism $w = (1 + r\rho + r^2\rho^2)1/\pi^2$ to $\pi^2 D = 0$, we get

$$(1 + r\rho + r^2\rho^2)D = 0\,.$$

Since $(1 + \rho + \rho^2)D = 0$ also, we obtain $\{(r-1)\rho + (r^2-1)\rho^2\}D = 0$ or $(r-1)\rho\{1 + (r+1)\rho\}D = 0$. Since D is a p-division point, $(p, r-1) = 1$ and $\rho \in \operatorname{Aut}(A)$, it follows that $\{1 + (r+1)\rho\}D = 0$ or $(r-\rho)D = r\{1 + (r+1)\rho\}D = 0$. Hence,

$$A[\pi] \subseteq \operatorname{Ker}(\varphi) \subseteq N \subseteq A[\pi^2] \cap A[\rho - r]\,.$$

By Lemmas 5.1 and 5.2, there is a $Q \in A[\pi^2] - A[\pi]$ such that $\rho(Q) = Q$ and $\rho(\pi Q) = r(\pi Q)$. Let $D = (a + b\pi)Q \in A[\rho - r]$, with $a, b \in \mathbf{F}_p$. Then $(a + b\pi)Q = (ar + br\pi)Q$, whence $a = ar$ and $a = 0$. Thus $D \in A[\pi]$ and $A[\pi^2] \cap A[\rho - r] = A[\pi]$. Hence, $\operatorname{Ker}(\varphi) = N = A[\pi]$. $\square$

COROLLARY 6.3. *The isogeny* φ: $A \to \prod_{j=0}^{2} \operatorname{Jac}(X_j)$ *factors as*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\pi"'] & \prod_{j=0}^{2} \operatorname{Jac}(X_j) \\ A \arrow[ur, "f"] & \end{tikzcd},$$

where f: $A \to \prod_{j=0}^{2} \operatorname{Jax}(X_j)$ *is an isomorphism of abelian varieties defined over* K.

Proof. We define an isomorphism f: $A \to \prod_{j=0}^{2} \operatorname{Jac}(X_j)$ of abelian varieties as follows. Given $D \in \operatorname{Pic}^0(F_A)$, let E be such that $\pi E = D$. E exists since π is an isogeny. Then we define $f(D) = \varphi(E)$. f is well-defined and injective by definition. In particular, f is a birational isomorphism of abelian varieties and hence an isomorphism of abelian varieties. $\square$

Let C be the Klein quartic curve over $\mathbf{C}$ with projective equation

$$X^3Y + Y^3Z + Z^3X = 0\,.$$

C has genus 3, $\operatorname{Aut}(C) \approx PSL(2, \mathbf{F}_7)$, and the morphism

$$F^7_{1,2,4} \longrightarrow C, \quad (x, y) \longrightarrow ((x-1)/y^2, \quad -(x-1)/y^3)$$

is a birational isomorphism. Let $\mathrm{Jac}(C)$ be the Jacobian of C. We will denote by σ and ρ the following automorphisms of C:

$$\sigma\colon (x, y) \longrightarrow (\zeta^4 x, \zeta^5 y), \quad \rho\colon (x, y) \longrightarrow (1/y, x/y),$$

where ζ is a primitive 7-th root of unity. Then by Proposition 2.1, we have the epimorphism

$$\mathbf{Q}[\sigma, \rho] \longrightarrow \mathrm{End}^0(\mathrm{Jac}(C)).$$

By Theorem 4.1 and Corollary 6.3, we have

COROLLARY 6.4. *Let* $W = 7^{-1}(1 + r\rho + r^2\rho^2)(\sigma - 1)^4 \in \mathbf{Q}[\sigma, \rho]$, *with* $r = 2$. *Then* $\mathrm{End}(\mathrm{Jac}(C)) = \mathrm{Im}(\mathbf{Z}[\sigma, \rho, W])$ *and* $\mathrm{Jac}(C)$ *is isomorphic to a cube of an elliptic curve* E.

Remarks. (1) From the Weierstrass equation for E computed in [10], we see that E is $J_0(49)$.

(2) As an application of Theorem 4.1, we give a second proof of the following result due to Prapavessi [10]: Let $\infty_1 = (1, 0, 0)$, $\mu_j = \zeta^j + \zeta^{-j}$ $(j \geq 0)$ and let $P = (\mu_1, \mu_3^{-1}, 1)$. Then $D = P + \rho P - 2\infty_1$ generates the kernel of π^3 over $\mathbf{Z}[\zeta]$. Prapavessi showed ([10], Lemma 2.1) that $\pi^3(D) = 0$. It remains to show that $\pi^2(D) \neq 0$. Let $\infty_2 = (0, 1, 0)$ and $\infty_3 = (0, 0, 1)$. Suppose, on the contrary, that $\pi^2(D) = 0$. Applying the endomorphism $(1 - r^2\rho)1/\pi$ of $\mathrm{Jac}(C)$ we obtain $(1 - r^2\rho)\pi D = 0$, or

$$\pi D = r^2\left\{\frac{\zeta^r - 1}{\pi}\right\}\pi\rho D = r^2\left\{r + \frac{r(r-1)}{2}\pi + \cdots\right\}\pi\rho D = \pi\rho D.$$

Since the group of π-division points on $\mathrm{Jac}(C)$ is generated by $\infty_i - \infty_j$ $(i \neq j)$, $\pi(P - \rho^2 P) = 0$ follows from $\pi(D - \rho D) = 0$. Hence there is a non-constant rational function g on C whose divisor is $\pi(P - \rho^2 P)$. In particular, $g\colon C \to \mathbf{P}^1$ is a double covering, and C is a hyperelliptic curve, which is a contradiction. This completes the proof that $\pi^2(D) \neq 0$.

(3) Our knowledge of the endomorphism ring of A allows us to deduce a result of Greenberg [5] for $A = J^p_{1,r,-(1+r)}$. We have noted that $w = (1 + r\rho + r^2\rho^2)1/\pi^2$ is an endomorphism of A which is defined over K. Thus if $D \in A(K)$, then it follows that $w(D) \in A(K)$. Let $Q \in A[\pi^3] - A[\pi^2]$ be such that $\rho(Q) = r^2 Q$. Setting $P = \pi^2 Q$, we have $w(P) = (1 + r\rho + r^2\rho^2)(Q) = 3Q$ is an element of $A(K)$. Let $\lambda, \mu \in \mathbf{Z}$ be such that $3\mu + p\lambda$

$= 1$. Then $Q = 3\mu Q \in A(K)$. Since $A[\pi^3]$ is a cyclic $\mathbf{Z}[\zeta]$-module with Q as a generator, it follows that $A[\pi^3] \subseteq A(K)$. We also remark that the p-part of $A(K)$ is of the form $A[\pi^{3l}]$ for some $l \geq 1$.

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