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ON SOME GENERALIZATIONS OF THEOREMS OF TODA AND WEISSENBORN TO DIFFERENTIAL POLYNOMIALS

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Dedicated to Professor Niro Yanagihara on his 60th birthday

§ 1. Introduction

We assume that the readers are familiar with the notations in Nevanlinna theory, see [2], [9].

Let f be a nonconstant meromorphic function in the plane. We say that a function $h(r)$, $0 \leqq r \leqq \infty$, *is* $S(r, f)$ if

$$h(r) = o(T(r, f))$$

as $r \to \infty$, possibly outside a set of finite linear measure.

A meromorphic function $a(z)$ is said to be a *small function for* f if

$$T(r, a) = S(r, f).$$

Throughout this paper, we denote by a, b_0, b_1, $\cdots$, a_0, a_1, $\cdots$ small meromorphic functions for f.

Let

$$\phi(z) = f^n + a_{n-1}f^{n-1} +, \cdots, + a_1 f + a_0 . \tag{1.1}$$

E. Mues and N. Steinmetz [8] proved the following Theorem.

THEOREM A. *Let f be a meromorphic function. Assume that ϕ given by* (1.1) *satisfies*

$$\overline{N}(r, 0; \phi) = S(r, f) \quad \textit{and} \quad \overline{N}(r, f) = S(r, f). \tag{1.2}$$

Then

$$\phi = (f + a_{n-1}/n)^n .$$

N. Toda [12] proved an extension of the Theorem A

THEOREM B. *Let $f(z)$ be a meromorphic function and ϕ be given by*

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(1.1). *If*

$$\limsup_{r\to\infty\ r\notin E} (\overline{N}(r, 0; \phi) + 2\overline{N}(r, f))/T(r, f) < 1/2, \tag{1.3}$$

then we have

$$\phi = (f + a_{n-1}/n)^n .$$

Recently, Weissesnborn [14] proved the following theorem:

THEOREM C. *Let f be a meromorphic function and ϕ be given by* (1.1). *Then we have that either*

$$\phi = (f + a_{n-1}/n)^n$$

or

$$T(r, f) \leqq \overline{N}\ (r, 0; \phi) + \overline{N}(r, f) + S(r, f). \tag{1.4}$$

In this note, we will extend these theorems to differential polynomials, instead of (mere) polynomial, of f.

We call, for a meromorphic function f,

$$M[f] = a(z)f^{n_0}(f')^{n_1} \cdots (f^{(m)})^{n_m}$$

as a *differential monomial* in f of *degree* $\gamma_M = n_0 + \cdots + n_m$ and of *weight* $\Gamma_M = n_0 + 2n_1 + \cdots + (m+1)n_m$. We call

$$P[f] = \sum_{\lambda\in I} M_\lambda = \sum_{\lambda\in I} a_\lambda(z)f^{n_0}(f')^{n_1} \cdots (f^{(m)})^{n_m}$$

as a differential polynomial in f, where a_λ are meromorphic functions and I is a finite set of multi-indices $\lambda = (n_0, n_1, \cdots, n_m)$ for which $a_\lambda \neq 0$ and $n_0, n_1, \cdots, n_m$ are nonnegative integers. We define the *degree* γ_P and *weight* Γ_P of P by

$$\gamma_P = \max_{\lambda\in I} \gamma_{M_\lambda} \quad \text{and} \quad \max_{\lambda\in I} \Gamma_{M_\lambda} .$$

If P is a differential polynomial, then P' denotes the differential polynomial which satisfies

$$P'[f(z)] = \frac{d}{dz} P[f(z)]$$

for any meromorphic function f. Note that $\gamma_{P'} = \gamma_P$.

Steinmetz [11] investigated the value distribution of some differential polynomials in f. His result is as follows: put

$$\Psi = f^n P[f] + Q[f], \tag{1.5}$$

where P and Q are differential polynomials in f. Then

THEOREM D. *Let f be meromorphic function and Ψ be given in* (1.5) *and $\Gamma_Q \leqq n-2$. If*

$$\bar{N}(r, 0; \Psi) = S(r, f),$$

then

$$m(r, f) + m(r, 0; f) + N_1(r, f) + N_1(r, 0; f) = S(r, f).$$

If, in (1.1), we replace f by $f - a_{n-1}/n$, then we can write ϕ in (1.1) in the form

$$\phi = f^n + Q[f], \quad Q[f] = b_{n-2}f^{n-2} + \cdots + b_1 f + b_0. \tag{1.6}$$

The form (1.6) for polynomial corresponds to the form (1.5) with $\Gamma_Q \leqq n-2$ for differential polynomial.

In consideration of this Theorem D due to Steinmetz, we will prove here the following Theorems:

THEOREM 1. *Let f be a meromorphic function and ϕ be given in* (1.6) *and $Q[f] \neq 0$. Then*

$$2T(r, f) \leqq \bar{N}(r, f) + \bar{N}(r, 0; f) + \bar{N}(r, 0; \phi) + S(r, f). \tag{1.7}$$

If $Q[0] \neq 0$, then

$$nT(r, f) \leqq \bar{N}(r, f) + \bar{N}(r, 0; f) + \bar{N}(r, 0; Q) + \bar{N}(r, 0; \phi) + S(r, f). \tag{1.8}$$

THEOREM 2. *Let f be a meromorphic function and Ψ be given in* (1.5). *We suppose $Q[f] \neq 0$ and $\Gamma_Q \leqq n-2$. Then we have*

$$2T(r, f) \leqq \bar{N}(r, f) + \bar{N}(r, 0; f) + (\gamma_P + 1)\bar{N}(r, 0; \Psi) + S(r, f). \tag{1.9}$$

If further $m(r, P) = S(r, f)$, then

$$2T(r, f) \leqq \bar{N}(r, f) + \bar{N}(r, 0; f) + \bar{N}(r, 0; \Psi) + S(r, f). \tag{1.10}$$

§ 2. Preliminary lemmas

LEMMA 1 ([2] [8] [11] [14]). *Let Q and Q^* be differential polynomials in f having coefficients a_j and a_k^*. Suppose that $m(r, a_j) = S(r, f)$ and $m(r, a_k^*) = S(r, f)$, but we don't require that $T(r, a_j) = S(r, f)$ and $T(r, a_k^*) = S(r, f)$. If $\gamma_Q \leqq n$ and*

$$f^n Q^*[f] = Q[f],$$

then

$$m(r, Q^*[f]) = S(r, f).$$

Remark. Clunie proved his lemma under the stronger hypothesis that $T(r, a_j) = S(r, f)$ and $T(r, a_k^*) = S(r, f)$. Mues and Steinmetz [8] remarked that Clunie's proof does also work under the weaker assumption stated above. In particular, there might be coefficients of the form f'/f or, more generally, Ψ'/Ψ where Ψ is the differential polynomial given by (1.0).

LEMMA 2. *If $P[f]$ is a differential polynomial and $\gamma_P = h$ then*

$$m(r, P) \leqq hm(r, f) + S(r, f). \tag{2.1}$$

Proof. Write

$$P[f] = P_h[f] + \cdots + P_0[f]$$

where $P_j[f]$ $(j = 0, 1, \cdots, h)$ are homogeneous polynomials with respect to $f, f', \cdots, f^{(m)}$, with degree j. $P_j[f]$ is the sum of a finite number of terms [see 1],

$$a(z)(f'/f)^{n_1} \cdots (f^{(m)}/f)^{n_m} \cdot f^j,$$

where $j = n_1 + \cdots + n_m$. Thus we can write

$$P[f] = R_h[f]f^h + \cdots + R_0[f],$$

where $R_j[f] = P_j[f]/f^j$ and hence

$$m(r, R_j[f]) = S(r, f), \qquad j = 0, 1, \cdots, h.$$

Therefore we have

$$\begin{aligned} m(r, P[f]) &\leqq hm(r, f) + \sum_{j=0}^{h} m(r, R_j; f) \\ &\leqq hm(r, f) + S(r, f). \end{aligned}$$

Remark. Yang [13] proved above lemma under the condition $N(r, f) = S(r, f)$.

§ 3. Proof of Theorems 1 and 2

Proof of *Theorem* 1. Write

$$\phi = f^n + f^m Q_1[f]$$

where

$$0 \leqq m \leqq n-2, \quad Q_1[0] \neq 0, \quad \gamma_{Q_1} = \gamma_Q - m \leqq n-m-2.$$

Put $\psi = f^{n-m}/Q_1$ and apply the second fundamenntal Theorem to ψ. Then we obtain

$$T(r, \psi) \leqq \bar{N}(r, \psi) + \bar{N}(r, 0; \psi) + \bar{N}(r, -1, \psi) + S(r; \psi). \tag{3.1}$$

Since ψ is a rational of f with degree $n-m$, we apply the Mokhon'ko's theorem [6].

$$T(r, \psi) = (n-m)T(r, f) + S(r, f). \tag{3.2}$$

Thus

$$S(r, \psi) = S(r, f). \tag{3.3}$$

Each term on the right side of (3.1) are estimated as follows:

$$\bar{N}(r, \psi) \leqq \bar{N}(r, 0; Q_1) + \bar{N}(r, f) + S(r, f), \tag{3.4}$$

$$\bar{N}(r, 0; \psi) \leqq (r, 0; f) + S(r, f), \tag{3.5}$$

$$\bar{N}(r, -1; \psi) \leqq \bar{N}(r, 0; \phi) + S(r, f), \tag{3.6}$$

$$\bar{N}(r, 0; Q_1) \leqq (n-m-2)T(r, f) + S(r, f). \tag{3.7}$$

From (3.1)–(3.6)

$$\begin{aligned}(n-m)T(r, f) \leqq \bar{N}(r, 0; Q_1) &+ \bar{N}(r, f) + \bar{N}(r, 0; f) \\ &+ \bar{N}(r, 0; \phi) + S(r, f).\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we obtain (1.7). If $Q[0] \neq 0$, that is $m = 0$, $Q_1 = Q$, then we get (1.8) by (3.8).

For the proof of Theorem 2, we follow some ideas given in [8], [11], [14].

Proof of Theorem 2. We may suppose $\psi \neq 0$, see [11]. Differentiating (1.5), we obtain

$$f^{n-1}A = B \tag{3.9}$$

with

$$A = (\Psi'/\Psi)fP - nf'P + fP' \tag{3.10}$$

$$B = Q' - (\Psi'/\Psi)Q. \tag{3.11}$$

By the Remark after the Lemma 1, we look at A and B as differential polynomials in f with coefficients having small proximity function and $\gamma_B \leqq n-2$.

We may suppose $A \neq 0$ [see 11]. By applying Lemma 1 we have

$$m(r, A) = S(r, f)\,, \tag{3.12}$$

$$m(r, Af) = S(r, f)\,, \tag{3.13}$$

hence

$$m(r, f) \leqq m(r, Af) - m(r, 0; A) \leqq m(r, 0; A) + S(r, f)\,. \tag{3.14}$$

We define $\omega(z_0, f)$ as follows; if z_0 is a pole of ν-th order for $f(z)$, then $\omega(z_0, f) = \nu$, and if z_0 is a regular point for $f(z)$, then $\omega(z_0, f) = 0$. Let z_0 be a pole of f and neither pole nor zero of coefficients of P and Q. Put $\omega(z_0, f) = p$ and $\omega(z_0, Q) = k$, $0 \leqq k \leqq p\Gamma_Q \leqq p\ (n-2)$. Write

$$Q(z) = R/(z-z_0)^k + \cdots\,, \quad R \neq 0 \tag{3.14}$$

hence for $k \geqq 1$

$$Q'(z) = -kR/(z-z_0)^{k+1} + \cdots \tag{3.15}$$

We have

$$\Psi'(z)/\Psi(z) = -n^*/(z-z_0) + \cdots\,, \quad (n^* \geqq n \geqq k+2)\,. \tag{3.16}$$

From (3.11), (3.14), (3.15) and (3.16)

$$B(z) = (n^* - k)R/(z-z_0)^{k+1} + \cdots$$

For $k = 0$, we have

$$\omega(z_0, B) = 1$$

Thus

$$\omega(z_0, B) \leqq k+1, \quad k \geqq 0\,. \tag{3.17}$$

If we have the development around z_0

$$A(z) = S(z-z_0)^\mu + \cdots\,, \quad \mu \in Z, \quad S \neq 0\,,$$

then from (3.9) and (3.17)

$$p(n-1) - \mu \leqq k+1 \leqq p(n-2)+1\,,$$

hence

$$p - 1 \leqq \mu . \tag{3.18}$$

Thus

$$\omega(z_0, f) - 1 \leqq \omega(z_0, 1/A) . \tag{3.19}$$

By (3.10) and (3.18), if z_0 is a pole of A and neither pole nor zero of coefficients of P and Q then, z may not be pole of f. Thus z_0 is a zero of Ψ. And we see from (3.10) $\omega(z_0, A)$ is at most one. Therefore,

$$\bar{N}(r, A) \leqq \bar{N}(r, 0; \Psi) + S(r, f) , \tag{3.20}$$

$$N_1(r, A) = S(r, f) . \tag{3.21}$$

From (3.10)

$$A = fPG \tag{3.22}$$

with

$$G = (\Psi'/\Psi) - n(f'/f) + (P'/P) . \tag{3.33}$$

Let z_1 be a zero of f and neither pole nor zero of coefficients of P and Q then $\omega(z_1, G)$ is at most one by (3.23). Thus

$$\omega(z_1, 1/f) - 1 \leqq \omega(z_1, 1/A) . \tag{3.24}$$

From (3.19) and (3.24)

$$N_1(r, f) + N_1(r, 0; f) \leqq N(r, 0; A) + S(r, f) . \tag{3.25}$$

From (3.22)

$$m(r, A/f) \leqq m(r, P) + m(r, G) \leqq m(r, P) + S(r, f) . \tag{3.26}$$

By the first fundamental theorem

$$\begin{aligned} m(r, f + (1/f)) &= T(r, (f^2 + 1)/f) - N(r, f + (1/f)) \\ &= 2T(r, f) - N(r, f) - N(r, 0; f) + O(1) \\ &= m(r, f) + m(r, 0; f) + O(1) , \end{aligned}$$

hence

$$\begin{aligned} m(r, f) + m(r, 0; f) &= m(r, f + (1/f)) + O(1) \\ &\leqq m\{r, A(f + (1/f))\} + m(r, 0; A) + O(1) \\ &\leqq m(r, Af) + m(r, A/f) + m(r, 0; A) + O(1) . \end{aligned} \tag{3.27}$$

From (3.13), (3.26), (3.27) and Lemma 2

$$m(r,f)+m(r,0,f)\leqq m(r,P)+m(r,0;A)+S(r,f)$$
$$\leqq hm(r,f)+m(r,0;A)+S(r,f),$$

from (3.14), we get

$$m(r,f)+m(r,0;f)\leqq(h+1)m(r,0;A)+S(r,f). \tag{3.28}$$

By the first fundamental Theorem, (3.28) (3.25), (3.20) and (3.21), we obtain

$$\begin{aligned}2T(r,f)&=m(r,f)+m(r,0;f)+N_1(r,f)+N_1(r,0;f)\\&\quad+\overline{N}(r,f)+\overline{N}(r,0;f)+O(1)\leqq(h+1)m(r,0;A)+N(r,0;A)\\&\quad+\overline{N}(r,f)+\overline{N}(r,0;f)+S(r,f)\leqq(h+1)T(r,A)+\overline{N}(r,f)\\&\quad+\overline{N}(r,0;f)+S(r,f)\leqq(h+1)\overline{N}(r,A)+(h+1)\{N_1(r,A)\\&\quad+m(r,A)\}+\overline{N}(r,f)+\overline{N}(r,0;f)+S(r,f)\\&\leqq(h+1)\overline{N}(r,0;\Psi)+\overline{N}(r,f)+\overline{N}(r,0;f)+S(r,f).\end{aligned}$$

From this proof, if $m(r,P)=S(r,f)$, then we may put $h=0$ in (1.9). Thus Theorem 2 is proved.

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