G. Nakamura
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THE EXISTENCE OF SYMMETRIC RIEMANN SURFACES DETERMINED BY CYCLIC GROUPS

GOU NAKAMURA

Abstract. Let $n > 1$, $m \geq 1$, $g \geq 3$ and γ be given integers. The purpose of this paper is to determine the relations of n, m, g and γ for the existence of the symmetric Riemann surfaces S of type (n, m) with genus g and species γ. If n is an odd prime, the relations are known in [3]. In the case that n is odd, we shall show the analogous result when $E(S)$ is isomorphic to a cyclic group $\mathbf{Z}_{2n}$ and when the quotient space $S/E(S)$ is orientable.

§1. Introduction

Let S be a compact Riemann surface. We denote by $E(S)$ the group of analytic homeomorphisms and anti-analytic homeomorphisms of S onto itself and by $A(S)$ its subgroup of analytic homeomorphisms. If $A(S)$ is isomorphic to a cyclic group $\mathbf{Z}_n$ of order n and the quotient space $S/A(S)$ is of genus m, then S is called a Riemann surface of type(n, m). An element T in $E(S)\setminus A(S)$ is called a symmetry on S if $T^2(= T \circ T) = I_S$ (the identity map). A compact Riemann surface with symmetries is said to be symmetric. For a symmetry T on S the quotient space $S/\langle T\rangle$ is a Klein surface. Let k be the number of boundary components of $S/\langle T\rangle$. Then we define the species $\mathrm{sp}(T)$ of T by

$$\mathrm{sp}(T) = \begin{cases} k & (\text{if } S/\langle T\rangle \text{ is orientable}), \\ -k & (\text{if } S/\langle T\rangle \text{ is non-orientable}). \end{cases}$$

In this paper we suppose that $E(S)$ is isomorphic to a cyclic group $\mathbf{Z}_{2n}$ of order $2n$. Then for such a symmetric Riemann surface S, the symmetry T on S is uniquely determined. Hence we define the species of S by that of T.

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symmetric Riemann surfaces S of type (n, m) with genus g and species γ. If n is an odd prime, the relations are known in [3]. In the case that n is odd, we shall show the analogous result when $E(S)$ is isomorphic to a cyclic group $\mathbf{Z}_{2n}$ and when the quotient space $S/E(S)$ is orientable.

§2. Non-Euclidean crystallographic groups

Let $H = \{z \in \mathbf{C} \mid \Im z > 0\}$ be the upper half plane. With each matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbf{R}$ and with $\det A = \pm 1$, we associate the mapping

$$f_A : H \to H \ ; \ z \mapsto \begin{cases} \dfrac{az+b}{cz+d} & \text{if } \det A = 1, \\ \dfrac{a\bar{z}+b}{c\bar{z}+d} & \text{if } \det A = -1. \end{cases}$$

Then $E(H) = \{f_A \mid \det A = \pm 1\}$ and $A(H) = \{f_A \mid \det A = 1\}$. We regard $E(H)$ as a topological space by means of the inclusion $E(H) \hookrightarrow PGL(2, \mathbf{R})$. A discrete subgroup Γ of $E(H)$ is called a non-Euclidean crystallographic group (shortly an NEC group) if the quotient H/Γ is compact. An NEC group Γ is called a Fuchsian group if $\Gamma \subset A(H)$, and a proper NEC group otherwise. For a proper NEC group Γ, $\Gamma^+ = \Gamma \cap A(H)$ is called the canonical Fuchsian group of Γ.

In general, each NEC group Γ is formed by the generators

$$\begin{array}{llll} x_i & \in \Gamma^+ & ; & i = 1, \cdots, r, \\ e_i & \in \Gamma^+ & ; & i = 1, \cdots, k, \\ c_{ij} & \in \Gamma \setminus \Gamma^+ & ; & i = 1, \cdots, k, \ \ j = 0, \cdots s_i, \\ a_i, b_i & \in \Gamma^+ & ; & i = 1, \cdots, g \ \text{ if } H/\Gamma \text{ is orientable}, \\ d_i & \in \Gamma \setminus \Gamma^+ & ; & i = i, \cdots, g \ \text{ if } H/\Gamma \text{ is non-orientable}, \end{array}$$

satisfying the relations

$$\begin{array}{lll} x_i^{m_i} = I_H & ; & i = 1, \cdots, r, \\ e_i^{-1} c_{i0} e_i c_{is_i} = I_H & ; & i = 1, \cdots, k, \\ c_{i,j-1}^2 = c_{ij}^2 = (c_{i,j-1} c_{ij})^{n_{ij}} = I_H & ; & i = 1, \cdots, k, \ \ j = 1, \cdots, s_i, \\ x_1 \cdots x_r e_1 \cdots e_k [a_1, b_1] \cdots [a_g, b_g] = I_H & & \text{if } H/\Gamma \text{ is orientable}, \\ x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_g^2 = I_H & & \text{if } H/\Gamma \text{ is non-orientable}, \end{array}$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$. We call x_i an elliptic element, c_{ij} a reflection of Γ. Then the signature $\sigma(\Gamma)$ of Γ is written by

$$(1) \quad \sigma(\Gamma) = (g; \pm; [m_1, \cdots, m_r]; \{(n_{11}, \cdots, n_{1s_1}), \cdots, (n_{k1}, \cdots, n_{ks_k})\}),$$

where "+" means that H/Γ is orientable, and "−" means that H/Γ is non-orientable. This "+" or "−" is called the sign of $\sigma(\Gamma)$ and denoted by $\mathrm{sign}(\sigma(\Gamma))$. We call g the genus, m_i the proper periods, n_{ij} the periods, and $(n_{i1}, \cdots, n_{is_i})$ the period-cycles of $\sigma(\Gamma)$. If there are no proper periods, we write $[-]$ in place of $[m_1, \cdots, m_r]$. If there are no periods in the period-cycle, we write $(-)$ in place of $(n_{i1}, n_{i2}, \cdots, n_{is_i})$. If there are no period-cycles, we write $\{-\}$ in place of $\{(n_{11}, \cdots, n_{1s_1}), \cdots, (n_{k1}, \cdots, n_{ks_k})\}$.

For an NEC group Γ with signature (1), the Gauss-Bonnet theorem shows that the non-Euclidean area $\mu(F)$ of a fundamental region F of Γ is given by

$$\mu(F) = 2\pi \left(\alpha g + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right) \right),$$

where $\alpha = 2$ if $\mathrm{sign}(\sigma(\Gamma)) =$ "+", $\alpha = 1$ if $\mathrm{sign}(\sigma(\Gamma)) =$ "−". This does not depend on the choice of fundamental regions. We define the area of $\sigma(\Gamma)$ by $\mu(F)/2\pi$ and denote it by $\mu(\Gamma)$.

Let Γ' be an NEC group and Γ a subgroup of Γ' with finite index. Then Γ is an NEC group, and the following formula (called the Riemann-Hurwitz relation) is fulfilled:

$$\frac{\mu(\Gamma)}{\mu(\Gamma')} = [\Gamma' : \Gamma].$$

§3. The main result

Let $m_1, m_2, \cdots, m_k$ be integers. We denote the least common multiple of $\{m_1, m_2, \cdots, m_k\}$ by $\mathrm{l.c.m.}\{m_1, m_2, \cdots, m_k\}$.

THEOREM 1. *Let $n > 1$ be an odd integer and $m \geq 1$, $g \geq 3$ and γ integers. Then there exists a symmetric Riemann surface S of type (n, m) with genus $g(S) = g$, $sp(S) = \gamma$, $E(S) \cong \mathbf{Z}_{2n}$ and with the orientable quotient $S/E(S)$ if and only if:*

There exist non-negative integers r, t and divisors $d_1, \cdots, d_{r+t} (\neq 1)$ of n and an integer $k \geq 1$ such that:

(a) *If $m = 1$, then $r \geq 2$. If $m = 2$, then $r \geq 1$.*

(b) $g = n \left(m - 1 + \sum_{i=1}^{r} \left(1 - \frac{1}{d_i}\right) \right) + 1.$

(c) *$m+1-k$ is even and non-negative.*

(d) $0 \leq t \leq k$.

(e) $\gamma = n\left(k - \sum_{i=1}^{t}\left(1 - \frac{1}{d_{r+i}}\right)\right)$ (≥ 0).

(f) *If $r+t>0$, then* l.c.m.$\{d_1, \cdots, d_{r+t}\} =$ l.c.m. $\{d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_{r+t}\}$ *for every i.*

(g) *If $k = m+1$, then* l.c.m.$\{d_1, \cdots, d_{r+t}\} = n$.

We note that the divisors $d_1, \cdots, d_{r+t}$ are not necessarily distinct.

If n is an odd prime p, our theorem is reduced to the following

COROLLARY 1. [3; Theorem 2.1] *There exists a symmetric Riemann surface S of type(p,m) with $g(S) = g$, $sp(S) = \gamma$, $E(S) \cong \mathbf{Z}_{2p}$ and with the orientable quotient $S/E(S)$ if and only if:*
There exist non-negative integers r, t and an integer $k \geq 1$ such that:

(a) *If $m = 1$, then $r \geq 2$. If $m = 2$, then $r \geq 1$.*

(b) $g = p(r+m-1) - r + 1$.

(c) *$m+1-k$ is even and non-negative.*

(d) $0 \leq t \leq k$.

(e) $\gamma = p(k-t) + t$.

(f) *If $r+t>0$, then $r+t \geq 2$.*

(g) *If $k = m+1$, then $r+t \neq 0$.*

§4. The proof of our theorem

We shall use the following lemma (see [4; Lemma 3.1.1]).

LEMMA 1. *Let m_1, m_2, $\cdots$, $m_k > 0$ be odd integers and N a (positive) multiple of $M =$*l.c.m.$\{m_1, m_2, \cdots, m_k\}$*. Then the following conditions are equivalent to each other.*

(1) *There exist $\xi_1, \cdots, \xi_k$ in $\mathbf{Z}_N$ such that $o(\xi_i) = m_i$ and $\xi_1 + \cdots + \xi_k = 0$ in $\mathbf{Z}_N$.*

(2) *For every* i, l.c.m.$\{m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_k\} = M$.

Proof of our theorem. First we shall show the "only if" part. By our assumption $g \geq 3$, H is the universal covering surface for S, so that there exists a torsion-free Fuchsian group Γ_S satisfying $S \cong H/\Gamma_S$. Then the signature of Γ_S is $\sigma(\Gamma_S) = (g; +; [-]; \{-\})$. We denote by N_S the normalizer of Γ_S in $E(H)$. We shall show that the signatures of N_S and $N_S^+ (= N_S \cap A(H))$ have the following forms with some non-negative integers r, k $(1 \leq k \leq m+1)$ and divisors $d_1, \cdots, d_r$ of n:

$$\sigma(N_S) = \Big(\frac{m+1-k}{2}; +; [d_1, d_2, \cdots, d_r]; \{\overbrace{(-), \cdots, (-)}^{k}\}\Big),$$
$$\sigma(N_S^+) = \Big(m; +; [d_1, d_1, d_2, d_2, \cdots, d_r, d_r]; \{-\}\Big).$$

We note that $d_1, \cdots, d_r$ are not necessarily distinct. Since $S/E(S) \cong (H/\Gamma_S)/(N_S/\Gamma_S) \cong H/N_S$ is orientable, we get sign$(\sigma(N_S))$ ="+". Let r be the number of elliptic elements in canonical generators of N_S. The orders of elliptic elements are divisors $(\neq 1)$ of n. We write them $d_1, \cdots, d_r$. Let k be the number of period-cycles of N_S. Since there exists a symmetry on S, N_S contains reflections. Hence it follows that $k \geq 1$. Since $N_S/\Gamma_S \cong E(S) \cong \mathbf{Z}_{2n}$, there exists an epimorphism

$$\eta : N_S \to \mathbf{Z}_{2n}$$

with $\ker(\eta) = \Gamma_S$. For every element u of order 2 in N_S, we get $\eta(u) = n$. Thus, for u, v in N_S of order 2, $\ker(\eta)$ contains uv. Since Γ_S is a torsion-free group, uv is not an element of finite order > 1. Hence there are no periods in any period-cycles of $\sigma(N_S)$. Since $S/A(S) \cong H/N_S^+$ and $S/A(S)$ has genus m, the genus of $\sigma(N_S^+)$ is equal to m. By Corollary 2.2.5 in [4], we get the required forms of $\sigma(N_S)$ and $\sigma(N_S^+)$.

We shall show the assertion (a). First we assume $m = 1$. The signature of N_S^+ is of form

$$\sigma(N_S^+) = (1; +; [d_1, d_1, \cdots, d_r, d_r]; \{-\}).$$

The area of $\sigma(N_S^+)$ is given by

$$\mu(N_S^+) = 2\sum_{i=1}^{r}\left(1 - \frac{1}{d_i}\right).$$

From $\mu(N_S^+) > 0$ it follows that $r \geq 1$. All signatures with respect to maximal Fuchsian groups are known in Theorems 1, 2 and 3 in [8]. From these known results it follows that in the case of $r = 1$, N_S^+ is not maximal, because $\sigma(N_S^+) = (1; +; [d, d]; \{-\})$ for some divisor $d (\neq 1)$ of n. Hence, by Theorem 1 in [8], there exists a Fuchsian group $\Gamma' \supset N_S^+$ satisfying

$$[\Gamma' : N_S^+] = 2 \quad \text{and} \quad \sigma(\Gamma') = (0; +; [2, 2, 2, 2, d]; \{-\}),$$

so that the generators of Γ' is represented by $y_1, \cdots, y_5$ with the relations

$$y_i^2 = I_H (1 \leq i \leq 4),\ y_5^d = I_H \text{ and } y_1 \cdots y_5 = I_H.$$

We see that Γ' includes Γ_S as a normal subgroup by the following way.

Let D_n be the dihedral group of order $2n$, namely,

$$D_n = \langle a, b \mid a^n = b^2 = (ab)^2 = e \text{ (unit element)} \rangle.$$

Since $N_S^+/\Gamma_S \cong A(S) \cong \mathbf{Z}_n \cong \langle a \rangle$, there exists an epimorphism $\theta : N_S^+ \to \mathbf{Z}_n$ with $\ker(\theta) = \Gamma_S$. By $[\Gamma' : N_S^+] = 2$, we can write $\Gamma' = N_S^+ \cup N_S^+ \gamma_1$ for some γ_1 in Γ'. Therefore for each y_i $(1 \leq i \leq 4)$ there exists y_i' in N_S^+ satisfying $y_i = y_i' \gamma_1$. We note that $y_5 \in N_S^+$. Then We can define an epimorphism $\varphi_1 : \Gamma' \to D_n$ satisfying

$$\begin{aligned} \varphi_1(y_i) &= \theta(y_i')b \quad \text{for} \quad 1 \leq i \leq 4, \\ \varphi_1(y_5) &= \theta(y_5). \end{aligned}$$

Since $\ker(\varphi_1) = \Gamma_S$, Γ_S is a normal subgroup of Γ'. Hence $r \geq 2$ must hold because N_S^+ is the normalizer of Γ_S in $A(H)$.

Next we assume m=2. The signature of N_S^+ is of form

$$\sigma(N_S^+) = (2; +; [d_1, d_1, \cdots, d_r, d_r]; \{-\}).$$

By Theorems 1 and 2 in [8], N_S^+ is not maximal in the case of $r = 0$, because $\sigma(N_S^+) = (2; +; [-]; \{-\})$. Then, by Theorem 1 in [8], there exists a Fuchsian group $\Gamma'' \supset N_S^+$ satisfying

$$[\Gamma'' : N_S^+] = 2 \quad \text{and} \quad \sigma(\Gamma'') = (0; +; [2, 2, 2, 2, 2, 2]; \{-\}),$$

so that the generators of Γ'' is represented by $z_1, \cdots, z_6$ with the relations $z_i^2 = z_1 \cdots z_6 = I_H$ $(1 \leq i \leq 6)$. Since $[\Gamma'' : N_S^+] = 2$, we can write $\Gamma'' = N_S^+ \cup N_S^+ \gamma_2$ for some γ_2 in Γ''. Therefore for each z_i there exists z_i'

in N_S^+ satisfying $z_i = z_i'\gamma_2$. We can define an epimorphism $\varphi_2 : \Gamma'' \to D_n$ satisfying

$$\varphi_2(z_i) = \theta(z_i')b \quad \text{for} \quad 1 \le i \le 6.$$

Since $\ker(\varphi_2) = \Gamma_S$, Γ_S is a normal subgroup of Γ''. Hence $r \ge 1$ must hold because N_S^+ is the normalizer of Γ_S in $A(H)$. Thus the assertion (a) holds.

We put $g' = (m+1-k)/2$. Then the set of canonical generators of N_S is represented by

$$\{a_i, b_i(1 \le i \le g'), x_j(1 \le j \le r), e_l, c_l = c_{l0}(1 \le l \le k)\},$$

with the relations

$$x_j^{d_j} = I_H \ (1 \le j \le r), \ \ e_l^{-1}c_l e_l c_l = c_l^2 = I_H \ (1 \le l \le k)$$

and

$$\prod_{j=1}^{r} x_j \prod_{l=1}^{k} e_l \prod_{i=1}^{g'} [a_i, b_i] = I_H.$$

We put

$$F = \{1 \le l \le k \ ; \ e_l \notin \Gamma_S\} \text{ and } t = \#F.$$

For each l in F we denote by f_l the order of $\eta(e_l)$ in $\mathbf{Z}_{2n}$, which is a divisor ($\neq 1$) of n. Then $d_1, \cdots, d_r, f_l$ ($l \in F$) are required divisors. The equality (b) is shown by the Riemann-Hurwitz relation $\mu(\Gamma_S) = [N_S : \Gamma_S]\mu(N_S)$, namely,

$$2g - 2 = 2n\left(m - 1 + \sum_{i=1}^{r}\left(1 - \frac{1}{d_i}\right)\right)$$

The assertion (c) follows from the genus of $\sigma(N_S)$. The assertion (d) follows from $t = \#F$.

We shall show the assertion (e). Let T be a symmetry on S. Since $\{I_S, T\}$ is a subgroup of $E(S) \cong N_S/\Gamma_S$, there exists a subgroup Γ_1 of N_S satisfying $\Gamma_1/\Gamma_S \cong \{I_S, T\}$. Then $\Gamma_1 = \eta^{-1}(\{0, n\})$. Since $H/\Gamma_1 \cong (H/\Gamma_S)/(\Gamma_1/\Gamma_S) \cong S/\langle T\rangle$, $|\mathrm{sp}(S)|$ is the number of period-cycles of $\sigma(\Gamma_1)$. Consequently we shall determine the signature of Γ_1. Since $[N_S : \Gamma_1]$ is odd, we get $\mathrm{sign}(\sigma(\Gamma_1)) = \mathrm{sign}(\sigma(N_S)) =$ "+" ([4;Theorem 2.1.2]). The order of $\Gamma_1 x_j$ in N_S/Γ_1 is equal to that of x_j in N_S. Hence there are no proper periods of $\sigma(\Gamma_1)$ ([4;Theorem 2.2.3]). Since $\sigma(N_S)$ does not have any period in all period-cycles, neither does $\sigma(\Gamma_1)$. For each l in F, the order of $\Gamma_1 e_l$

in N_S/Γ_1 is equal to f_l, so that by using Theorem 2.4.2 in [4] the number k_1 of period-cycles of $\sigma(\Gamma_1)$ is given by

$$k_1 = n(k-t) + \sum_{l \in F} \frac{n}{f_l} = n\left(k - \sum_{l \in F}\left(1 - \frac{1}{f_l}\right)\right).$$

Hence the signature of Γ_1 is given by

$$\sigma(\Gamma_1) = (g_1; +; [-]; \{\overbrace{(-), \cdots, (-)}^{k_1}\}),$$

where $g_1 = (g - k_1 + 1)/2$. Since $\mathrm{sign}(\sigma(\Gamma_1)) =$ "+", $S/\langle T \rangle$ is orientable, so that $\gamma = k_1$. Hence the assertion (e) holds.

If $r + t > 0$, we put $M = \mathrm{l.c.m.}\{d_1, \cdots, d_r, f_l\ (l \in F)\}$. Then

$$\langle \eta(x_j)(1 \leq j \leq r), \eta(e_l)(l \in F) \rangle \cong \mathbf{Z}_M.$$

The canonical relation $\prod_{j=1}^r x_j \prod_{l=1}^k e_l \prod_{i=1}^{g'} [a_i, b_i] = I_H$ implies $\sum_{j=1}^r \eta(x_j) + \sum_{l \in F} \eta(e_l) = 0$ in $\mathbf{Z}_{2n}$, so that we can take elements ξ_j $(1 \leq j \leq r)$, ε_l $(l \in F)$ in $\mathbf{Z}_M$ satisfying $o(\xi_j) = d_j$, $o(\varepsilon_l) = f_l$ and $\sum_{j=1}^r \xi_j + \sum_{l \in F} \varepsilon_l = 0$. Therefore the assertion (f) follows from Lemma 1.

We shall show the assertion (g). If $k = m + 1$ then the set of canonical generators of N_S is represented by

$$\{x_j\ (1 \leq j \leq r),\ e_l,\ c_l = c_{l0}\ (1 \leq l \leq k)\}$$

with the relations

$$x_j^{d_j} = I_H\ (1 \leq j \leq r),\ e_l^{-1} c_l e_l c_l = c_l^2 = I_H\ (1 \leq l \leq k)$$

and

$$\prod_{j=1}^r x_j \prod_{l=1}^k e_l = I_H.$$

Since $\eta : N_S \to \mathbf{Z}_{2n}$ is surjective, the image of η,

$$\mathrm{Im}(\eta) = \langle \eta(x_j)\ (1 \leq j \leq r),\ \eta(e_l),\ \eta(c_l)\ (1 \leq l \leq k) \rangle,$$

contains elements of order $2n$. Since $\eta(c_l)$ $(1 \leq l \leq k)$ are elements of order 2, it follows that $\mathrm{l.c.m.}\{d_1, \cdots, d_r, f_l\ (l \in F)\} = n$. Thus the assertion (g) holds. Hence the proof of "only if" part is completely achieved.

Conversely we shall show the "if" part. Let $n, m, g, \gamma, r, t, d_1, \cdots, d_{r+t}$ and k be given numbers satisfying conditions (a) to (g). We put

$$\sigma = (g'; +; [d_1, \cdots, d_r]; \{\overbrace{(-), \cdots, (-)}^{k}\}),$$

where $g' = (m + 1 - k)/2$. By (c), g' is a non-negative integer. Since the area $\mu(\sigma) = m - 1 + \sum_{j=1}^{r}(1 - 1/d_j)$ is positive by (b), there exist NEC groups with signature σ. By Corollary 2.2.5 in [4] the canonical Fuchsian groups of such NEC groups have the signature

$$\sigma^+ = (m; +; [d_1, d_1, \cdots, d_r, d_r]; \{-\}).$$

From (a) it follows that

$$\sigma^+ \neq (1; +; [d_i, d_i]; \{-\}) \text{ and } \sigma^+ \neq (2; +; [-]; \{-\}).$$

Therefore, by Theorems 1 and 2 in [8], there exists a maximal Fuchsian group with signature σ^+, so that we have a maximal NEC group with signature σ. We denote it by N.

Let $\{a_i, b_i (1 \leq i \leq g'), x_j (1 \leq j \leq r), e_l, c_l = c_{l0} (1 \leq l \leq k)\}$ be the set of canonical generators of N satisfying

$$x_j^{d_j} = I_H \ (1 \leq j \leq r), \ e_l^{-1} c_l e_l c_l = c_l^2 = I_H \quad (1 \leq l \leq k)$$

and

$$\prod_{j=1}^{r} x_j \prod_{l=1}^{k} e_l \prod_{i=1}^{g'} [a_i, b_i] = I_H.$$

Assume $r + t > 0$. By the condition (f) and Lemma 1 there exist ξ_j in $\mathbf{Z}_{2n}$ of order d_j $(1 \leq j \leq r + t)$ such that

$$\sum_{j=1}^{r+t} \xi_j = 0 \text{ in } \mathbf{Z}_{2n}.$$

We can define an epimorphism $\eta : N \to \mathbf{Z}_{2n}$ satisfying

$$\begin{aligned}
&\eta(a_1) = \eta(b_1) = 2 \ (\text{if } g' \geq 1), \ \eta(a_i) = \eta(b_i) = 0 \ (2 \leq i \leq g'), \\
&\eta(x_j) = \xi_j \ (1 \leq j \leq r, \ \text{if } r \neq 0), \\
&\eta(c_l) = n \ (1 \leq l \leq k), \\
&\eta(e_l) = \begin{cases} \xi_{r+l} & (1 \leq l \leq t, \ \text{if } t \neq 0 \ ), \\ 0 & (t + 1 \leq l \leq k). \end{cases}
\end{aligned}$$

Because η is compatible with the relations in N, that is,

$$\begin{aligned}
x_j^{d_j} = I_H \quad &\Rightarrow \quad \eta(x_j^{d_j}) = d_j\xi_j = 0 \ (1 \le j \le r),\\
c_l^2 = I_H \quad &\Rightarrow \quad \eta(c_l^2) = 2n \ (1 \le l \le k),\\
e_l^{-1}c_le_lc_l = I_H \quad &\Rightarrow \quad \eta(e_l^{-1}c_le_lc_l) = 0 \ (1 \le l \le k),\\
\textstyle\prod_{j=1}^r x_j \prod_{l=1}^k e_l \prod_{i=1}^{g'}[a_i, b_i] = I_H \quad &\Rightarrow \quad \eta(\textstyle\prod_{j=1}^r x_j \prod_{l=1}^k e_l \prod_{i=1}^{g'}[a_i, b_i])\\
&\qquad = \textstyle\sum_{j=1}^{r+t} \xi_j = 0.
\end{aligned}$$

We shall show that η is surjective. Since $k \ge 1$, $\mathrm{Im}(\eta)$ contains $\eta(c_1)$ of order 2. Therefore it is sufficient to show that $\mathrm{Im}(\eta)$ contains an element of order n. If $g' \ge 1$, then $\eta(a_1)$ and $\eta(b_1)$ are of order n by the definition. If $g' = 0$, that is, $k = m + 1$, then by (g) there exist elements of order n in $\mathrm{Im}(\eta)$. Thus $\mathrm{Im}(\eta) = \mathbf{Z}_{2n}$.

We put

$$\Gamma = \ker(\eta) \text{ and } S = H/\Gamma.$$

Then Γ is an NEC group.

We shall show that S is a required Riemann surface. By the definition of η, there exist no elliptic elements and orientation-reversing ones in Γ, so that the genus of $\sigma(\Gamma)$ is equal to g by the Riemann-Hurwitz relation $\mu(\Gamma) = 2n\mu(N)$. Therefore Γ is a Fuchsian group of signature $\sigma(\Gamma) = (g; +; [-]; \{-\})$. Hence S is a compact Riemann surface of genus g. Since N is maximal and includes Γ as a normal subgroup, N is the normalizer of Γ in $E(H)$. Therefore $E(S) \cong N/\Gamma \cong \mathbf{Z}_{2n}$. We put $\Gamma_2 = \eta^{-1}(\{0, n\})$. Since Γ_2/Γ is a subgroup of order 2 in N/Γ, there exists a symmetry T on S such that

$$\Gamma_2/\Gamma \cong \{I_S, T\} \subset E(S).$$

Thus S is symmetric. From $[E(S) : A(S)] = 2$ it follows that $A(S) \cong \mathbf{Z}_n$. The genus of $\sigma(N^+)$ is equal to $2g' + k - 1 = m$, so that the genus of $S/A(S) \cong H/N^+$ is equal to m. Thus S is of type (n, m). The orientability of $S/E(S)$ is derived from $S/E(S) \cong H/N$ and $\mathrm{sign}(\sigma(N)) =$"+".

We shall show $\mathrm{sp}(S) = \gamma$. Note the form of $\sigma(\Gamma_1)$ given in the "only if" part. Similarly we obtain

$$\sigma(\Gamma_2) = (g_2; +; [-]; \{\overbrace{(-), \cdots, (-)}^{k_2}\})$$

and

$$k_2 = n(k - t) + \sum_{l=1}^{t} \frac{n}{d_{r+l}} = n\left(k - \sum_{l=1}^{t}\left(1 - \frac{1}{d_{r+l}}\right)\right).$$

Since $S/\langle T\rangle \cong (H/\Gamma)/(\Gamma_2/\Gamma) \cong H/\Gamma_2$, we have $\mathrm{sp}(S) = k_2 = \gamma$. Hence S is a symmetric Riemann surface of type (n,m) with $g(S) = g$, $\mathrm{sp}(S) = \gamma$, $E(S) \cong \mathbf{Z}_{2n}$ and with the orientable quotient $S/E(S)$. The proof of "if" part is completely achieved.

COROLLARY 2. *If* $\sum_{i=1}^{r}(1 - 1/d_i) = \sum_{i=1}^{t}(1 - 1/d_{r+i})$ *in the above theorem, then*

$$g(S) + k(S/\langle T\rangle) - 1 = \#A(S)\,(g(S/A(S)) + k(S/E(S)) - 1)\,,$$

where $k(X)$ *denotes the number of boundary components of* X.

Proof. By (b) and (e), we get $g + \gamma - 1 = n(m + k - 1)$.

§5. Examples

We shall show the simplest examples on our theorem.

EXAMPLE 1. In the case of $n = 9$ and $m = 1$, our theorem is reduced to the following:
There exists a symmetric Riemann surface S of type $(9,1)$ with $g(S) = g$, $\mathrm{sp}(S) = \gamma$, $E(S) \cong \mathbf{Z}_{18}$ and with the orientable quotient $S/E(S)$ if and only if there exist non-negative integers r_1, r_2, t_1, t_2, $\overbrace{3,\cdots,3}^{r_1+t_1}$ and $\overbrace{9,\cdots,9}^{r_2+t_2}$ such that :

(1) $r_1 + r_2 \geq 2$.

(2) $g = 6r_1 + 8r_2 + 1$.

(3) $0 \leq t_1 + t_2 \leq 2$.

(4) $\gamma = 2(9 - 3t_1 - 4t_2)$.

(5) We put $\mathbf{r} = (r_1, r_2)$ and $\mathbf{t} = (t_1, t_2)$, then

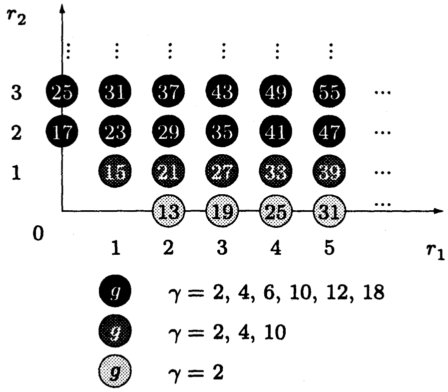
(5.1) $\mathbf{r} = (s, 0),\ s \geq 2 \Rightarrow \mathbf{t} = (0, 2)$,

(5.2) $\mathbf{r} = (s, 1),\ s \geq 1 \Rightarrow \mathbf{t} = (0, 1), (1, 1), (0, 2)$.

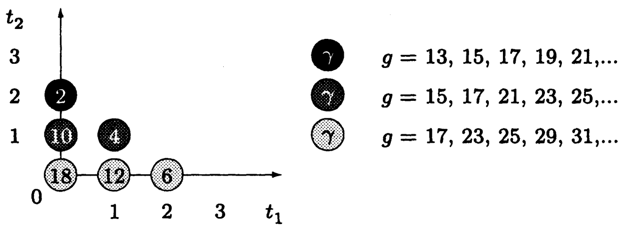
Then the possible genera g and species γ are listed as follows:

g	13	15	17	19	21	23	25	27	29	31	$\cdots$
	2	2	2	2	2	2	2	2	2	2	
		4	4		4	4	4	4	4	4	
γ			6			6	6		6	6	$\cdots$
		10	10		10	10	10	10	10	10	
			12			12	12		12	12	
			18			18	18		18	18	

The following figure illustrates the relation of g, γ and $\mathbf{r}$.



The following figure illustrates the relation of g, γ and $\mathbf{t}$.



The possible g and γ satisfying the equality in Corollary 2 are the following

$$\begin{aligned} g &= 15 \quad \gamma = 4 \quad (\mathbf{r} = \mathbf{t} = (1,1)), \\ g &= 17 \quad \gamma = 2 \quad (\mathbf{r} = \mathbf{t} = (0,2)). \end{aligned}$$

EXAMPLE 2. In the case of $n = 15$ and $m = 1$, our theorem is reduced to the following:
There exists a symmetric Riemann surface S of type $(15,1)$ with $g(S) = g$, $\mathrm{sp}(S) = \gamma$, $E(S) \cong \mathbf{Z}_{30}$ and with the orientable quotient $S/E(S)$ if and

only if there exist non-negative integers $r_1, r_2, r_3, t_1, t_2, t_3$, $\overbrace{3,\cdots,3}^{r_1+t_1}$, $\overbrace{5,\cdots,5}^{r_2+t_2}$ and $\overbrace{15,\cdots,15}^{r_3+t_3}$ such that:

(1) $r_1 + r_2 + r_3 \geq 2$.

(2) $g = 10r_1 + 12r_2 + 14r_3 + 1$.

(3) $0 \leq t_1 + t_2 + t_3 \leq 2$.

(4) $\gamma = 2(15 - 5t_1 - 6t_2 - 7t_3)$.

(5) We put $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{t} = (t_1, t_2, t_3)$, then

(5.1) $\mathbf{r} = (s, 0, 0),\ s \geq 2 \Rightarrow \mathbf{t} = (0, 2, 0), (0, 1, 1), (0, 0, 2)$,

(5.2) $\mathbf{r} = (0, s, 0),\ s \geq 2 \Rightarrow \mathbf{t} = (2, 0, 0), (1, 0, 1), (0, 0, 2)$,

(5.3) $\mathbf{r} = (1, 1, 0) \qquad \Rightarrow \mathbf{t} = (1, 1, 0), (1, 0, 1), (0, 1, 1),$
$(0, 0, 1), (0, 0, 2)$,

(5.4) $\mathbf{r} = (1, s, 0),\ s \geq 2 \Rightarrow \mathbf{t} \neq (0, u, 0),\ u \geq 0$,

(5.5) $\mathbf{r} = (s, 1, 0),\ s \geq 2 \Rightarrow \mathbf{t} \neq (u, 0, 0),\ u \geq 0$,

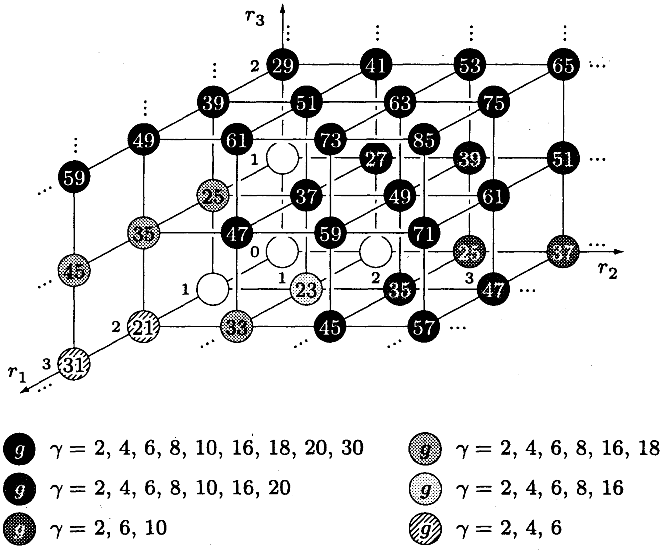
(5.6) $\mathbf{r} = (s, 0, 1),\ s \geq 1 \Rightarrow \mathbf{t} \neq (u, 0, 0),\ u \geq 0$,

(5.7) $\mathbf{r} = (0, s, 1),\ s \geq 1 \Rightarrow \mathbf{t} \neq (0, u, 0),\ u \geq 0$.

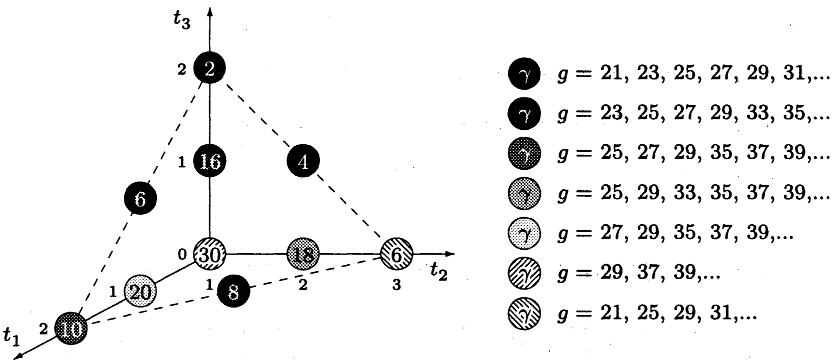
Then the possible genera g and species γ are listed as follows:

g	21	23	25	27	29	31	33	35	37	39	$\cdots$
	2	2	2	2	2	2	2	2	2	2	
	4	4	4	4	4	4	4	4	4	4	
	6	6	6	6	6	6	6	6	6	6	
		8	8	8	8		8	8	8	8	
γ			10	10	10			10	10	10	$\cdots$
		16	16	16	16		16	16	16	16	
			18		18		18	18	18	18	
				20	20			20	20	20	
					30				30	30	

The following figure illustrates the relation of g, γ, and $\mathbf{r}$.



The following figure illustrates the relation of g, γ and $\mathbf{t}$.



The possible g and γ satisfying the equality in Corollary 2 are the following

$$
\begin{array}{lll}
g = 23 & \gamma = 8 & (\mathbf{r} = \mathbf{t} = (1,1,0)), \\
g = 25 & \gamma = 6 & (\mathbf{r} = \mathbf{t} = (1,0,1)), \\
 & & (\mathbf{r} = (1,0,1),\ \mathbf{t} = (0,2,0)), \\
 & & (\mathbf{r} = (0,2,0),\ \mathbf{t} = (1,0,1)), \\
g = 27 & \gamma = 4 & (\mathbf{r} = \mathbf{t} = (0,1,1)), \\
g = 29 & \gamma = 2 & (\mathbf{r} = \mathbf{t} = (0,0,2)).
\end{array}
$$

References

[1] N. l. Alling and N. Greenleaf, Foundations of the theory of Klein surfaces, Lecture Notes in Math., Vol. 219, Springer-Verlag, Berlin-New York, 1971.

[2] E. Bujalance, *Normal subgroups of NEC groups*, Math. Z., **178** (1981), 331–341.

[3] E. Bujalance, A. F. Costa and J. M. Gamboa, Real parts of complex algebraic curves, Lecture Notes in Math., Vol. 1420, Springer, Berlin, 1990, pp. 81–110.

[4] E. Bujalance, J. J. Etayo, J. M. Gamboa and G. Gromadzki, Automorphism groups of compact bordered Klein surfaces, Lecture Notes in Math., Vol. 1439, Springer-Verlag, Berlin, 1990.

[5] E. Bujalance and D. Singerman, *The symmetry type of a Riemann surface*, Proc. London Math. Soc., **51,** No. 3 (1985), 501–519.

[6] A. M. Macbeath, *The classification of non-Euclidean plane crystallographic groups*, Canad. J. Math., **19** (1967), 1192–1205.

[7] C. Maclachlan, *Maximal normal Fuchsian groups*, Illinois J. Math., **15** (1971), 104–113.

[8] D. Singerman, *Finitely maximal Fuchsian groups*, J. London Math. Soc., **6,** No. 2 (1972), 29–38.

[9] D. Singerman, *On the structure of non-Euclidean crystallographic groups*, Proc. Cambridge Philos. Soc., **76** (1974), 233–240.

[10] H. C. Wilkie, *On non-Euclidean crystallographic groups*, Math. Z., **91** (1966), 87–102.

Graduate School of Human Informatics
Nagoya University
Chikusa-ku, Nagoya 464-8601
Japan
nakamura@math.human.nagoya-u.ac.jp