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Schur's theorem for naturally reductive Riemannian S-manifolds

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Abstract

One considers the theorem of Schur for the class of naturally reductive Riemannian S-manifolds. This class includes the nearly Kähler manifolds and the naturally reductive locally s-regular spaces.

1. Introduction

The simplest examples of Riemannian and Kähler manifolds are those of constant sectional and constant holomorphic sectional curvature. Their classification relies on the important theorem of Schur. It plays a similar role in the classification of some other classes of manifolds. For example, it is proved in [9] that Schur's theorem is still valid for the class of nearly Kähler manifolds.

These nearly Kähler manifolds have remarkable properties. See, for example, [2] - [5]. In [3] a lot of examples appear as naturally reductive 3-symmetric spaces. The latter form a subclass of the k-symmetric spaces and the (locally) s-regular manifolds, introduced in [1] and studied extensively in [7]. These locally homogeneous spaces generalize the locally symmetric spaces.

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In [8] the first author extended this further and introduced the notion of Riemannian S-manifold. In particular, since the natural reductivity plays a key role in many properties, he concentrated on the subclass of naturally reductive S-manifolds. This led to a new characterization theorem, in terms of the curvature, for the naturally reductive locally s-regular manifolds. It extends the similar one obtained in [3] for the naturally reductive nearly Kähler manifolds.

The Riemannian S-manifolds are endowed in a natural way with some smooth distributions which are equipped with an almost Hermitian structure. The main purpose of this paper is to discuss Schur's theorem for the corresponding holomorphic sectional curvatures. Hopefully it will be a step towards a classification of these spaces.

2. Preliminaries

In the whole paper (M,g) denotes a connected, smooth, finite-dimensional Riemannian manifold with Levi Civita connection ∇ and associated Riemann curvature tensor R defined by

$$R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

where $X, Y \in \mathcal{X}(M)$, the Lie algebra of smooth vector fields on M.

We start by recalling the definition of the class of manifolds we will consider and we collect some results from [8].

Definition 1. A *Riemannian S-manifold* (M,g,S) is a Riemannian manifold (M,g) together with a (1,1)-tensor field S such that g and ∇S are S-invariant, that is,

$$g(SX,SY) = g(X,Y) \qquad , \qquad (\nabla_{SX}S)SY = S(\nabla_X S)Y$$

for all X, Y in $\mathcal{X}(M)$, and $I - S$ is non-singular.

There is no scarcity of *examples*. Of course, any Riemannian manifold is a $(-I)$-manifold but this aspect is of no interest for our considerations. Further, any Kähler manifold is a J-manifold. Since a quasi-Kähler manifold is an almost Hermitian manifold (M,g,J) such that

$$(\nabla_X J)Y + (\nabla_{JX}J)JY = 0,$$

it is a $(-\frac{1}{2}I + \frac{\sqrt{3}}{2}J)$-manifold. Locally s-regular manifolds M are endowed with a so-called s-structure and they are S-manifolds where $S_m = s_{m*}(m)$ for all $m \in M$ (see [1], [7] for more details). Finally, other examples may be constructed by taking products.

Now, on any (M,g,S) we define a tensor field D of type (1,2) by

$$D(X,Y) = D_X Y = (\nabla_{(I-S)^{-1}X}S)S^{-1}Y,$$

for all $X, Y \in \mathcal{X}(M)$. Then a connection $\bar{\nabla}$ is defined by

$$\bar{\nabla}_X Y = \nabla_X Y - D_X Y,$$

$X, Y \in \mathcal{X}(M)$. It follows easily that $\bar{\nabla}$ is a metric connection, or equivalently

$$g(D_X Y, Z) + g(Y, D_X Z) = 0$$

for all $X, Y, Z \in \mathcal{X}(M)$. Moreover, $\bar{\nabla} S = 0$. Hence, the eigenvalues of S, regarded as a field of orthogonal endomorphisms, are constant. Thus, the eigenvalues of S have the form

$$e^{\pm i\theta_1} = c_1 \pm is_1, ..., e^{\pm i\theta_r} = c_r \pm is_r$$

where $0 < \theta_1$,..., $\theta_r < \pi$, together with -1 as the only possible real eigenvalue.

We make the following

Assumption. In the rest of the paper we assume that -1 does not occur as an eigenvalue for any (M, g, S) under consideration.

Next, associated with $\theta_1, ..., \theta_r$, smooth distributions $\mathcal{D}_i, i = 1, ..., r$ are defined by

$$\mathcal{D}_i = \ker(S^2 - 2c_i S + I).$$

It follows that any $X \in \mathcal{X}(M)$ has a unique decomposition into a sum of *distribution vector fields*, that is, $X = X_1 + ... + X_r$ where $X_i \in \mathcal{D}_i, i = 1, ..., r$. Moreover, we define smooth *projection tensor fields* I_i by $I_i X = X_i$. Finally, an *almost complex structure* J is defined by

$$JX = \sum_{i=1}^{r} \frac{1}{s_i}(S - c_i I) X_i$$

and g is then *almost Hermitian.*

The following useful lemma follows from the S-invariance of ∇S (see [8, Lemma 2.5]):

Lemma 1. *For any i, j, k either*

i) $I_i D_{Y_k} X_j = 0$ for all X_j, X_k, or

ii) $\cos\theta_k = \cos(\theta_i + \alpha_{ijk}\theta_j)$ where the only possibilities are

a) $\alpha_{ijk} = 1$ if $\theta_i + \theta_j + \theta_k = \pi$ or $\theta_k = \theta_i + \theta_j$,

and

b) $\alpha_{ijk} = -1$ if $\theta_j = \theta_k + \theta_i$ or $\theta_i = \theta_k + \theta_j$.

In case ii) we have

$$I_i(J D_{Y_k} X_j + \alpha_{ijk} D_{Y_k} J X_j) = 0 \tag{1}$$

for all X_j, Y_k.

Now, we consider a special class of S-manifolds.

Definition 2. A Riemannian S-manifold is said to be *naturally reductive* if the tensor field T defined by

$$T(X,Y,Z) = g(D_X Y, Z),$$

$X,Y,Z \in \mathcal{X}(M)$, is skew-symmetric.

Note that, since $\bar{\nabla}$ is metric, the condition of skew-symmetry is equivalent to

$$D_X X = 0$$

for all $X \in \mathcal{X}(M)$.

Nearly Kähler manifolds are almost Hermitian manifolds (M, g, J) such that

$$(\nabla_X J)X = 0$$

for all $X \in \mathcal{X}(M)$. So, since they are also quasi-Kählerian, they provide examples of naturally reductive $(-\frac{1}{2}I + \frac{\sqrt{3}}{2}J)$-manifolds.

In [3] a characterization of nearly Kähler (that is, naturally reductive) 3-symmetric spaces is given by means of ∇R. The main result of [8] is an extension of this criterion to the class of naturally reductive locally s-regular manifolds. We have ([8, Theorem 2.2]) :

Theorem 1. *Let (M,g,S) be a naturally reductive S-manifold with associated eigenspace distributions $\mathcal{D}_1, ..., \mathcal{D}_r$. Then (M,g,S) is a locally s-regular manifold with symmetry tensor field S if and only if*

$$(\nabla_{X_i} R)(X_i, JX_i, X_i, JX_i) = 0$$

for each $X_i \in \mathcal{D}_i, i = 1, ..., r$.

From the proof of [8, Theorem 2.2] we have the following relations for R and the curvature tensor $\bar{R}$ of $\bar{\nabla}$. First,

$$\begin{aligned} g(\bar{R}(Z,W)Y,X) &= \bar{R}(X,Y,Z,W) \\ &= R(X,Y,Z,W) + g(D_X Z, D_Y W) - g(D_X W, D_Y Z) \\ &\quad - 2g(D_X Y, D_Z W). \end{aligned} \tag{2}$$

Also

$$\bar{R}(SX,SY,Z,W) = \bar{R}(X,Y,Z,W) \tag{3}$$

and

$$(\bar{\nabla}_V \bar{R})(SX,SY,Z,W) = (\bar{\nabla}_V \bar{R})(X,Y,Z,W) \tag{4}$$

for all $X, Y, Z, V, W \in \mathcal{X}(M)$. From (3) and (4) we see that

$$\bar{R}(X_h, Y_j, Z_k, W_\ell) = 0 \quad \text{unless} \quad X_h, Y_j, Z_k, W_\ell \in \mathcal{D}_i \quad \text{for some } i \tag{5}$$

and similarly

$$(\bar{\nabla}_{V_m}\bar{R})(X_h, Y_j, Z_k, W_\ell) = 0 \quad \text{unless} \quad X_h, Y_j, Z_k, W_\ell, V_m \in \mathcal{D}_i \quad \text{for some } i. \tag{6}$$

Moreover, for each $i = 1, \dots, r$ we have

$$\bar{R}(JX_i, JY_i, Z_i, W_i) = \bar{R}(X_i, Y_i, Z_i, W_i) \tag{7}$$

and

$$(\bar{\nabla}_{V_i}\bar{R})(JX_i, JY_i, Z_i, W_i) = (\bar{\nabla}_{V_i}\bar{R})(X_i, Y_i, Z_i, W_i). \tag{8}$$

3. The theorem of Schur for naturally reductive S-manifolds

Let $\mathcal{D}_i$ be an eigenspace distribution on (M, g, S). We say that $\mathcal{D}_i$ has constant holomorphic sectional curvature K_i if K_i is a smooth function on M such that at each point $p \in M$ the sectional curvature $K_i(p)$ of every J-invariant two-plane P at p contained in $\mathcal{D}_i$ takes the value $K_i(p)$. We prove

Theorem 2. *Let (M, g, S) be a naturally reductive S-manifold with almost complex structure J as defined above and suppose $\mathcal{D}_i$ is an eigenspace distribution of dimension > 2 which has constant holomorphic sectional curvature K_i. Then K_i is constant on M.*

Proof. First, we define tensor fields A, B of type (0,4) by

$$A(X,Y,Z,W) = 2g(D(X,Y), D(Z,W)) + g(D(X,Z), D(Y,W)) - g(D(X,W), D(Y,Z))$$

and

$$\begin{aligned} B(X,Y,Z,W) =& g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + g(X,JZ)g(Y,JW) \\ &- g(X,JW)g(Y,JZ) + 2g(X,JY)g(Z,JW) \end{aligned}$$

for all $X, Y, Z, W \in \mathcal{X}(M)$. Then A and B satisfy the usual Riemannian curvature tensor identities, that is,

a) $T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z)$;

b) $T(X,Y,Z,W) = T(Z,W,X,Y)$;

c) $T(X,Y,Z,W) + T(X,Z,W,Y) + T(X,W,Y,Z) = 0$.

In addition, we have

$$B(JX, JY, Z, W) = B(X, Y, Z, W) \tag{9}$$

and

$$B(X, JX, X, JX) = 4(g(X, X))^2. \tag{10}$$

Also, from Lemma 1 and the skew-symmetry of D, we have

$$D(X_i, JX_i) = 0$$

from which

$$A(X_i, Y_i, Z_i, W_i) - A(JX_i, JY_i, Z_i, W_i) = 4g(D(X_i, Y_i), D(Z_i, W_i)) \tag{11}$$

and

$$A(X_i, JX_i, X_i, JX_i) = 0 \tag{12}$$

for all $X_i, Y_i, Z_i, W_i \in \mathcal{D}_i$.

Next, we note from (2) and (7) that

$$\bar{R}(X_i, JX_i, X_i, JX_i) = R(X_i, JX_i, X_i, JX_i) \tag{13}$$

and

$$R(JX_i, JY_i, Z_i, W_i) - R(X_i, Y_i, Z_i, W_i) = -4g(D(X_i, Y_i), D(Z_i, W_i)). \tag{14}$$

Now, we define the (0,4)-tensor T_i by

$$T_i(X, Y, Z, W) = R(X, Y, Z, W) - A(X, Y, Z, W) - \frac{1}{4}K_i B(X, Y, Z, W).$$

Then T satisfies the curvature identities (a), (b) and (c). Also, for all $X_i, Y_i, Z_i, W_i \in \mathcal{D}_i$, we get

$$T_i(X_i, Y_i, Z_i, W_i) = T_i(JX_i, JY_i, Z_i, W_i)$$

and from the conditions of the theorem

$$T_i(X_i, JX_i, X_i, JX_i) = 0.$$

This implies [6, Chap. IX, p. 166] that $T_i = 0$ when restricted to $\mathcal{D}_i$. Thus

$$R(X_i, Y_i, Z_i, W_i) = \frac{1}{4}K_i B(X_i, Y_i, Z_i, W_i) + A(X_i, Y_i, Z_i, W_i). \tag{15}$$

Also from (2) we have

$$\bar{R}(X_i, Y_i, Z_i, W_i) = \frac{1}{4}K_i B(X_i, Y_i, Z_i, W_i) + 2g(D(X_i, Z_i), D(Y_i, W_i)) \\ - 2g(D(X_i, W_i), D(Y_i, Z_i)). \tag{16}$$

Again from [8], we know that ∇S and $\nabla^2 S$ are S-invariant and it follows from [1] that $\bar{\nabla}(\nabla S) = 0$. So $\bar{\nabla} D = 0$. Now, $X_i \in \mathcal{D}_i$ if and only if

$$(S^2 - 2c_i S + I)X_i = 0$$

and since $\bar{\nabla} S = 0$, we see that, for all $V \in \mathcal{X}(M)$, $\bar{\nabla}_V X_i \in \mathcal{D}_i$. Hence, from (16) we have

$$(\bar{\nabla}_V \bar{R})(X_i, Y_i, Z_i, W_i) = \frac{1}{4} V(K_i) B(X_i, Y_i, Z_i, W_i). \tag{17}$$

Since the torsion tensor field $\bar{T}$ of $\bar{\nabla}$ is defined by

$$\bar{T}(X, Y) = -2D(X, Y),$$

we get, from (17) and the second Bianchi identity,

$$\mathfrak{S}\{(\bar{\nabla}_{V_i}\bar{R})(X_i, Y_i, Z_i, W_i) - 2\bar{R}(X_i, Y_i, D(V_i, Z_i), W_i)\} = 0 \tag{18}$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to V_i, Z_i, W_i. Equivalently, we have from (17) and (18)

$$\mathfrak{S}\{\frac{1}{4} V_i(K_i) B(X_i, Y_i, Z_i, W_i) - 2\bar{R}(X_i, Y_i, D(V_i, Z_i), W_i)\} = 0. \tag{19}$$

Next, for any point $p \in M$ choose $V_i \in T_p M$ and choose a unit vector X_i such that X_i and JX_i are orthogonal to V_i. Then it follows easily from (16) that (19) reduces to $V_i(K_i) = 0$. Further, let $V_j \in T_p M$ for $j \neq i$. Then, from (6), (10) and (17), we also get $V_j(K_i) = 0$. Thus,

$$V(K_i) = 0$$

for all $V \in T_p M$. So, K_i is constant on the connected M as required.

Using this result we finally prove

Theorem 3. *Let (M, g, S) be a naturally reductive S-manifold with almost complex structure as defined above and suppose each $\mathcal{D}_i, i = 1, ..., r$ has dimension > 2 and has constant holomorphic sectional curvature K_i. Then (M, g) is a locally s-regular manifold with symmetry field S.*

Proof. Under the hypotheses, (6), (17) and Theorem 2 yield $\bar{\nabla}\bar{R} = 0$ on M. Hence, from (2) we have $\bar{\nabla} R = 0$ since $\bar{\nabla} g = \bar{\nabla} D = 0$. But then

$$\nabla_X R = D_X R$$

for $X \in \mathcal{X}(M)$ and it follows that ∇R is S-invariant. Then the results from [8] imply that (M, g, S) is a locally s-regular manifold with symmetry tensor field S.

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